Advanced Quantum Field Theory SS 2019

Prof. Andreas Weiler (TUM); Dr. Ennio Salvioni, Dr. Javi Serra

https://www.t75.ph.tum.de/teaching/ss19-quantum-field-theory-ii/

Sheet 6: Goldstone bosons, effective potential (28.06.2019)

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1 Goldstone decay constant

Consider the following Lagrangian for a complex scalar field Φ,

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - \mu^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 \,. \tag{1}$$

We choose $\mu^2 < 0$ so that the degenerate minima of the potential are

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} e^{i\theta}, \qquad \theta \in (0, 2\pi] \tag{2}$$

which defines v (the factor of $\sqrt{2}$ is inserted by convention). Parametrize the fields as

$$\Phi = e^{i\chi(x)/v} \tfrac{1}{\sqrt{2}} (v + \rho(x)), \tag{3}$$

where χ is identified with the massless Goldstone boson of the spontaneously broken $U(1)$ symmetry.

a) Calculate the Noether current j^μ in terms of the ρ and χ fields.

b) Define the decay constant F_χ of the Goldstone boson via

$$\langle \chi(k) | j^\mu(x) | 0 \rangle = i F_\chi k^\mu e^{+ikx}, \tag{4}$$

where $|\chi(k)\rangle$ is the state of a single Goldstone boson with four momentum k^μ, and $|0\rangle$ is the $\theta = 0$ vacuum. Compute F_χ at tree level, that is, by using the result of **a**) for the current and employing free field theory to evaluate the matrix element. A nonvanishing value of F_χ indicates that the corresponding current is spontaneously broken.

c) Discuss how this result would be modified by higher-order corrections.

2 Effective potential

a) Consider the effective potential in (0 + 1)-dimensional spacetime:

$$V_{\text{eff}}(\varphi) = V(\varphi) + \frac{\hbar}{2} \int \frac{dk_E}{2\pi} \log \frac{k_E^2 + V''(\varphi)}{k_E^2} + O(\hbar^2) \,. \tag{5}$$

Do we need counter-terms? Since (0+1)-dimensional field theory is just quantum mechanics, evaluate the integral and show that V_{eff} is in complete accord with your knowledge of quantum mechanics.

b) Compute the one-loop Coleman-Weinberg effective potential $V_{\text{eff}}(\phi)$ in the Φ^4 theory (in 1 + 3 dimensions),

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi^4 \,, \tag{6}$$

by explicitly summing up the 1PI n-point functions for vanishing external momenta. Start with

$$V_{\text{eff}} - V_0 = -\sum_{n=1} \frac{1}{n!}\phi^n \tilde{\Gamma}^{(n)}(p_1 = 0, \ldots, p_n = 0) \tag{7}$$

where $\tilde{\Gamma}^{(n)}(p_i = 0)$ denotes de 1PI n-point function in momentum space with all external momenta set to zero and V_0 is a field-independent contribution, which does not have to be computed. Then draw the relevant one-loop diagrams and show that the symmetry factor (defined as the number of contractions that lead to the same diagram) is given by $(4!)^{n/2}(n-1)!/2^{n/2}$.

3 [*central tutorial*] The Gross-Neveu model

The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2}g_0^2(\bar{\psi}_i\psi_i)^2 \tag{8}$$

with $i = 1, \ldots, N$. The kinetic term of two-dimensional fermions is built from matrices γ_μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma_0 = \sigma_2\,, \quad \gamma_1 = i\sigma_1 \tag{9}$$

where σ_i are Pauli sigma matrices. One can define

$$\gamma_5 = \gamma_0\gamma_1 = \sigma_3\,, \tag{10}$$

which anticommutes with γ_μ.

a) Show that this theory is invariant with respect to the transformation $\psi_i \to \gamma_5\psi_i$ and that this symmetry forbids a fermion mass.

b) Show that this theory is renormalizable (at the level of dimensional analysis).

c) Show that the functional integral for this theory can be represented in the following form:

$$\int \mathcal{D}\psi \exp\left[i\int d^2x\mathcal{L}\right] = \int \mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int d^2x\left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g_0^2}\sigma^2\right\}\right] \tag{11}$$

where $\sigma(x)$ (not to be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

d) Compute the leading correction to the effective potential for σ by integrating over the fermion fields ψ_i. You will encounter the determinant of a Dirac operator; to evaluate this determinant go to momentum space. This contribution requires a renormalization of g_0^2. Renormalize by minimal substraction.

e) Ignoring two-loop and higher-order contributions, minimize this potential. Show that the σ field acquires a vacuum expectation value which breaks the symmetry of part *(a)*. Convince yourself that this result does not depend on the particular renormalization condition chosen.

f) Note that the effective potential derived in part *(d)* depends on g_0 and N according to the form

$$V_{\text{eff}}(\sigma) = N \cdot f(g_0^2 N) \tag{12}$$

Convince yourself that in the limit $N \to \infty$ with $g_0 N^2$ fixed our analysis in part *(d)*, based on the stationary point of the effective action (constant σ), is justified.