Advanced Quantum Field Theory SS 2019

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https://www.t75.ph.tum.de/teaching/ss19-quantum-field-theory-ii/

Sheet 5: Gauge invariance from the bottom up, spontaneous symmetry breaking (14.06.2019)

1 Spin-3 kinetic term

Construct the free kinetic Lagrangian for a spin-3 particle by:

a) Decompose the tensor product of Lorentz representations

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) \tag{1}$$

into its irreducible representations and decompose it further to its irreducible representations of spin $so(3)$. Show then why a spin-3 particle can be embedded in a tensor $\phi_{\mu\nu\sigma}$ symmetric under the exchange of any pair of indices.

b) Identify the (linear) gauge transformations of $\phi_{\mu\nu\sigma}$ that realize its Stueckelberg decomposition into massless fields.
Hint: Impose the condition that the trace of the entire Stueckelberg field vanishes: $\eta^{\mu\nu}\xi_{\mu\nu}=0$.

c) Derive the coefficients of the most general kinetic Lagrangian for $\phi_{\mu\nu\rho}$,

$$\mathcal{L}_{kin} = a(\partial_\sigma\phi_{\mu\nu\rho})^2 + b(\partial^\mu\phi_{\mu\nu\rho})^2 + c(\partial_\mu\phi_\nu)^2 + d(\partial^\mu\phi_\mu)^2 + e\partial^\mu\phi^\rho\partial^\nu\phi_{\mu\nu\rho}\,, \quad \phi_\mu \equiv \phi_\mu{}^\nu{}_\nu \tag{2}$$

by requiring that it does not give rise to (problematic) kinetic terms for the Stueckelberg fields.

2 $O(3)$ sigma model

Consider the $O(3)$-invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - V(\vec{\phi}), \qquad V(\vec{\phi}) = \mu^2\phi^i\phi^i + \lambda(\phi^i\phi^i)^2, \tag{3}$$

where ϕ^i $(i=1,2,3)$ are real scalar fields.

a) Calculate the Noether currents associated with the $O(3)$ symmetry.

b) Assuming that $\mu^2<0$ and $\lambda>0$, minimize V and show that the vacua $\langle\vec{\phi}\rangle$ lie on a (2-)sphere, whose radius f is determined by μ^2 and λ. By convention, take $\langle\vec{\phi}\rangle$ to lie along the direction of ϕ^3. Which generators leave the vacuum invariant? Writing $\phi^3 = f+\eta(x)$, find the masses of $\phi^{1,2}$ and η.

c) Parametrize the fields as

$$\vec{\phi} = e^{i\frac{\pi^a(x)}{f}X^a}\begin{pmatrix}0\\0\\f+h(x)\end{pmatrix}, \qquad \pi^a(x)X^a = -i\begin{pmatrix} & & \pi^1\\ & & \pi^2\\ -\pi^1 & -\pi^2 & \end{pmatrix} \tag{4}$$

where X^a (a = 1, 2) are the broken generators. Derive the infinitesimal transformations of the fields, and use them to express the Noether currents in terms of π^a and h.

d) Now add to the potential a term $\delta V = a\phi_3$ that explicitly breaks the $O(3)$ symmetry. Minimize $V + \delta V$ using the perturbative ansatz

$$\langle\phi^3\rangle = \langle\phi^3\rangle_0 + a\langle\phi^3\rangle_1 + O(a^2), \tag{5}$$

and find the expressions of $\langle\phi^3\rangle_0$ and $\langle\phi^3\rangle_1$. Two solutions exist for the former, but only one corresponds to an actual minimum of the potential. Identify the minimum, expand around it using $\phi^3 = \langle\phi^3\rangle + \eta(x)$ and determine the masses of $\phi^{1,2}$ and η at $O(a)$. Compare the results to those you obtained in point **b**).