# Advanced Quantum Field Theory SS 2019

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https://www.t75.ph.tum.de/teaching/ss19-quantum-field-theory-ii/

**Sheet 4: Gravity as gauge theory, gauge interactions** (07.06.2019)

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## 1 Vanishing of torsion in pure gravity

Consider the Einstein action

$$S_E = \frac{1}{16\pi G}\int d^4x\, E\, R_{mn}{}^{mn}\,, \tag{1}$$

where $E = \det(e_\mu^m)$. In this problem we prove that the variation of $S_E$ with respect to the connection implies vanishing torsion.

**a)** By calculating the variation of $S_E$ with respect to the connection, show that in the absence of matter the equations of motion take the form

$$D_\mu[E(e_m^\mu e_n^\rho - e_n^\mu e_m^\rho)] = 0\,. \tag{2}$$

**b)** Prove that Eq. (2) can be equivalently written in the form

$$e_q^\rho S_{mn}{}^q \,-\, e_m^\rho S_{qn}{}^q \,+\, e_n^\rho S_{qm}{}^q \,=\, 0\,, \tag{3}$$

and show that this last equation implies that the torsion coefficients vanish identically, $S_{mn}{}^q\, = 0\,.$

## 2 Einstein's equations

Derive the equations of motion of pure gravity

$$S_E = \frac{1}{16\pi G}\int d^4x\, E\, R_{mn}{}^{mn}\,, \tag{4}$$

by demanding a vanishing variation of the action with respect to the vierbein, $e_\mu^m$.

Recall that the Riemann tensor can be written as

$$R_{mn}{}^{pq} = (e_m^\mu e_n^\rho - e_n^\mu e_m^\rho)(\partial_\mu \omega_\rho{}^{pq} - \omega_\mu{}^{rp}\omega_{\rho r}{}^q)\,. \tag{5}$$

As intermediate steps, you will need to show that the variation of $E = \det(e_\mu^m)$ is given by

$$\delta E = E e_m^\mu \delta e_\mu^m\,, \tag{6}$$

as well as the identity

$$\delta e_m^\mu = -e_m^\lambda e_q^\mu \delta e_\lambda^q\,. \tag{7}$$

## 3 Spin-1 and Spin-2 interactions

**a)** Derive the leading interactions of a spin-1 $U(1)$ gauge field, $A_\mu$, with a complex scalar, $\phi$, by requiring that upon the transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \pi(x) \tag{8}$$

such interactions do not depend on the longitudinal mode $\pi$. Note the complex scalar will need to transform as well,

$$\phi \to \phi + i\, a\, \pi\, \phi + b\, \pi^2\, \phi + \cdots\,, \tag{9}$$

where the parameters $a, b, \ldots$ will need to be determined as well.

*Hint*: Start from the interaction

$$\mathcal{L} \supset -iA_\mu(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \tag{10}$$

and derive the complete set of interactions iteratively.

**b)** Derive the leading interactions of a spin-2 gauge field, $h_{\mu\nu}$, with a real scalar, $\phi$, by requiring that upon the transformation

$$h_{\mu\nu} \to h'_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu\,, \quad h'_{\mu\nu} = h_{\mu\nu} + \pi^\alpha \partial_\alpha h_{\mu\nu} + (\partial_\mu \pi^\alpha) h_{\alpha\nu} + (\partial_\nu \pi^\alpha) h_{\mu\alpha} \tag{11}$$

such interactions do not depend on the vector mode $\pi_\mu$ at linear order. Note the scalar will need to transform as well,

$$\phi \to \phi + \pi^\alpha \partial_\alpha \phi\,. \tag{12}$$

*Hint*: Start from the interaction

$$\mathcal{L} \supset \frac{1}{2} h \phi\,, \tag{13}$$

where $h \equiv h^\mu_\mu$, show that the Lagrangian must at least contain

$$\mathcal{L} \supset \phi + \frac{1}{2} h \phi \tag{14}$$

in order to cancel terms with two fields, and determine the complete set of interactions by iteratively cancelling terms with increasing number of $h$ fields.

Recall that both Eq. (11) and Eq. (12) follow from general coordinate transformations $x \to x + \pi$,

$$h_{\mu\nu} \to (\delta^\alpha_\mu + \partial_\mu \pi^\alpha)(\delta^\beta_\nu + \partial_\nu \pi^\beta)[\eta_{\alpha\beta} + h_{\alpha\beta}(x^\gamma + \pi^\gamma)] - \eta_{\mu\nu}\,, \tag{15}$$

$$\phi \to \phi(x^\alpha + \pi^\alpha)\,, \tag{16}$$

at linear order in $\pi$.