# Advanced Quantum Field Theory SS 2019

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https://www.t75.ph.tum.de/teaching/ss19-quantum-field-theory-ii/

**Sheet 1: Non-abelian gauge invariance** (10.05.2019)

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## 1 Geometrical origin of the Abelian field strength

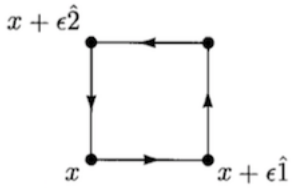
Consider the comparator $U(y,x)$, a scalar quantity that depends on two spacetime points $x$ and $y$ and transforms as $U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$ under a local $U(1)$ transformation acting on matter fields as $\psi(x) \to e^{i\alpha(x)}\psi(x)$. In this picture the gauge vector field is identified with the connection $A_\mu(x)$. Here we use $U(y,x)$ to derive geometrically the form of the locally invariant kinetic term for $A_\mu$.

**a)** Assuming that the comparator is a pure phase, i.e. $U(y,x) = e^{i\phi(y,x)}$, and that in addition it satisfies $[U(x,y)]^\dagger = U(y,x)$, show that

$$U(x+\epsilon n, x) = \exp\big[-ie\epsilon n^\mu A_\mu(x+\tfrac{\epsilon}{2}n) + O(\epsilon^3)\big] \tag{1}$$

for small $\epsilon$, where $n^\mu$ is an arbitrary vector and the constant $e$ was extracted arbitrarily.

**b)** Define $\mathbf{U}(x)$ as the product of the four comparators calculated along the following loop




where $\hat{1},\hat{2}$ are the unit vectors in the $(1,2)$ plane. Prove that $\mathbf{U}(x)$ is invariant under local transformations. Then, using the result in **a)**, show that

$$\mathbf{U}(x) = 1 - ie\epsilon^2\big[\partial_1 A_2(x) - \partial_2 A_1(x)\big] + O(\epsilon^3). \tag{2}$$

Conclude that the object

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

is locally invariant, and therefore suited to construct the kinetic term for the vector field.

## 2 Adjoint representation and gauge transformations

**a)** Consider a Lie algebra

$$[T^a, T^b] = if^{abc}T^c, \qquad a,b,c = 1,\ldots,k \tag{4}$$

where the $f^{abc}$ are totally antisymmetric real coefficients. The $f^{abc}$ can be regarded as $k$ $(k\times k)$ matrices, $(\mathbf{f}^a)_{bc} = qf^{abc}$ where $q$ is a purely imaginary number. Using the Jacobi identity show that these $k\times k$ matrices obey, for an appropriate choice of $q$, the same Lie algebra as the $T^a$.

**b)** Show that if $\omega$ transforms in the adjoint representation of $SU(N)$, its covariant derivative is given by $\mathbf{D}_\mu\omega = \partial_\mu\omega + i[\mathbf{A}_\mu,\omega]$ where $\mathbf{A}_\mu$ is the matrix of gauge fields, and transforms in the same way as $\omega$.

**c)** Show from the gauge transformation properties of $\mathbf{A}_\mu$ that the field strength $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + i[\mathbf{A}_\mu,\mathbf{A}_\nu]$ does indeed transform as the adjoint of $SU(N)$.

## 3 Winding number

Consider a map from $S^1 \to S^1$. Given a smooth map $U(\phi)$, its winding number can be written as

$$n = \frac{i}{2\pi} \int_0^{2\pi} d\phi \, U \partial_\phi U^\dagger. \tag{5}$$

**a)** Show that e.g. $U(\phi) = e^{in\phi}$ has winding number $n$.

**b)** Assuming that $UU^\dagger = 1$ (as for gauge transformations), we will show that $n$ is invariant under any smooth transformation. Consider an infinitesimal transformation $U \to U + \delta U$. Show that $\delta U^\dagger = -U^{\dagger 2} \delta U$.

**c)** Use this result to show that $\delta(U \partial_\phi U^\dagger) = -\partial_\phi (U^\dagger \delta U)$.

**d)** Use this to show that $\delta n = 0$. Since any smooth deformation can be generated by compounding infinitesimal ones, this proves that $n$ is invariant under any smooth transformation.