Lecture @ TUM   summer 2018   by B. Bellazzini
(25-26/07/2018)

Aspects of (Micro)Causality in QFT

- introductory lecture for PhD students in High-energy physics
- quite heuristic, little rigor
- for simplicity discuss only scalar operators

→ Goal: illustrate how causality can be used (along w/ unitarity...) to constrain S-matrix elements at low-energy, i.e. EFT's.

(→ secondary goal: show that causality is an essential ingredient in QFT's.)

By (micro-)causality in a relativistic QFT we usually mean that commutators of local operators at spacelike distance vanish, i.e.

$$(1) \qquad [\,\mathcal{O}(x_1)\,,\ \mathcal{O}(x_2)\,] = 0 \qquad \text{for} \quad (x_1 - x_2)^2 < 0$$

(↖ denoted as x_{12}^2 below as well as $x_1^0 - x_2^0 = x_{12}^0 \ldots$)

Let's go through some examples where causality in the form (1) is relevant (of course spin-statistics, CPT,...)

(a) Lorentz invariance of perturbative S-matrix

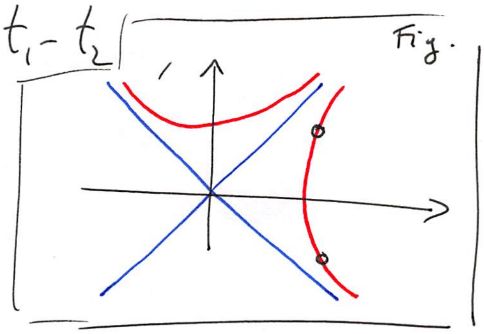
In perturbation theory the Dyson series $S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \ldots dt_n \, \overset{\text{time ordering}}{T}\big(V(t_1) \ldots V(t_n)\big)$ is a formal solution for the asymptotic time evolution operator

$U(\infty, -\infty) \equiv S$ with $\frac{d}{dt} U(t,t_0) = V(t)\, U(t,t_0)$ in the interacting picture

Locality suggests interactions at a point, $V(t) = \int d^3x\, H(t,x)$, for some Hamiltonian density $H(x) = H(t,x)$ so that the Dyson series is almost trivially Lorentz invariant, namely

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n \, T\left(H(x_1) \dots H(x_n) \right) ,$$

The trouble being that time ordering is not Lorentz invariant. Indeed, for spacelike separated points ($x_{12}^2 < 0$), a sufficiently large boost can reverse the sign of $t_{12} = t_1 - t_2$, as illustrated in the figure ⟶



But this problem is solved precisely by the causality condition (1), $[O(x_1), O(x_2)] = 0$ for $x_{12}^2 < 0$, since at spacelike separated points the operators commute and any ordering is irrelevant.

- This is an example where it was important that (1) holds as operator statement, that is for any type of states $\langle \alpha | [\ ,\ ] | \beta \rangle$ one wishes to consider.

Below we see two examples where we restrict to $|\alpha\rangle = |0\rangle$, the single most important state of the system, the vacuum. Let's see how (1) come to be true on that state.

(b) Källén-Lehmann

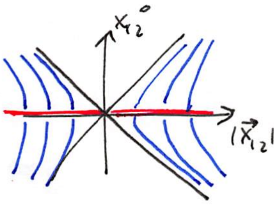
One of the reasons why causality (1) holds on vacuum states is because of the Källén-Lehmann spectral representation. Indeed, for a free-field $\phi(x)$ the (1) is trivial (e.g. from canonical quantization)

$$[\phi(x_1), \phi(x_2)] = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\vec{x}_{12}}}{2\omega(k)} \left(e^{-i\omega(k)x_{12}^0} - e^{i\omega(k)x_{12}^0}\right)$$

(2)

$$= \int \frac{d^4k}{(2\pi)^4} (2\pi)\delta^4(k^2-m^2)\,\epsilon(k^0)\,e^{-ikx_{12}} \equiv i\Delta_{m^2}(x_{12})$$

given that at equal time, $x_{12}^0 = 0 \Rightarrow [\phi(x_1), \phi(x_2)]\Big|_{x_{12}^0} = 0$
$\Rightarrow$ vanish at spacelike distance by Lorentz invariance



⌈ Incidentally, $[\phi(x_1), \phi(x_2)]$ solves $(\Box + m^2)\Delta_{m^2}(x_{12}) = 0$ with boundary conditions $\partial_{x_1^0}\Delta_{m^2}(x_{12})\Big|_{x_{12}^0=0} = -\delta^3(\vec{x}_{12})$, $\Delta_{m^2}(x_{12})\Big|_{x_{12}^0=0} = 0$ ⌋

Now, these properties are readily inherited by an interacting field $\vartheta(x)$ via Källén-Lehmann

$$\langle 0|[\vartheta(x_1), \vartheta(x_2)]|0\rangle = \int_0^\infty d\mu^2\, \rho(\mu^2)\, i\Delta_{\mu^2}(x_{12}) \qquad (3)$$

with $\boxed{\rho(\mu^2) > 0}$ the positive density of states

⌈ Recall: $\langle 0|\vartheta(x_1)\vartheta(x_2)|0\rangle = \sum_{m,\sigma} e^{-ip_m x_{12}} |\langle 0|\vartheta(0)|m,\sigma\rangle|^2 = \sum \int \frac{d^4p}{(2\pi)^4} (2\pi)^4\delta^4(p-p_m)\, e^{-ipx_{12}} |\langle 0|\vartheta(0)|m,\sigma\rangle|^2$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ipx_{12}}\, \theta(p^0)\, \rho(p^2) \quad \text{where} \quad \theta(p^0)\rho(p^2) \equiv \sum (2\pi)^4\delta^4(p-p_m)|\langle 0|\vartheta(0)|m,\sigma\rangle|^2$$

which is Lorentz invariant. By inserting $\int d\mu^2\,\delta(p^2-\mu^2) = 1$ and changing order

$\rightarrow \int_0^\infty d\mu^2\,\rho(\mu^2) \int \frac{d^4p}{(2\pi)^4} e^{-ipx_{12}}\theta(p^0)\delta(p^2-\mu^2) \Rightarrow \langle 0|[\vartheta(x_1), \vartheta(x_2)]|0\rangle \underset{(2)}{=} \int_0^\infty d\mu^2\,\rho(\mu^2)\, i\Delta_{\mu^2}(x_{12})$ ⌋

(c) Euclidean 2pt-function

Causality condition (1) is intrinsically formulated in Minkoski space. However, it's very instructive to see how it is actually built-in and inherited from the Euclidean definitions of correlators that are Wick rotated.

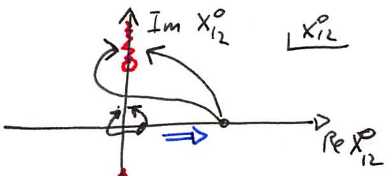
In the Euclidean, fields commute $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\dots\rangle^E = \int D\mathcal{O}\, e^{-S_E[\mathcal{O}]}\mathcal{O}(x_1)\mathcal{O}(x_2)\dots$ as long as $x_1 \neq x_2$ where the singularity are located.

For the 2pt function, the SO(4) invariance give $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle^E = F(x_{12}^{0\,2}+\vec{x}_{12}^{\,2})$ and singularity appear only at the Euclidean origin $x_{12}^0 = \vec{x}_{12} = 0$.

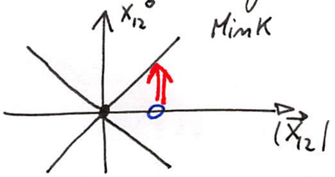
Let's take a CFT 2pt-function as explicit example

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle^E = \frac{1}{(x_{12}^{0\,2}+\vec{x}_{12}^{\,2})^\Delta}\,.$$

For complex time x_{12}^0 at fixed $\vec{x}_{12}$ things become more interesting:



For x_{12}^0 close to the origin (with $\vec{x}_{12}\neq 0$) we encounter no singularity by Wick rotation $x_{12}^0 \to i x_{12}^0$ that moves to imaginary time. But, as $|x_{12}^0|$ reaches $|\vec{x}_{12}|$ we hit the Lightcone and singularity and ambiguities appear.



Reaching the point at the lightcone $|\vec{x}_{12}^0| = |\vec{x}_{12}|$ or inside depends on the path: from right gives T-ordered correlators, from left anti-T-ordered correlators

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle^E \nearrow \langle T\,\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \quad \text{say } x_{12}^0>0$$

$$\searrow \langle \bar{T}\,\cancel{\mathcal{O}}\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle$$

The discontinuity is the now non-vanishing commutator inside the lightcone. Outside the lightcone there was no ambiguity and the commutator vanishes as inherited by regularity of the Euclidean correlator. Moral: from Euclid to Mink we move from outside the lightcone (away from origin) into the lightcone.

(d) Linear Response Theory

We have seen that causality (1) is implied by Kallen-Lehmann and in fact trivially implemented in an Euclidean approach. It was important in order to recover Lorentz invariance of S-matrix. Let' see now another example where causality (1) is crucial for consistency with the relation of cause and effect, hence justifying the name "causality" itself.

We want to see the VEV of $\mathcal{O}(x)$ as result of turning-on a localized perturbance (or source) $J(x)$:

$$H(\mathcal{O}(x)) \longrightarrow H(\mathcal{O}(x)) + J(x)\,\mathcal{O}(x)$$

In the interacting-picture we get (equal time VEV, rather than in-out S-matrix elements)

$$(4)\quad \underbrace{\langle \mathcal{O}(x)\rangle_J}_{\substack{\text{Heisenberg pict.}\\ \text{with vacuum at } J\neq 0}} = \underbrace{\langle 0|\, U^{+}(x^0,-\infty)\, \mathcal{O}(x)\, U(x^0,-\infty)|0\rangle}_{\substack{\text{interacting picture with}\\ \text{vacuum at } J=0}} = \langle 0|\underset{\substack{\uparrow\\ \text{anti time ordering}}}{\bar{T}} e^{i\int_{-\infty}^{x^0} dy^0 d^3y\, J(y)\mathcal{O}(y)}\, \mathcal{O}(x)\, T e^{-i\int_{-\infty}^{x^0} dy^0 d^3y\, J}|0\rangle$$

$$= \langle \mathcal{O}(x)\rangle_{J=0} - i\int_{-\infty}^{x^0} dy^0 \int d^3y\, \langle [\mathcal{O}(x), \mathcal{O}(y)]\rangle_{J=0}\, J(y) + \ldots$$

This is nothing but the Kubo Formula for linear response (say $\langle \mathcal{O}(x)\rangle$ is some electric current induced by an external electric field $E=J$, this way one calculates the resistivity)

What's interesting about this formula is the causal structure

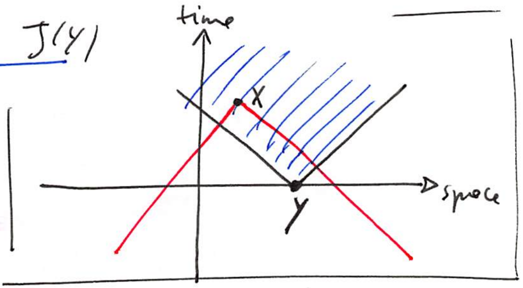
We can rewrite it as $\boxed{\langle \mathcal{O}(x)\rangle_J = \langle \mathcal{O}(x)\rangle_{J=0} \cancel{+} i\int d^4x\, G^R(x-y)\, J(y) + \dots}$

where we introduced the retarded Green-function

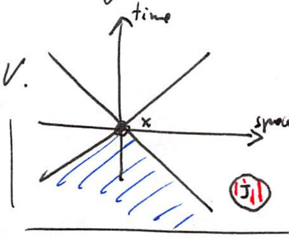
(5) $$\boxed{G^R(x-y) = \cancel{i}\ \theta(x^0-y^0)\langle[\mathcal{O}(x), \mathcal{O}(y)]\rangle_{J=0}}$$

~~(assuming translat. invariance left J=0 for simplicity)~~

Notice that $G^R(x-y)$ vanishes everywhere except in the future lightcone of the perturbance $J(y)$



Likewise, the vev at x receive contributions only by sources that have support in its past lightcone.

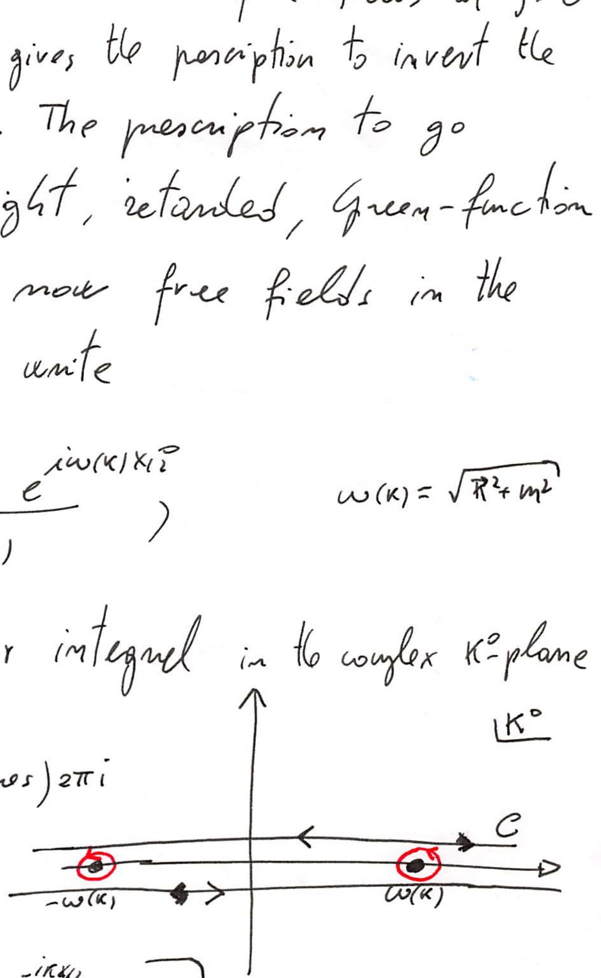


A source J(x) outside the past lightcone of x will return zero vev.
The theory is causal and perturbances travel subluminal in the sense that nothing leaks outside the lightcone.
The G^R is the Green-function because for perfectly localized source $J(y)=\delta^4(y)$ one gets $\delta\langle\mathcal{O}\rangle = G$, and moreover often works with free fields at $J=0$ so that indeed $(\Box+m^2)G = -i\delta^4$, provided one gives the prescription to invert the differential operator, as we are going to do now. The prescription to go around the poles is what makes G^R the right, retarded, Green-function for this problem. More specifically, Take now free fields in the vacuum: we have seen that in (P3) we could write

(6) $$[\mathcal{O}(x_1), \mathcal{O}(x_2)] = \int \frac{d^3K}{(2\pi)^3} e^{i\vec{K}\vec{x}_{12}} \left(\frac{e^{-i\omega(K)x^0_{12}} - e^{i\omega(K)x^0_{12}}}{2\omega(K)} \right) \qquad \omega(K) = \sqrt{\vec{K}^2+m^2}$$

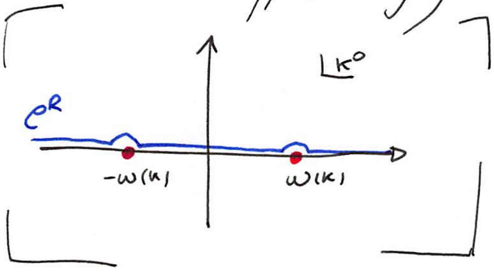
that we can obtain from the following contour integral in the complex K^0-plane

(7) $$\int_C \frac{d^4K}{(2\pi)^4} e^{-iKx_{12}} \frac{-i}{K^2-m^2} = (\text{sum of two residues})\,2\pi i$$

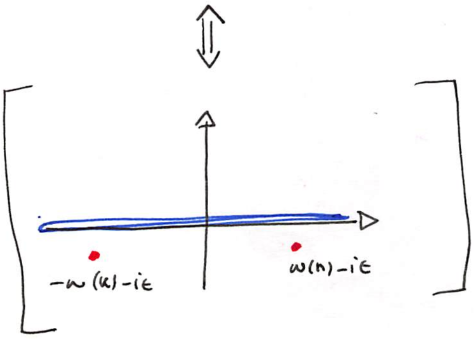


$$\left[K^2-m^2 = (K^0+\omega(\vec{K}))(K^0-\omega(\vec{K})) \Rightarrow \lim_{K^0\to\omega}(K^0-\omega)\frac{e^{-iKx_{12}}}{K^2-m^2} + \lim_{K^0\to-\omega}(K^0+\omega)\frac{e^{-iKx_{12}}}{K^2-m^2} = \dots\right]$$

Now, the retarded green function $\theta(x_1^0-x_2^0)\langle[\mathcal{O}(x_1),\mathcal{O}(x_2)]\rangle$ satisfies the same equation of motion → has the same kernel but integrated over a different contour (that is, poles moved around differently)



$$\rightarrow \int_{C^R}\frac{d^4k}{(2\pi)^4}\, e^{-ikx_{12}}\,\frac{-i}{k^2-m^2} = \int\frac{d^3k}{(2\pi)^3}\, e^{+i\vec{k}\vec{x}_{12}}\int\frac{dk^0}{(2\pi)}\, e^{-ik^0x_{12}^0}\,\frac{-i}{(k^0-\omega+i\epsilon)(k^0+\omega+i\epsilon)}$$

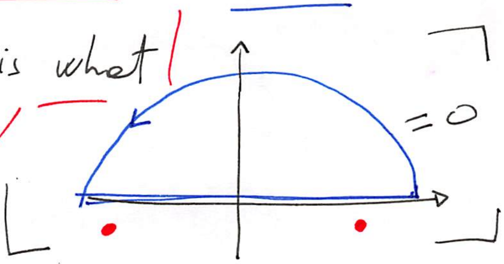


$$= \theta(x_{12}^0)\,\underbrace{\text{(close contour on lower plane)}}_{\text{sum of 2 residues}} + \theta(-x_{12}^0)\,\underbrace{\text{(close contour on upper plane)}}_{=\,0}$$

as it is analytic there!!

$$= \theta(x_{12}^0)\langle[\mathcal{O}(x_1),\mathcal{O}(x_2)]\rangle \qquad (8)$$

The analyticity in the upper complex k^0-plane is what gives the correct ~~one~~ causal structure.



A couple of remarks:

- we have worked with free fields at $J=0$, but this is not important as one can see applying directly Källen-Lehmann $\Delta_{m^2}(x_{12}) \to \int d\mu^2\rho(\mu^2)\Delta_{\mu^2}$ and $G^R_{m^2}(x_{12}) \to \int d\mu^2\rho(\mu^2)G^R_{\mu^2}(x_{12})$

[caveat: K.L. may, in principle, not be converging; one would need to take subtractions when $\Delta_{\mathcal{O}}$ = scale-dim. of $\mathcal{O}$ is larger = than 2]

- We have studied this problem in the vacuum, ~~but the~~ how things change in other states given that microcausality holds for any state?

→ For example, what does happen if we are in a thermal bath at finite temperature T? This is actually simple for a free theory at $J=0$ since $\langle\beta|[\mathcal{O}(x_1),\mathcal{O}(x_2)]|\beta\rangle_{J=0} = \langle[\mathcal{O}(x_1),\mathcal{O}(x_2)]\rangle_{\substack{J=0\\T=0}}$ as it is well known.

(P8)

This is so because because finite temperature is nothing but ~~than~~ a different representation (than Fock $a_k|0\rangle$) of the same canonical commutation relation $\begin{cases}[a_k, a_p] = 0 \\ [a_k, a_p^{\dagger}] = (2\pi)^3 \delta^3(k-p)\end{cases}$

$$\Rightarrow \langle\beta|[\mathcal{O}(x_1), \mathcal{O}(x_2)]|\beta\rangle = \langle\beta=\infty|[\mathcal{O}(x_1), \mathcal{O}(x_2)]|\beta=\infty\rangle$$

$(\beta = 1/T)$

even though $\langle\beta|\mathcal{O}(x_1)\,\mathcal{O}(x_2)|\beta\rangle$ is different than at zero temperature

As result, the commutator still vanishes at spacelike distance.

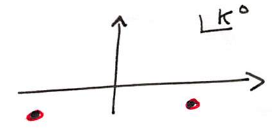
$$\left[\ \langle\beta|a_k^{\dagger} a_p|\beta\rangle \equiv \frac{\mathrm{Tr}[e^{-\beta H} a_k^{\dagger} a_p]}{\mathrm{Tr}[e^{-\beta H}]} \underset{t=i\beta}{=} \frac{\mathrm{Tr}[a_k^{\dagger} e^{-\beta H} a_p]}{\mathrm{Tr}[e^{-\beta H}]} e^{-\beta\omega(k)} \underset{\text{cyclicity}}{=} e^{-\beta\omega(k)} \langle\beta|a_p a_k^{\dagger}|\beta\rangle\ \right]$$

$$\left(\begin{array}{l} e^{itH} a_k e^{-itH} = e^{i\omega(k)t} a_k \\ H = \int \frac{d^3k}{(2\pi)^3}\, \omega(k)\, a_k^{\dagger} a_k \end{array} \right)$$

$$\underset{[a,a^{\dagger}]=\delta}{=} e^{-\beta\omega(k)} \left[(2\pi)^3 \delta^3(k-p) + \langle\beta|a_k^{\dagger} a_p|\beta\rangle \right]$$

$$\Rightarrow \begin{cases} \langle\beta|a_k^{\dagger} a_p|\beta\rangle = \dfrac{1}{e^{\beta\omega(k)}-1} (2\pi)^3 \delta^3(k-p) \\ \langle\beta|a_p a_k^{\dagger}|\beta\rangle = \dfrac{1}{1-e^{-\beta\omega(k)}} (2\pi)^3 \delta^3(k-p) \\ \langle\beta|[a_k, a_p^{\dagger}]|\beta\rangle = (2\pi)^3 \delta^3(k-p) \end{cases}$$

$\Rightarrow$ same analytic structure for $[\mathcal{O}, \mathcal{O}]$



and hence, same causal structure.

It's interesting trying to calculate $\langle\beta|[\mathcal{O}, \mathcal{O}]|\beta\rangle$ from the Matsubara KK mode compactifying along a complex time direction (given that $\langle T\,\mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle_{\beta} = \langle T\,\mathcal{O}(x_1^0 + i\beta, \vec{x}_1)\,\mathcal{O}(x_2^0, \vec{x}_2)\rangle_{\beta}$)

This should be totally analogous to the spatial compactification studied in 0703.1483, once we had interactions in the bulk.

- suppose now we are in a medium (breaking Lorentz spontaneously like before but now at $T=0$) whose only effect is changing the refraction index

(9) $\boxed{n(\omega)\,\omega = k}$ (← for a massless field for simplicity)

This means that the Green function takes the following form

$$G^R(x_{12}) = \theta(x_{12}^0)\langle m|\,[\mathcal{O}(x_1), \mathcal{O}(x_2)]\,|m\rangle$$

(10)
$$= \theta(x_{12}^0)\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\vec{x}_{12}}\,\frac{e^{-i\omega(k)x_{12}^0} - e^{i\omega(k)x_{12}^0}}{2\omega(k)}$$
($\omega(k)$ solution of (9))

$$= \int\frac{d^3k}{(2\pi)^3}\int\frac{dk^0}{(2\pi)}\, e^{-ikx_{12}}\,\frac{-i}{(k^0+i\epsilon)^2 - \omega^2(k)}$$

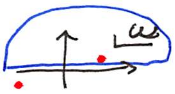
That is, same causal/analytic structure in the k^0-plane (because of $\theta(x_{12}^0)$) but with a modified dispersion relation $\omega=\omega(k)$ which breaks Lorentz. (again spontaneously; this technically means $\omega(k\to\infty) = k$, i.e. $n(\omega\to\infty)\to 1$)
This is causal, clearly, because of $\theta(x_{12}^0)$ (or the analytic structure), but does it vanish as well outside the vacuum-determined lightcone? Do signal propagate inside that lightcone or can leak out for weird $n(\omega)$?
Let's get rid of the angular variables in (10), $\int\frac{d^3k}{(2\pi)^3} = \int\frac{d\Omega}{4\pi}\int_0^\infty\frac{\vec{k}^2 d|\vec{k}|}{2\pi^2}$ $\quad(|\vec{x}_{12}| \equiv z)$

$$G^R(x_{12}) = \int_0^\infty\frac{|\vec{k}|^2 d|\vec{k}|}{(2\pi)^2}\,\frac{-i}{|\vec{k}|z}\left(e^{+i|\vec{k}|z} - e^{-i|\vec{k}|z}\right)\int\frac{dk^0}{2\pi}\, e^{-ik^0x_{12}^0}\,\frac{-i}{(k^0+i\epsilon)^2 - \omega^2(|k|)}$$

(11)
$$= \frac{1}{z}\int_{-\infty}^{+\infty}\frac{dk\; k}{(2\pi)^2}\int\frac{dk^0}{(2\pi)}\,\frac{e^{ikz - ik^0x_{12}^0}}{(k^0+i\epsilon)^2 - \omega^2(k)}$$

now we change variables since $\omega(k)$ satisfies (9), $k = n(\omega)\,\omega$ $\quad dk = d\omega\,[n'\omega + n]$
↓

$$\Rightarrow G^R(x_{12}) = -\frac{1}{r}\int_{-\infty}^{+\infty}\frac{dk^0}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{(2\pi)^2}\, n(\omega)\,\omega\,[n(\omega)+n'(\omega)\,\omega]\,\frac{e^{-ik^0x^0_{12}+in(\omega)\omega r}}{(k^0+i\epsilon)^2-\omega^2} \qquad (12)$$



We can now perform the integral over $d\omega$ assuming: (*) $n(\omega)$ analytic in the upper complex ω-plane

$\Rightarrow$ only $\omega = k^0+i\epsilon$ contribute by closing contour in the upper part of complex ω-plane

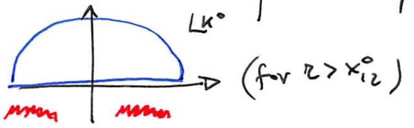
$$\Rightarrow G^R(x_{12}) = +\frac{i}{2r}\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)^2}\, n(k^0)\,[n(k^0)+n'(k^0)\,k^0]\; e^{-ik^0(x^0_{12}-n(k^0)r)} \qquad (13)$$

Notice that we know, by construction, that $G^R(x_{12})$ vanish for $x^0_{12}<0$ which is nicely consistent with having required analyticity of $n(k^0)$ in the upper complex plane, and it justifies in retrospect the assumption (*).

There is now one extra requirement: Lorentz invariance should be recovered at $k^0 \to \infty$, that is

$$(**) \qquad \boxed{n(k^0) \xrightarrow[k^0\to\infty]{} 1} \qquad (14)$$

With this, we can close again in the upper k^0-plane when $r > x^0_{12}$, that is for spacelike distance, which returns zero!



$$\Rightarrow \boxed{\text{analyticity of } n(k^0)} + \boxed{n \xrightarrow[k^0\to\infty]{} 1} \text{ have implied that}$$

even in this Lorentz-breaking state the Green function vanish outside the future (Lorentz-determined) lightcone.

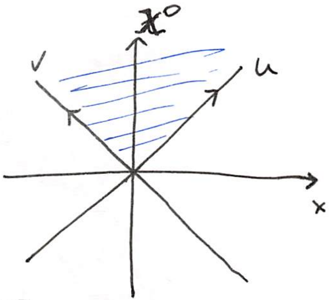
We can generalize a little more the previous discussion asking what are the conditions under which a general Green function is (·) causal and (··) vanish outside lightcone, even in Lorentz breaking bkg., e.g. with $G(x_{12})$ controlled by, say, a differential equation

$$(15) \qquad \boxed{P(\partial_t, \partial_x)\, G(x_{12}) = \delta^4(x_{12})}$$

For (·), just take $G(x_{12}) = G^R(x_{12}) \propto \theta(x_{12}^0) \Rightarrow \int e^{-ik^0 x_{12}^0} G(x_{12}^0, \vec{x}_{12})$ analytic in the upper k^0-plane.

For (··), let's take the Fourier Tr. w/ respect to $\vec{k}$ as well

$$G^R(x_{12}) = \int \frac{dk^0}{(2\pi)} \int \frac{d^3k}{(2\pi)^3}\, e^{-ik^0 x_{12}^0 + i\vec{k}\vec{x}_{12}}\, \hat{G}^R(k^0, \vec{k})$$



Let's restrict to $D = 1+1$ for simplicity and define

$\begin{cases} v = x^0 - x \\ u = x^0 + x \end{cases}$ the coordinates along the lightcone so that the boundary is jus $v =$ [illegible] and $u = 0$

$$G^R(x_{12}) = \int \frac{dk^0}{(2\pi)} \int \frac{dk}{(2\pi)}\, e^{-i\left(\frac{k^0 - k}{2}\right)u_{12} - i\left(\frac{k^0 + k}{2}\right)v_{12}}\, \hat{G}^R(k^0, k)$$

$$\left(\begin{array}{l} k^0 - k \equiv k_v \\ k^0 + k \equiv k_u \\ e^{-\frac{i}{2}(k_v u + k_u v)} \end{array}\right)$$

We want $G^R(x_{12}) = 0$ for $\boxed{v_{12} < 0 \;\text{ or }\; u_{12} < 0}$

$\Rightarrow$ $\hat{G}^R(k^0, k)$ is analytic in the region $\boxed{\begin{cases} \mathrm{Im}(k^0 - k) > 0 \\ \mathrm{Im}(k^0 + k) > 0 \end{cases}}$

that is $\boxed{\mathrm{Im}\, k^0 > |\mathrm{Im}\, k|}$ (16)

("forward tube", i.e. $\mathrm{Im}\, k^\mu$ is positive timelike vector)

Now, this condition is automatically satisfied by a $G = \frac{-i}{m^2(k^0)\, k^{0\,2} - \vec{k}^2}$ requiring that $m(k^0)$ has indeed no singularity in the upper complex k^0-plane and $m \xrightarrow[k^0 \to \infty]{} 1$. (see 0709.1483 for detailed proof)

(e) Refraction index and Forward Scattering

We have seen a direct connection between causality / locality and analyticity of Green functions and refraction index.
This was coming from the properties of $[\mathcal{O}(x_1), \mathcal{O}(x_2)]$ even in non-vacuum states.
We push forward in this direction and look at the scattering (interactions on, this time) of particle in a medium, and derive the beautiful connection between Refraction Index ⟺ Forward Elastic Scattering Ampl.
This is of course well known in the neutrino-physics community since MSW-effect can be seen as the statement that

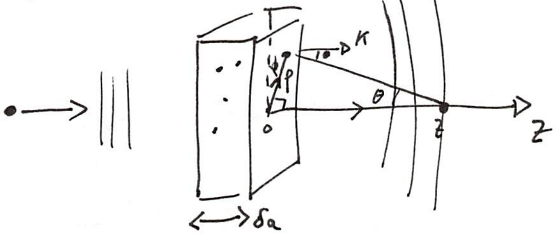
$$\boxed{n - 1 \propto \frac{\text{forward Amplitude}}{E^2}\Big|_{\text{LAB fr.}}} = \frac{2\pi}{E^2} N f(E, \theta = 0)$$

← forward amp.

↑ number density of medium.

(for ~~[illegible]~~ medium where $n \simeq 1$ anyways)

This relation (see e.g. 9206209 and Fermi's original deriv.) comes about because the single scatterers of the medium interfere and average out except for the forward direction that is singled out by the beam, and where all contribution sum up coherently creating a phase shift. Except for the forward directions, the ~~total~~ differential cross-section is just the sum due to each particle's medium.
But for $\cos\theta = 1$ we have instead to sum over the amplitudes,



each of the type $\psi_K(z) = e^{-i\omega t}\left[e^{i\vec{K}\vec{z}} + f(\omega,\theta)\frac{e^{ikr}}{r}\right]$

that is

$$(17) \quad \psi_E(z) = e^{-iEt}\left[e^{iKz} + N\cdot\delta a \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\infty} d\rho\, \rho \underbrace{\frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}}_{\frac{1}{ik}\frac{d}{d\rho}(e^{ik\sqrt{}})} f\left(K, \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}\right)\right]$$

$(K \simeq E)$

↑ number density

We can integrate now (17) by parts given that $\frac{d}{d\rho} e^{ik\sqrt{\rho^2+z^2}} = \frac{ik e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} \rho$,

$$\psi_k(z) = e^{-i\epsilon t}\Big[e^{ikz} - \frac{i2\pi N\delta a}{k} \underbrace{e^{ik\sqrt{\rho^2+z^2}} f(k, \cos\theta = \tfrac{z}{\sqrt{z^2+\rho^2}})}_{\substack{\text{throw away } \rho=\infty \\ \text{because it averages out} \\ (\text{or formally } \underset{\epsilon\to 0}{k \to k+i\epsilon})}} \Big|_0^{\rho=\infty} + \underbrace{\frac{2\pi i N\delta a}{k}\int_0^\infty d\rho\, e^{ik\sqrt{\rho^2+z^2}} \frac{-\rho\cos\theta}{(z^2+\rho^2)}\frac{\partial f}{\partial\cos\theta}}_{\substack{\text{negligible for} \\ kz \gg 1 \ (\text{if } \frac{\partial f}{\partial\cos\theta} \text{ regular}) \\ (\text{this means } n(k)\simeq 1)}} \Big]$$

$$\Rightarrow \psi_k(z) = e^{-i\epsilon t}\, e^{ikz}\Big[1 + \frac{2\pi i N\delta a}{k} f(k, \cos\theta = 1) \Big]$$

(see Fermi's book on Nuclear Physics)

Integrating over $a/_{\delta a} = j$ consecutive slabs of material, $\lim_{j\to\infty}(1 + \frac{\#}{j})^j = e^{\#}$ and since it holds up to back of the edge of slab

↳ means $e^{\frac{2\pi i N a}{k} f} \underset{z\simeq a}{\simeq} e^{ikz\left(\frac{2\pi N f}{k^2}\right)}$

$$\Rightarrow \boxed{n = 1 + \frac{2\pi N}{k^2} f(k, \cos\theta = 1)} \quad (18)$$

↓

[a more consistent treatment inside medium is possible, see Foldy 1945: $n^2 = 1 + \frac{4\pi}{k^2} N f(k,\theta=0)$]

We can reexpress it in terms of scattering amplitude $M(E, \cos\theta)$ given that $\left.\frac{d\sigma}{d\Omega}\right|_{lab} = |f|^2 = \frac{1}{64\pi^2}\frac{1}{E_{cm}^2}|M|^2 = \frac{1}{64\pi^2 m^2}|M|^2$

← c.m. energy for heavy target masses $m \gg E$

$(s = m^2 + 2mE)$

$$\Rightarrow \boxed{n - 1 = \frac{N}{4E^2 m_{target}} M(E, \cos\theta = 1)} \quad (19)$$

- This relation says that the <u>analytical</u> properties of n are inherited by the <u>forward scattering amplitude</u> $M(E)/_{E^2}$
- $\underset{E\to\infty}{n \to 1} \Rightarrow M(E)$ is <u>polynomially bounded</u> (order polynomial known $= 2$)

Both this fact are actually true in general, and not just in the approximation of large target masses. The moral is that <u>causality</u> says something on <u>S-matrix elements</u>.

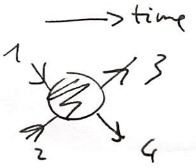
(Notice also that <u>$\mathrm{Im}\, n \propto \mathrm{Im} f_{\theta=0} > 0$</u> so that there is positive damping of amplitudes $e^{-\mathrm{Im}\, n z}$ by <u>optical th. = unitarity</u>

(f) $\boxed{\text{Causality} \Longleftrightarrow \text{Analyticity S-matrix}}$

Let's keep going in the exploration of causality/analyticity link for scattering amplitudes. The key, is to write even in-out correlators, that is S-matrix elements, in terms of retarded (or advanced) commutators, like for in-in correlators in linear response theory. This is possible thanks to the LSZ reduction: write

$$\boxed{T\vartheta(x_1)\,\vartheta(x_2) = \theta(x^0_{12})\,[\vartheta(x_1),\vartheta(x_2)] + \vartheta(x_2)\,\vartheta(x_1)} \tag{20}$$

and the 2 → 2 connected amplitude reads



$$\langle 1,2|\, S-\mathbb{1}\,|3,4\rangle = \prod_{LSZ_{1,3}} \langle 2|\, T\vartheta(x_1)\,\vartheta(x_3)\,|4\rangle$$

$$(21)\qquad = \prod_{LSZ_{1,3}} \theta(x^0_{13})\langle 2|\,[\vartheta(1),\vartheta(3)]\,|4\rangle + \underbrace{\prod_{LSZ_{1,3}}\langle 2|\,\vartheta_3\,\vartheta_1\,|4\rangle}_{\overset{\shortparallel}{0}\ \text{on-shell}}$$

The catch is that the last term vanishes on-shell, indeed, since $\prod_{LSZ_3}$ project on outgoing-$|3\rangle$-state and $\prod_{LSZ_1}$ on ingoing $|1\rangle$-state

$$\sum_m \prod_{LSZ_{1,3}} \langle 2|\,\vartheta_3\,|m\rangle\langle m|\,\vartheta_1\,|4\rangle = \sum_m \langle 2|S-\mathbb{1}|3\,m\rangle\langle 1\,m|\,S-\mathbb{1}\,|4\rangle$$

$$(22)\qquad = 2\pi i \sum_m \delta^4(p_2-p_3-p_m)\,\mathcal{M}_{2\to 3m}\;\delta^4(p_4-p_1-p_m)\,\mathcal{M}_{1m\to 4}$$

But these clearly vanish because we are assuming asymptotic stable particles, $\mathcal{M}_{2\to\text{any}} = 0$ on-shell.

reminder: $\prod_{LSZ_1} = \lim_{p_1^2\to m_1^2}\int d^4x\; e^{-ip_1 x}\, i(\Box_x+m_1^2)$ and $\prod_{LSZ_3} = \lim_{p_3^2\to m_3^2}\int d^4x\; e^{+ip_3 x}\, i(\Box_x+m_3^2)$

↑ |in⟩ has positive energy ↑ |out⟩ has negative energy p15

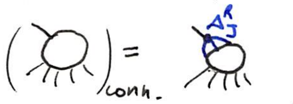
The connected S-metrix element is therefore

$$(23)\qquad \langle 1,2|\, S-\mathbb{1}\,|3,4\rangle = \prod_{LSZ_{1,3}} \theta(x^0_{13})\langle 2|\,[\vartheta(1),\vartheta(3)]\,|4\rangle$$

where we see the appearance of a retarded green function.
We can move the $\Box_x$ inside the LSZ-projector past $\theta(x^0_{13})$ because we are interested only in the analytical properties of (23), whereas $\Box_x$ acting on $\theta(x^0)$ would produce equal-time commutators which (again by (1)) vanish unless when $x_1 = x_3$. They corrispond to $P(\partial)\,\delta^4(x_1-x_3)$ whose Fourier tr. inside $\prod_{LSZ}$ returns polynomials hence local quantity we can safely discard (corresponds to adding local operators to the lagrangian, that is changing renormalization scheme). It's the same thing as going from T-ordered product to the covariant T^*-ordered product.

$$(24)\qquad \langle 1,2|\, S-\mathbb{1}\,|3,4\rangle = -\int d^4x_1\, e^{-ip_1x_1}\int d^4x_3\, e^{+ip_3x_3}\,\theta(x^0_{13})\langle 2|\,[J(1),J(3)]\,|4\rangle$$

where $J(x) \equiv (\Box_x + m^2)\,\vartheta(x)$

Schematically corresponds to remove external free propagator at pole with the retarded commutator (on-shell) ()

By translation invariant the integral in (24) brings a $(2\pi)^4\delta^4(\Sigma p_i - \Sigma p_o)$ which is factored out in the definition of $\mathcal{M}_{12\to34}$, namely
$S = \mathbb{1} + (2\pi)^4 i\,\delta^4(\Sigma p_i - \Sigma p_o)\,\mathcal{M}$.
Specializing it to elastic Forward amplitude, $\begin{matrix}4\longrightarrow 2\\ 3\longrightarrow 1\end{matrix}$ we get

$$\boxed{\mathcal{M}_{12\to12}(k_1) = i\int d^4x\, e^{ik_1x}\,\theta(x^0)\,\langle 2|\,[J(x),J(0)]\,|2\rangle}\qquad(25)$$

(k_1: mom. particle 1)

The (25) and its Fourier inverse are our Master's Formula:

$$(26)\begin{cases} M_{12\to12}(K) = i\,\hat{G}^R(K) \\ G^R(x) = -i\int e^{-iKx} M_{12\to12}(K)\frac{d^4K}{(2\pi)^4} = \theta(x^0)\langle 2|[J(x),J(0)]|2\rangle \end{cases}$$

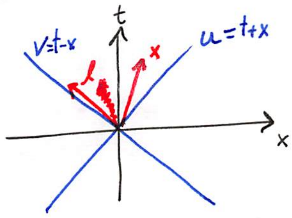
The Forward elastic amp. is written as the F.T. of a retarded commutator of the sources $J(x)$,

Clearly, $M_{12\to12}(K)$ is thus analytic ~~[illegible]~~ in the "forward cone"

Take e.g. 1,3 massless particles for simplicity (2, 4 massive instead)

$\Rightarrow K^\mu = E\,\ell^\mu$ for $\ell^2=0$ and $\ell^0=1$ (say $\ell^\mu=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$)

Then (25) tells us that indeed $G^R(x)$ vanishes outside the future lightcone $\begin{cases}x^2>0\\x^0>0\end{cases}$ since $K\cdot X = E\underbrace{(\ell\cdot x)}_{>0}$



$\Rightarrow E\to E+iE_I$ makes (25) convergent for $\boxed{E_I>0}$ $\hookrightarrow$ (doesn't have to have the same ℓ^μ)

$$M_{12\to12}(E+iE_I) = i\int d^4x\, e^{iE(\ell x)}\,\underline{\underline{e^{-E_I(\ell x)}}}\,\theta(x^0)\langle 2|[J(x),J(0)]|2\rangle$$

as long as $\langle\ \rangle$ grows less than a linear exponential of E_I (e.g. for Tempered-distributions that are polynomially bounded)

$\Rightarrow$ $\boxed{M_{12\to12}(E) \text{ is the real boundary of an analytic function for}}$ $\boxed{\operatorname{Im}E>0}$

[vice versa: $G^R(x)\sim -i\int dK_u dK_v\, e^{-\frac{i}{2}(K_u V+K_v u)} M_{12\to12}(K)$ should vanish for

either $u<0$ or $V<0$ $\Rightarrow\begin{cases}\operatorname{Im}K_u>0\\ \operatorname{Im}K_v>0\end{cases}\Leftrightarrow\begin{matrix}\operatorname{Im}K^0>|\operatorname{Im}K|\\ \operatorname{Im}K^0>0\end{matrix}$]

~~Notice that $(K^1+K^2)^2 = m_2^2+2EM_2 = s$ in the frame where $K_2=(m,\vec{0})$ @ rest~~

~~$\Rightarrow$ $\boxed{M_{12\to12}(s) \text{ analytic in the upper s-plane}}$~~

- $M_{12\to 12}(E)$ is analytic in the upper E-plane
- what about the "u-channel" $M_{\bar{1}2\to\bar{1}2}(E)$?

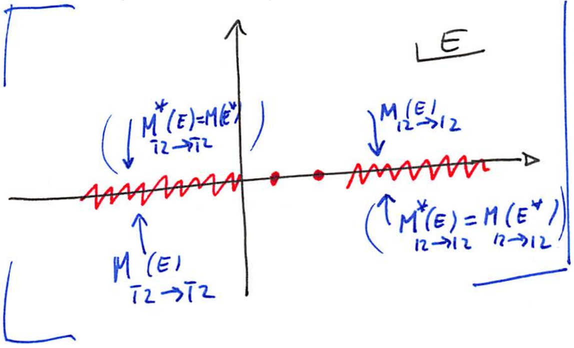
That is, what if we started with the anti-particles $M_{\bar{3}2\to\bar{1}4}(E)$? ← in our case, particles are real $\bar{1}=1$

$\begin{pmatrix}\bar{3}\to 1\\ 2\to 4\end{pmatrix}$ We could repeat the argument, which now would involve the advanced Green-function and find

$$(27)\qquad \boxed{M_{\bar{1}2\to\bar{1}2}(E) = M_{12\to 12}(-E)}$$

known as Crossing Symmetry

with $M_{\bar{1}2\to\bar{1}2}$ clearly analytic in the lower E-plane, as appropriate for antiparticles. So, in fact a single analytic function everywhere, except on the Real axis, defines both amplitudes



Notice that

$$(28)\qquad \begin{cases}(K_1+K_2)^2 \equiv s = m_2^2+2Em_2\\ (K_1-K_2)^2 = u = m_2^2-2Em_2\\ (t=0=(K_1-K_3)^2)\end{cases}$$

⟹ we can think of M as analytic in the s-plane (cutting away the real axis) and $E\to -E$ as just

$\boxed{s \leftrightarrow u}$ exchange under crossing.

$\bar{M}(\bar{E})$ is analytic in the rest if M real for some real E and cont.

- What about approaching the real axis from below but for $E>0$? [This comes from a little math theorem known as Schwarz reflection Principle see Fig. above

(9) Unitarity & Positivity

- summarizing, $M(s)$ is <u>analytic</u> everywhere in the complex s-plane except on the <u>real axis</u>. (recall, we are at <u>$t=0$</u> and <u>elastic</u>)
  ↪ particle production, stable particles, ...
- "unphysical" values of boundary values are related by <u>crossing symmetry</u> and <u>Schwarz-reflection</u> principle to physical boundary values. We can thus work with a single master $M(s) = M_{12\to12}(s)$ such that

$$(29)\quad \boxed{\begin{cases} M^*(s) = M(s^*) \\ M_{12\to12}(u=\Sigma_i m_i^2 - s) = M_{\bar{1}2\to\bar{1}2}(s) \end{cases}} \quad (t=0)$$

- Let's exploit now this analyticity of the forward elastic amplitude in connection with the last ingredient: Unitarity $\boxed{S^\dagger S = S S^\dagger = \mathbb{1}}$ (30)

⇒ For <u>2→2</u> <u>connected</u> (forward elastic) amplitudes says

$$(31)\quad \boxed{\begin{cases} \text{Im}\, M_{12\to12}(s) = s\sqrt{1-4m^2/s}\; \sigma_{12\to TOT}(s) \\ \text{Im}\, M_{\bar{1}2\to\bar{1}2}(s) = s\sqrt{1-4m^2/s}\; \sigma_{\bar{1}2\to TOT}(s) \end{cases}} \quad (\text{s-physical})$$

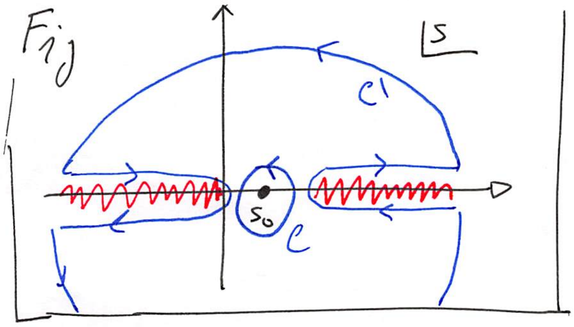
← (specialized to four identical particles of mass m, for simplicity)

where, because of Sch.refl., is about a Discontinuity across physical (real) values of s:

$$(32)\quad M_{12\to12}(s^{real}+i\epsilon) - M^*_{12\to12}(s^{real}+i\epsilon) = M_{12\to12}(s^{real}+i\epsilon) - M_{12\to12}(s^{real}-i\epsilon)$$

- Now we are ready to learn something universal about amplitudes

Consider the $2\to 2$ amplitude (forward elastic) for identical particles of mass m, the analytic structure being in Fig



(for example if they are the lightest stable particles)

By Cauchy integral theorem we get that

$$\boxed{\frac{1}{2\pi i}\oint_{C}\frac{M_{12\to 12}(s)}{(s-s_0)^3} = \frac{1}{2!}\frac{\partial^2 M_{12\to 12}}{\partial s^2}\bigg|_{\substack{s=s_0\\(t=0)}}} \qquad (33)$$

Deforming to the contour C',

$$\frac{1}{2!}\frac{\partial^2 M_{12\to 12}}{\partial s^2}\bigg|_{s=s_0} = \frac{1}{2\pi i}\left\{\int_{4m^2}^{\infty}\frac{ds}{(s-s_0)^3}\underbrace{\left[M_{12\to 12}(s+i\epsilon)-M_{12\to 12}(s-i\epsilon)\right]}_{\mathrm{Disc}\,M \overset{}{\underset{\text{Sch. ref.}}{=}} 2i\,\mathrm{Im}\,M} + \int_{-\infty}^{0}\frac{ds}{(s-s_0)^3}\underbrace{\left[M_{12\to 12}(s+i\epsilon)-M(s-i\epsilon)\right]}_{\substack{\mathrm{Disc}\,M\\ \shortparallel \\ 2i\,\mathrm{Im}\,M}}\right.$$

+(big circle @ infinity)

integral over physical energy

unphysical energy.

mapped to physical energy by crossing

$$\Rightarrow \frac{1}{2!}\frac{\partial^2 M_{12\to 12}}{\partial s^2}\bigg|_{s_0} = \frac{1}{\pi}\int_{4m^2} ds\left\{\frac{\mathrm{Im}\,M_{12\to 12}(s)}{(s-s_0)^3} + \frac{\mathrm{Im}\,M_{\bar{1}2\to\bar{1}2}(s)}{(s-4m^2+s_0)^3}\right\}$$

+(big circle)

We have obtained a dispersion relation, which using unitarity is

$$\boxed{\frac{1}{2!}\frac{\partial^2 M_{12\to 12}}{\partial s^2}\bigg|_{s_0} = \frac{1}{\pi}\int_{4m^2}^{\infty} ds\, s\sqrt{1-4m^2/s}\left[\frac{\sigma_{12\to TOT}(s)}{(s-s_0)^3} + \frac{\sigma_{\bar{1}2\to TOT}(s)}{(s-4m^2+s_0)^3}\right] + \text{big circle @ infinity}} \qquad (34)$$

IR / UV connection!!

- The big circle at infinity vanishes as it is taken that $\boxed{\frac{|M(s)|}{s^2} \xrightarrow[s\to\infty]{} 0 \qquad \text{Froissart Bound}}$ (← For any gapped theory) (justifies why looking @ $\frac{\partial^2 M}{\partial s^2}$)

which indeed makes the disp. relation (34) convergent

$$\underline{\sigma_{12\to TOT}(s\to\infty) \sim \log^2 s} \Rightarrow \text{integrand} \sim \frac{\log^2 s}{s^2}.$$

One way to understand this bound is recalling the index of refractiona, $n \xrightarrow[E\to\infty]{} 1$ required $\frac{M(s)}{s^2} \to 0$.

- All in all, we get a convergent dispersive relation that implies the following positivity

$$(35) \quad \boxed{\left.\frac{\partial^2 M_{12\to 12}}{\partial s^2}\right|_{\substack{s=s_0 \\ t=0}} > 0} \quad \text{as long as } \underline{0 \le s_0 < 4m^2}$$

Amplitude's positivity

[homework: what happens for odd number of subtr.?]

- This is a non-trivial constraint on the EFT for these particles, since s_0 is in IR and $t=0$.

For example, from $\mathcal{L}^{EFT} = \frac{1}{2}(\partial\pi)^2 - \frac{m^2\pi^2}{2} + \lambda(\partial\pi)^4 + \dots$ we get

$$(36) \quad \boxed{\lambda > 0}$$

whereas $\lambda = 0$ or $\lambda < 0$ would seem consistent: instead there is no consistent UV-completion that would give $\lambda \le 0$.

Fermionic Example

We have worked so far only with spin-0 particles and scalar fields, but the arguments actually generalize to arbitrary spin (see 1605.06111), including fermions even though they anticommute, as long as one keeps working in the forward elastic limit.

Let's see an example with spin-1/2 particles in the limit $m \to 0$, considering

$$\mathcal{L} = \chi^{\dagger} i \not{\partial} \chi - \frac{g_*^2}{\Lambda^4} \chi^{\dagger 2} \Box \chi^2 + \ldots$$

That is a theory with leading interactions starting at $O(p^4)$

[there is one unique such op., up to Fierz id and fields redef. requiring chiral symmetry $\chi \to e^{i\alpha} \chi$]

The forward elastic amplitude reads

$$\mathcal{M}_{\chi\chi \to \chi\chi}(s, t \to 0) = 4 \frac{g_*^2}{\Lambda^4} s^2 \implies$$

$\frac{g_*^2}{\Lambda^4}$, the coeff. of $\chi^{\dagger 2} \Box \chi^2$ must be positive

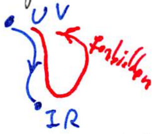
This is verified, e.g., in the Goldstino Lagrangian where

$\frac{g_*^2}{\Lambda^4} = \frac{1}{F^2}$ where $[F] = 2$ is the susy decay constant

and clearly $\Sigma_\alpha \| Q_\alpha |0\rangle \|^2 = 4 \langle 0 | H | 0 \rangle = 4F^2 > 0$.

General lessons & Conclusions

- (micro-) causality ⇒ analyticity of Green-functions
- ↗ is an op. statement ⇒ it can be exploited on various states
  ↳ the smarter the state the more info one gets
  (we have seen e.g. medium → index of ref.
  scatt. states → positivity)
- Using it on (plane wave) scattering states $|k\rangle$ ⇒ positivity
  (adding unitarity)
  ⇓
  in turn, constraints on EFT's
  and even QFT
  [e.g. $\lambda > 0$ we have seen implies
  irreversibility of RG-flow



  because of Komargodski-Schwimmer
  proof that $\lambda = a_{UV} - a_{IR} > 0$ in a certain setup
  see 1107.3987]
- There is a swampland even in QFT

not just in string theory

- just tip of an iceberg:
  - using smarter scattering states $|J\, L\, m\rangle$ ⇒ partial waves : more info & constraints
  - put bound on how soft amplitude can be 1605.06111
  - generalization to arbitrary spin →
  - Beyond Positivity & Beyond Forward elastic sc. (e.g. 1710.02539, 0912.4258, 1706.02712)
  - BSM applications (fermion compositeness 1706.03070; ZZ & Zγ @ LHC 1806.09640)
  - using states in CFT's (see T. Hartman et al. 1509.00014, 1610.05308)

$$\langle\psi|\, C,\ T|\psi\rangle$$

↖ shockwave states or generated by primary $T_{\mu\nu}\dots$

⇓

$$\text{ANEC} = \int du \langle T_{uu}\rangle \geq 0 \quad \left(\text{more generally, } \int \underset{\substack{\uparrow\\ \text{null param.}}}{d\lambda} \langle T_{\mu\nu}\rangle \underset{\substack{\uparrow\\ \text{null tangent}}}{u^\mu} u^\nu\right)$$

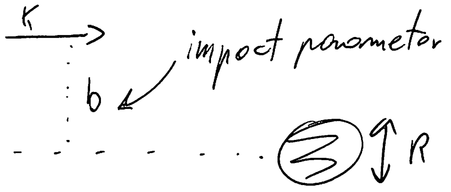
bounds on C_{TTT} ...

  - Bounds on higher spins (work in progress)
  - Weak-gravity conjecture (speculation)

. ne comments :

Crude argument for Froissart Bound (beside refraction index)

Eq 19 p13 with $n \xrightarrow[k\to\infty]{} 1$



← picture expected to be valid at high energy, particle-like behavior

$\ell = k \cdot b$

partial waves: $S_\ell^{elastic} \begin{cases} \longrightarrow \mathbb{1} & b = \frac{\ell}{k} \gg R \leftarrow \text{trivial} \\ \longrightarrow 0 & b = \frac{\ell}{k} \ll R \end{cases}$

$\uparrow$ fully inelastic

$\Rightarrow$ optical theorem: $\sigma^{TOT} \sim \frac{2\pi}{k^2} \sum_{\ell}^{\infty} (2\ell+1) \, \mathrm{Re}(\mathbb{1} - S_\ell^{elastic})$

$$= \frac{2\pi}{k^2} \sum_{\ell < kR} (2\ell+1) \simeq \frac{(2\pi)}{k^2} (kR)^2 = \underline{\underline{2\pi R^2}}$$

$\propto$ geometrical elastic cross-section

(R may depend on energy weakly, e.g.

Yukawa supp. $e^{-mR} \sim E^{\#} \Rightarrow R \sim \frac{1}{m} \ln E$

$\Rightarrow \sigma^{TOT} \longrightarrow \frac{1}{m^2} \log^2 E$ as observed.)

see also naive argument in hep-th/0602178

Useful Refs:

- Weinberg's QFT-1 chpt. 10.2, 10.3, 10.7, 10.8
- Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi hep-th/0602178
- Dubovsky, Nicolis, Trincherini, Villadoro 0709.1483
- B.B. 1605.06111
- Bogoliubov-Shirkov (chp 10), Itzykson-Zuber (chp 5), Bjorken-Drell (chpt 18)