c) Quantization of the vibrating string

before we quantize, we rescale the field ϕ:

<u>def.</u> $\tilde{\phi}(x,t) = \sqrt{\rho}\, \phi(x,t)$

then

$$\mathcal{L}(\partial_t \tilde{\phi}, \partial_x \tilde{\phi}) = \frac{1}{2}(\partial_t \tilde{\phi})^2 - \frac{P}{2\rho}(\partial_x \tilde{\phi})^2 - V(\tilde{\phi})$$

$c^2 = \frac{P}{\rho}$

allow for additional potential
e.g. $V(\tilde{\phi}) = \frac{1}{2} m^2 \tilde{\phi}^2$

and drop the tilde

$$\mathcal{L}(\partial_t \phi, \partial_x \phi, \phi) = \frac{1}{2}(\partial_t \phi)^2 - \frac{c^2}{2}(\partial_x \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$c = 1$: $\frac{1}{2}\left[(\partial_t \phi)^2 - (\partial_x \phi)^2\right] = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$

using $\partial_\mu = \frac{\partial}{\partial x^\mu}$ i.e. $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x}$

and $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

<u>e.o.m.</u>

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \frac{\partial}{\partial t}(\partial_t \phi) = \partial_t^2 \phi$$

$$\frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} = \frac{\partial}{\partial x}(-c^2 \partial_x \phi) = -c^2 \partial_x^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

then

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Rightarrow \boxed{\partial_t^2 \phi - c^2 \partial_x^2 \phi + m^2 \phi = 0} \qquad (10)$$

and $\boxed{\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi}$

quantization

1) replace fields by field operators

$$\phi(x,t) \longrightarrow \hat{\phi}(x,t)$$

$$\pi(x,t) \longrightarrow \hat{\pi}(x,t)$$

2) impose commutation relations:

$$\{A,B\} \longrightarrow -i[\hat{A},\hat{B}] \quad \text{where } \hbar = 1$$

i.e.

$$\boxed{\begin{aligned} [\hat{\phi}(x,t), \hat{\phi}(y,t)] &= [\hat{\pi}(x,t), \hat{\pi}(y,t)] = 0 \\ [\hat{\phi}(x,t), \hat{\pi}(y,t)] &= i\delta(x-y) \end{aligned}}$$

generalization of δ_{ij}

<u>def</u>. Hamiltonian operator

$$\hat{H}(t) = \int_{x_1}^{x_2} dx \underbrace{\left[\hat{\pi}(x,t)\dot{\hat{\phi}}(x,t) - \mathcal{L}(\hat{\phi}(x,t), \dot{\hat{\phi}}(x,t), \partial_x\phi(x,t), t)\right]}_{=\hat{\mathcal{H}}(x,t)}$$

<u>time evolution of $\hat{\phi}$ and $\hat{\pi}$:</u>

from $\dot{q}_k = \{q_k, H\} \Longrightarrow$

$\dot{p}_k = \{p_k, H\} \Longrightarrow$

$$\boxed{\begin{aligned} \dot{\hat{\phi}} &= -i[\hat{\phi},\hat{H}] \\ \dot{\hat{\pi}} &= -i[\hat{\pi},\hat{H}] \end{aligned}}$$

now, <u>apply to vibrating string</u>

first, find <u>classical solutions</u> of e.o.m. eq. (10)

<u>ansatz</u>

$$\phi(x,t) = e^{\pm i(kx - \omega_k t)} \qquad \text{for both signs } \pm$$

<u>check</u>

$$\partial_t \phi(x,t) = \mp i\omega_k \phi(x,t)$$
$$\partial_t^2 \phi(x,t) = -\omega_k^2 \phi(x,t)$$
$$\partial_x \phi(x,t) = \pm i k \phi(x,t)$$
$$\partial_x^2 \phi(x,t) = -k^2 \phi(x,t)$$

$$\overset{\text{eq.}(10)}{\Longrightarrow} \quad (-\omega_k^2 + c^2 k^2 + m^2)\,\phi(x,t) = 0$$

$$\Rightarrow \quad -\omega_k^2 + c^2 k^2 + m^2 \overset{!}{=} 0$$

thus

$$\omega_k = +\sqrt{c^2 k^2 + m^2}$$

(dispersion relation)

we can choose only + signs and take

$$e^{i(kx - \omega_k t)} \quad \text{and} \quad e^{-i(kx - \omega_k t)}$$

as basis for all solutions of eq. (10)

$k \;\hat{=}\;$ momentum

$\omega_k \;\hat{=}\;$ energy

<u>simplifying assumption</u>:

$$\phi(0,t) = \phi(x_L, t) \qquad \text{(periodic boundary condition)}$$

$\Rightarrow$ momentum k is quantized:

$$k = \frac{2\pi n}{x_L} \quad \text{for } n \in \mathbb{Z}$$

then:

most general solution of eq. (10)

$$\phi(x,t) = \sum_{k} \frac{1}{\sqrt{2\omega_k x_L}} \left(a_k e^{ikx - i\omega_k t} + \underbrace{a_k^+ e^{-ikx + i\omega_k t}}_{\text{complex conjugate of first term to make } \phi(x,t) \text{ real}} \right)$$

$k = \frac{2\pi n}{x_L}, \; n \in \mathbb{Z}$

and

$$\pi(x,t) = \partial_t \phi(x,t)$$
$$= -i \sum_k \sqrt{\frac{\omega_k}{2x_L}} \left(a_k e^{ikx - i\omega_k t} - a_k^+ e^{-ikx + i\omega_k t} \right) \quad (11)$$

plan for the following

- express a_k and a_k^+ by $\phi(x,t)$ and $\pi(x,t)$
- promote fields to operators

$$\begin{matrix} \phi \to \hat{\phi} \\ \pi \to \hat{\pi} \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} a_k \to \hat{a}_k \\ a_k^+ \to \hat{a}_k^+ \end{matrix}$$

- compute $[\hat{a}_k, \hat{a}_e^+]$, $[\hat{a}_k, \hat{a}_e]$ and $[\hat{a}_k^+, \hat{a}_e^+]$

first, a small computation that will be needed:

$$\int_0^{x_L} dx \, e^{i(k-e)x} \overset{\text{if } k \neq e}{=} -\frac{i}{k-e} e^{i(k-e)x} \Big|_0^{x_L}$$

$$= -\frac{i}{k-e} \left(\underbrace{e^{i(k-e)x_L}}_{= e^{\frac{2\pi i x_L}{x_L}(n-m)} = 1} - 1 \right) = 0$$

$$\Rightarrow \boxed{\int_0^{x_L} dx \, e^{i(k-e)x} = \delta_{ke} x_L \quad (12)}$$

if $k = e$: $\int_0^{x_L} dx \, e^0 = x_L$

in order to express $a_k^{(+)}$ by ϕ and π we will need 4 short computations (S0) – (S3):

(S0) $\displaystyle\int_0^{x_L} dx\; e^{-ilx + i\omega_l t}\,\phi(x,t)$ $\qquad l = \frac{2\pi m}{x_L},\ m \in \mathbb{Z}$

$$= \int_0^{x_L} dx \sum_k \frac{1}{\sqrt{2\omega_k x_L}} \Big(a_k \underbrace{e^{i(k-l)x}}_{\xrightarrow{\text{eq. }(12)}\ \delta_{k,l}\, x_L} e^{-i(\omega_k-\omega_l)t}$$

$$+ a_k^+ \underbrace{e^{-i(k+l)x}}_{\xrightarrow{\text{eq. }(12)}\ \delta_{k,-l}\, x_L} e^{i(\omega_k+\omega_l)t} \Big)$$

$$= \sqrt{\frac{x_L}{2\omega_l}} \left(a_l + a_{-l}^+ e^{2i\omega_l t} \right) \qquad \text{using } \omega_l = \omega_{-l}$$

$$\Rightarrow \int_0^{x_L} dx\; e^{-ilx + i\omega_l t}\,\phi(x,t) = \sqrt{\frac{x_L}{2\omega_l}} \left(a_l + a_{-l}^+ e^{2i\omega_l t} \right)$$

(S1) $\displaystyle\int_0^{x_L} dx\; e^{-ilx + i\omega_l t}\,\pi(x,t)$ $\qquad$ using eq. (11)

$$= -i \int_0^{x_L} dx \sum_k \sqrt{\frac{\omega_k}{2x_L}} \Big(a_k \underbrace{e^{+i(k-l)x}}_{\xrightarrow{\text{eq. }(12)}\ \delta_{k,l}\, x_L} e^{i(\omega_l-\omega_k)t}$$

$$- a_k^+ \underbrace{e^{-i(k+l)x}}_{\xrightarrow{\text{eq. }(12)}\ \delta_{k,-l}\, x_L} e^{i(\omega_l+\omega_k)t} \Big)$$

$$= \sqrt{\frac{\omega_l x_L}{2}} \left(-i a_l + i a_{-l}^+ e^{2i\omega_l t} \right)$$

$$\Rightarrow \int_0^{x_L} dx\; e^{-ilx + i\omega_l t}\,\pi(x,t) = \sqrt{\frac{\omega_l x_L}{2}} \left(-i a_l + i a_{-l}^+ e^{2i\omega_l t} \right)$$

(52) $$\int_0^{x_L} dx\, e^{ilx - i\omega_l t}\,\phi(x,t) = \sqrt{\frac{x_L}{2\omega_l}}\left(a_{-l}\, e^{-2i\omega_l t} + a_l^+\right)$$

(53) $$\int_0^{x_L} dx\, e^{ilx - i\omega_l t}\,\pi(x,t) = \sqrt{\frac{\omega_l x_L}{2}}\left(-i a_{-l}\, e^{-2i\omega_l t} + i a_l^+\right)$$

then

$$\int_0^{x_L} dx\, e^{-ilx + i\omega_l t}\big(\underbrace{\omega_l \phi(x,t)}_{=\omega_l \cdot \text{eq.}(50)} + \underbrace{i\pi(x,t)}_{= i\, \text{eq.}(51)}\big)$$

$$= \omega_l\sqrt{\frac{x_L}{2\omega_l}}\left(a_l + a_{-l}^+ e^{2i\omega_l t}\right) + i\sqrt{\frac{\omega_l x_L}{2}}\left(-i a_l + i a_{-l}^+ e^{2i\omega_l t}\right)$$

$$= \sqrt{\frac{\omega_l x_L}{2}}\left(a_l + a_{-l}^+ e^{2i\omega_l t} + a_l - a_{-l}^+ e^{2i\omega_l t}\right)$$

$$= \sqrt{2\omega_l x_L}\; a_l$$

$$\Rightarrow \boxed{a_l = \frac{1}{\sqrt{2\omega_l x_L}}\int_0^{x_L} dx\, e^{-ilx + i\omega_l t}\left(\omega_l \phi(x,t) + i\pi(x,t)\right)}$$

and

$$\int_0^{x_L} dx\, e^{ilx - i\omega_l t}\big(\underbrace{\omega_l \cdot \phi(x,t)}_{=\omega_l\, \text{eq.}(52)} - \underbrace{i\pi(x,t)}_{=(-i)\, \text{eq.}(53)}\big)$$

$$= \sqrt{\frac{\omega_l x_L}{2}}\left(a_{-l}\, e^{-2i\omega_l t} + a_l^+ - a_{-l}\, e^{-2i\omega_l t} + a_l^+\right)$$

$$= \sqrt{2\omega_l x_L}\; a_l^+$$

$$\Rightarrow \boxed{a_l^+ = \frac{1}{\sqrt{2\omega_l x_L}}\int_0^{x_L} e^{ilx - i\omega_l t}\left(\omega_l \phi(x,t) - i\pi(x,t)\right)}$$

promote fields to operators

Compute commutators

$$[\hat{a}_k, \hat{a}_l^+] = \frac{1}{2x_L\sqrt{\omega_k \omega_l}} \int_0^{x_L} dx \int_0^{x_L} dy \; e^{-ikx+ily+i(\omega_k-\omega_l)t}$$

$$\times \underbrace{\left[\omega_k \hat{\phi}(x,t) + i\hat{\pi}(x,t), \omega_l \hat{\phi}(y,t) - i\hat{\pi}(y,t)\right]}$$

$$= \omega_k\omega_l \underbrace{[\hat{\phi}(x,t), \hat{\phi}(y,t)]}_{=0} - i\omega_k \underbrace{[\hat{\phi}(x,t), \hat{\pi}(y,t)]}_{=i\delta(x-y)}$$

$$+ i\omega_l \underbrace{[\hat{\pi}(x,t), \hat{\phi}(y,t)]}_{=-i\delta(y-x)} + \underbrace{[\hat{\pi}(x,t), \hat{\pi}(y,t)]}_{=0}$$

$$= \frac{1}{2x_L\sqrt{\omega_k \omega_l}} \int_0^{x_L} dx \int_0^{x_L} dy \; e^{-ikx+ily+i(\omega_k-\omega_l)t}$$

$$\times \left(\omega_k \delta(x-y) + \omega_l \delta(x-y)\right)$$

$$= \frac{1}{2x_L\sqrt{\omega_k\omega_l}} \int_0^{x_L} dx \; \underbrace{e^{-i(k-l)+i(\omega_k-\omega_l)t}}_{\xrightarrow{eq. (12)} \delta_{k,l} x_L} (\omega_k + \omega_l)$$

$$= \frac{2\omega_k x_L}{2x_L\sqrt{\omega_k\omega_k}} \delta_{kl} = \delta_{kl}$$

$$\Longrightarrow \boxed{[\hat{a}_k, \hat{a}_l^+] = \delta_{kl}} \qquad (13)$$

Similarly:

$$\boxed{[\hat{a}_k, \hat{a}_l] = [\hat{a}_k^+, \hat{a}_l^+] = 0}$$

a_k^+ creation op.

a_l annihilation op.

Hamilton operator

$$\hat{\mathcal{H}}(x,t) = \hat{\pi}(x,t)\underbrace{\dot{\hat{\phi}}(x,t)}_{=\hat{\pi}(x,t)} - \mathcal{L} = \hat{\pi}(x,t)^2 - \mathcal{L}$$

$$= \hat{\pi}(x,t)^2 - \left(\frac{1}{2}\hat{\pi}(x,t)^2 - \frac{c^2}{2}(\partial_x\hat{\phi}(x,t))^2 - \frac{1}{2}m^2\hat{\phi}(x,t)^2\right)$$

$$= \frac{1}{2}\hat{\pi}(x,t)^2 + \frac{c^2}{2}(\partial_x\hat{\phi}(x,t))^2 + \frac{1}{2}m^2\hat{\phi}(x,t)^2$$

then

$$\hat{H} = \int_0^{x_L} dx\, \hat{\mathcal{H}}(x,t)$$

Compute $\hat{H}$ in terms of $\hat{a}_k$ and $\hat{a}_k^+$:

$$\hat{\phi}(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k x_L}}\left(\hat{a}_k e^{ikx - i\omega_k t} + \hat{a}_k^+ e^{-ikx + i\omega_k t}\right)$$

$$\hat{\pi}(x,t) = -i\sum_k \underbrace{\frac{\omega_k}{\sqrt{2\omega_k x_L}}}_{=\sqrt{\frac{\omega_k}{2x_L}}}\left(\hat{a}_k e^{ikx - i\omega_k t} - \hat{a}_k^+ e^{-ikx + i\omega_k t}\right)$$

$$\partial_x\hat{\phi}(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k x_L}}\left(ik\,\hat{a}_k e^{ikx - i\omega_k t} - ik\,\hat{a}_k^+ e^{-ikx + i\omega_k t}\right)$$

$$= i\sum_k \frac{k}{\sqrt{2\omega_k x_L}}\left(\hat{a}_k e^{ikx - i\omega_k t} - \hat{a}_k^+ e^{-ikx + i\omega_k t}\right)$$

then

$$\hat{\pi}(x,t)^2 = -\frac{1}{2x_L}\sum_{k,l}\sqrt{\omega_k\omega_l}\left[\hat{a}_k e^{ikx - i\omega_k t} - \hat{a}_k^+ e^{-ikx + i\omega_k t}\right] \times \left[\hat{a}_l e^{ilx - i\omega_l t} - \hat{a}_l^+ e^{-ilx + i\omega_l t}\right]$$

$$\Rightarrow \int_0^{x_L} dx\, \hat{\pi}(x,t)^2 = -\frac{1}{2x_L}\sum_{k,l}\sqrt{\omega_k\omega_l}\int_0^{x_L} dx\Big[\hat{a}_k\hat{a}_l e^{i(k+l)x} e^{-i(\omega_k+\omega_l)t} - \hat{a}_k\hat{a}_l^+ e^{i(k-l)x} e^{-i(\omega_k-\omega_l)t}$$

...

$$\ldots \quad - \hat{a}_k^+ \hat{a}_\ell e^{-i(k-\ell)x} e^{i(\omega_k-\omega_\ell)t}$$
$$+ \hat{a}_k^+ \hat{a}_\ell^+ e^{-i(k+\ell)x} e^{i(\omega_k+\omega_\ell)t}\Big]$$

$$= -\frac{1}{2x_L}\sum_{k,\ell}\sqrt{\omega_k\omega_\ell}\Big[\hat{a}_k\hat{a}_\ell\,\delta_{k,-\ell}\,x_L\,e^{-i(\omega_k+\omega_\ell)t}$$
$$-\hat{a}_k\hat{a}_\ell^+\,\delta_{k,\ell}\,x_L\,e^{-i(\omega_k-\omega_\ell)t}$$
$$-\hat{a}_k^+\hat{a}_\ell\,\delta_{k,\ell}\,x_L\,e^{i(\omega_k-\omega_\ell)t}$$
$$+\hat{a}_k^+\hat{a}_\ell^+\,\delta_{k,-\ell}\,x_L\,e^{i(\omega_k+\omega_\ell)t}\Big]$$

$$= -\frac{1}{2}\sum_k \omega_k\Big[\hat{a}_k\hat{a}_{-k}\,e^{-2i\omega_k t} - \hat{a}_k\hat{a}_k^+ - \hat{a}_k^+\hat{a}_k$$
$$+\hat{a}_k^+\hat{a}_{-k}^+\,e^{2i\omega_k t}\Big]$$

and

$$\left(\partial_x\hat{\phi}(x,t)\right)^2 = -\frac{1}{2x_L}\sum_{k,\ell}\frac{k\ell}{\sqrt{\omega_k\omega_\ell}}\Big[\hat{a}_k e^{ikx-i\omega_k t} - \hat{a}_k^+ e^{-ikx+i\omega_k t}\Big]$$
$$\times\Big[\hat{a}_\ell e^{i\ell x-i\omega_\ell t} - \hat{a}_\ell^+ e^{-i\ell x+i\omega_\ell t}\Big]$$

$$\Rightarrow \int_0^{x_L} dx\left(\partial_x\hat{\phi}(x,t)\right)^2 = -\frac{1}{2x_L}\sum_{k,\ell}\frac{k\ell}{\sqrt{\omega_k\omega_\ell}}\int_0^{x_L} dx\Big[\hat{a}_k\hat{a}_\ell e^{i(k+\ell)x} e^{-i(\omega_k+\omega_\ell)t}$$
$$-\hat{a}_k\hat{a}_\ell^+ e^{i(k-\ell)x} e^{-i(\omega_k-\omega_\ell)t}$$
$$-\hat{a}_k^+\hat{a}_\ell e^{-i(k-\ell)x} e^{i(\omega_k-\omega_\ell)t}$$
$$+\hat{a}_k^+\hat{a}_\ell^+ e^{-i(k+\ell)x} e^{i(\omega_k+\omega_\ell)t}\Big]$$

$$= -\frac{1}{2x_L}\sum_{k,\ell}\frac{k\ell}{\sqrt{\omega_k\omega_\ell}}\Big[\hat{a}_k\hat{a}_\ell\,\delta_{k,-\ell}\,x_L\,e^{-i(\omega_k+\omega_\ell)t}$$
$$-\hat{a}_k\hat{a}_\ell^+\,\delta_{k,\ell}\,x_L\,e^{-i(\omega_k-\omega_\ell)t}$$

...

$$\ldots - \hat{a}^+_k \hat{a}_l \delta_{k,l} x_L e^{i(\omega_k - \omega_l)t}$$

$$+ \hat{a}^+_k \hat{a}^+_l \delta_{k,-l} x_L e^{i(\omega_k + \omega_l)t} \Big]$$

$$= -\frac{\cancel{x_L}}{2\cancel{x_L}} \sum_k \frac{k^2}{\omega_k} \Big[-\hat{a}_k \hat{a}_{-k} e^{-2i\omega_k t} - \hat{a}_k \hat{a}^+_k$$

$$- \hat{a}^+_k \hat{a}_k - \hat{a}^+_k \hat{a}^+_{-k} e^{2i\omega_k t} \Big]$$

$$= \frac{1}{2} \sum_k \frac{k^2}{\omega_k} \Big[\hat{a}_k \hat{a}_{-k} e^{-2i\omega_k t} + \hat{a}_k \hat{a}^+_k + \hat{a}^+_k \hat{a}_k$$

$$+ \hat{a}^+_k \hat{a}^+_{-k} e^{2i\omega_k t} \Big]$$

<u>and</u>

$$\hat{\phi}(x,t)^2 = \frac{1}{2x_L} \sum_{k,l} \frac{1}{\sqrt{\omega_k \omega_l}} \Big[\hat{a}_k e^{ikx - i\omega_k t} + \hat{a}^+_k e^{-ikx + i\omega_k t} \Big]$$

$$\times \Big[\hat{a}_l e^{ilx - i\omega_l t} + \hat{a}^+_l e^{-ilx + i\omega_l t} \Big]$$

$$\Rightarrow \int_0^{x_L} dx\, \hat{\phi}(x,t)^2 = \frac{1}{2x_L} \sum_{k,l} \frac{1}{\sqrt{\omega_k \omega_l}} \int_0^{x_L} dx \Big[\hat{a}_k \hat{a}_l e^{i(k+l)x} e^{-i(\omega_k + \omega_l)t}$$

$$+ \hat{a}_k \hat{a}^+_l e^{i(k-l)x} e^{-i(\omega_k - \omega_l)t}$$

$$+ \hat{a}^+_k \hat{a}_l e^{-i(k-l)x} e^{i(\omega_k - \omega_l)t}$$

$$+ \hat{a}^+_k \hat{a}^+_l e^{-i(k+l)x} e^{i(\omega_k + \omega_l)t} \Big]$$

$$= \frac{\cancel{x_L}}{2\cancel{x_L}} \sum_{k,l} \frac{1}{\sqrt{\omega_k \omega_l}} \Big[\hat{a}_k \hat{a}_l \delta_{k,-l} e^{-i(\omega_k + \omega_l)t}$$

$$+ \hat{a}_k \hat{a}^+_l \delta_{k,l} e^{-i(\omega_k - \omega_l)t}$$

$$+ \hat{a}^+_k \hat{a}_l \delta_{k,l} e^{i(\omega_k - \omega_l)t}$$

$$+ \hat{a}^+_k \hat{a}^+_l \delta_{k,-l} e^{i(\omega_k + \omega_l)t} \Big]$$

$$= \frac{1}{2}\sum_k \frac{1}{\omega_k}\Big[\hat a_k \hat a_{-k} e^{-2i\omega_k t} + \hat a_k \hat a_{-k}^+ + \hat a_k^+ \hat a_k$$
$$+ \hat a_k^+ \hat a_{-k}^+ e^{2i\omega_k t}\Big]$$

<u>now, compute $\hat H$</u>

$$\hat H = -\frac{1}{4}\sum_k \omega_k \Big[\hat a_k \hat a_{-k} e^{-2i\omega_k t} - \hat a_k \hat a_k^+ - \hat a_k^+ \hat a_k + \hat a_k^+ \hat a_{-k}^+ e^{2i\omega_k t}\Big]$$
$$+ \frac{c^2}{4}\sum_k \frac{k^2}{\omega_k}\Big[\text{---''---} + \text{---''---} + \text{---''---} + \text{---''---}\Big]$$
$$+ \frac{m^2}{4}\sum_k \frac{1}{\omega_k}\Big[\text{---''---} + \text{---''---} + \text{---''---} + \text{---''---}\Big]$$

$$= \frac{1}{4}\sum_k \Big[\hat a_k \hat a_{-k} e^{-2i\omega_k t}\Big(-\omega_k + \underbrace{\frac{c^2k^2}{\omega_k} + \frac{m^2}{\omega_k}}_{=\omega_k}\Big)$$
$$+ \hat a_k \hat a_k^+ \Big(\omega_k + \underbrace{\frac{c^2k^2}{\omega_k} + \frac{m^2}{\omega_k}}_{=\omega_k}\Big)$$
$$+ \hat a_k^+ \hat a_k \Big(\omega_k + \underbrace{\frac{c^2k^2}{\omega_k} + \frac{m^2}{\omega_k}}_{=\omega_k}\Big)$$
$$+ \hat a_k^+ \hat a_{-k}^+ e^{2i\omega_k t}\Big(-\omega_k + \underbrace{\frac{c^2k^2}{\omega_k} + \frac{m^2}{\omega_k}}_{=\omega_k}\Big)\Big]$$

used dispersion relation
$$\omega_k^2 = c^2k^2 + m^2$$

$$\Rightarrow \boxed{\hat H = \frac{1}{2}\sum_k \omega_k\big(\hat a_k \hat a_k^+ + \hat a_k^+ \hat a_k\big)}$$ (without discussing "normal ordering")

use $\hat a_k \hat a_k^+ = \hat a_k^+ \hat a_k + 1$ from eq. (13)

$$\Rightarrow \boxed{\hat H = \sum_k \omega_k \hat a_k^+ \hat a_k} + \underbrace{\sum_k \omega_k \frac{1}{2}}_{\text{remove term, see lecture...}} \qquad (14)$$

Noether's theorem for translational invariance

$\Longrightarrow$ <u>total momentum operator</u> $\hat{p}$

<u>def.</u> $$\hat{p} = -\frac{1}{2}\int_0^{x_L} dx \left[\dot{\hat{\phi}}(x,t)\, \partial_x \hat{\phi}(x,t) + \partial_x \hat{\phi}(x,t)\, \dot{\hat{\phi}}(x,t) \right]$$

first part:

$$\int_0^{x_L} dx \underbrace{\dot{\hat{\phi}}(x,t)}_{k} \underbrace{\partial_x \hat{\phi}(x,t)}_{l} = \frac{1}{2x_L}\int_0^{x_L} dx \sum_{k,l} \sqrt{\frac{\omega_k}{\omega_l}}\, l$$

$$\times \left[\hat{a}_k e^{ikx - i\omega_k t} + \hat{a}_k^+ e^{-ikx + i\omega_k t} \right]$$

$$\times \left[\hat{a}_l e^{ilx - i\omega_l t} - \hat{a}_l^+ e^{-ilx + i\omega_l t} \right]$$

$$= \frac{1}{2x_L} \sum_{k,l} \sqrt{\frac{\omega_k}{\omega_l}}\, l \int_0^{x_L} dx \Big[\hat{a}_k \hat{a}_l e^{i(k+l)x} e^{-i(\omega_k + \omega_l)t}$$

$$- \hat{a}_k \hat{a}_l^+ e^{i(k-l)x} e^{-i(\omega_k - \omega_l)t}$$

$$+ \hat{a}_k^+ \hat{a}_l e^{-i(k-l)x} e^{i(\omega_k - \omega_l)t}$$

$$+ \hat{a}_k^+ \hat{a}_l^+ e^{-i(k+l)x} e^{i(\omega_k + \omega_l)t} \Big]$$

$$= \frac{\cancel{x_L}}{2\cancel{x_L}} \sum_{k,l} \sqrt{\frac{\omega_k}{\omega_l}}\, l \Big[\hat{a}_k \hat{a}_l \delta_{k,-l} e^{-i(\omega_k + \omega_l)t}$$

$$- \hat{a}_k \hat{a}_l^+ \delta_{k,l} e^{-i(\omega_k - \omega_l)t}$$

$$+ \hat{a}_k^+ \hat{a}_l \delta_{k,l} e^{i(\omega_k - \omega_l)t}$$

$$+ \hat{a}_k^+ \hat{a}_l^+ \delta_{k,-l} e^{i(\omega_k + \omega_l)t} \Big]$$

$$= \frac{1}{2}\sum_k k \left[-\hat{a}_k \hat{a}_{-k} e^{-2i\omega_k t} - \hat{a}_k \hat{a}_k^+ - \hat{a}_k^+ \hat{a}_k - \hat{a}_k^+ \hat{a}_{-k}^+ e^{2i\omega_k t} \right]$$

second part:

$$\int_0^{x_L} dx \underbrace{\partial_x \hat{\phi}(x,t)}_{k} \underbrace{\dot{\hat{\phi}}(x,t)}_{\omega} = \frac{1}{2}\sum_k k \left[\hat{a}_k \hat{a}_{-k} e^{-2i\omega_k t} - \hat{a}_k \hat{a}_k^+ - \hat{a}_k^+ \hat{a}_k + \hat{a}_k^+ \hat{a}_{-k}^+ e^{2i\omega_k t} \right]$$

$$\Rightarrow \hat{p} = -\frac{1}{4}\sum_k k \left(-2\hat{a}_k \hat{a}_k^+ - 2\hat{a}_k^+ \hat{a}_k \right)$$

$$= \frac{1}{2}\sum_k k \left(\hat{a}_k \hat{a}_k^+ + \hat{a}_k^+ \hat{a}_k \right)$$

use $\hat{a}_k \hat{a}_k^+ = \hat{a}_k^+ \hat{a}_k + 1$ from eq. (13)

$$\Rightarrow \boxed{\hat{p} = \sum_k k \, \hat{a}_k^+ \hat{a}_k} \underbrace{+\frac{1}{2}\sum_k k}_{=0} \qquad (15)$$

thus, we found eq. (14) and (15)

$$\boxed{\begin{aligned} \hat{p} &= \sum_k k \, \hat{n}_k \\ \hat{H} &= \sum_k \omega_k \hat{n}_k \\ \hat{n}_k &= \hat{a}_k^+ \hat{a}_k \end{aligned}}$$

total momentum operator

total energy operator

number operator of scalars with momentum $k = \frac{2\pi}{x_L} n, \; n \in \mathbb{Z}$

next task: compute algebra of $\hat{a}_k, \hat{a}_k^+, \hat{n}_k, \hat{p}$ & $\hat{H}$

Compute $[\hat{n}_k, \hat{n}_l]$:

if $k = l$: $[\hat{n}_k, \hat{n}_k] = 0$

if $k \neq l$: $[\hat{n}_k, \hat{n}_l] = [\hat{a}_k^+ \hat{a}_k, \hat{a}_l^+ \hat{a}_l] = 0$

using $[\hat{a}_k, \hat{a}_l] = [\hat{a}_k, \hat{a}_l^+] = [\hat{a}_k^+, \hat{a}_l^+] = 0$ for $k \neq l$

$$\Rightarrow \boxed{[\hat{n}_k, \hat{n}_l] = 0}$$

Compute $[\hat{p}, \hat{n}_k]$:

$$[\hat{p}, \hat{n}_k] = \sum_l l \underbrace{[\hat{n}_l, \hat{n}_k]}_{=0} = 0$$ using eq. (15)

Compute $[\hat{H}, \hat{n}_k]$:

$$[\hat{H}, \hat{n}_k] = \sum_l \omega_l \underbrace{[\hat{n}_l, \hat{n}_k]}_{=0} = 0$$ using eq. (14)

Compute $[\hat{H}, \hat{p}]$:

$$[\hat{H}, \hat{p}] = \sum_{k,l} \omega_k l \underbrace{[\hat{n}_k, \hat{n}_l]}_{=0} = 0$$

$\Rightarrow$ $\boxed{\hat{n}_k, \hat{p} \text{ and } \hat{H} \text{ commute with eachother}}$

there are simult. eigenstates of $\hat{n}_k, \hat{p}$ and $\hat{H}$ specified by eigenvalues n_k of $\hat{n}_k$:

$$|\{n_l\}_{l = \frac{2\pi}{L} n,\, n \in \mathbb{Z}}\rangle = |\{n_l\}\rangle$$ $\Rightarrow$ Fock space

such that

$$\hat{n}_k |\{n_l\}\rangle = n_k |\{n_l\}\rangle$$

then

$$\hat{H}|\{n_l\}\rangle = \sum_k \omega_k \hat{n}_k |\{n_l\}\rangle = \sum_k \omega_k n_k |\{n_l\}\rangle$$

$$= E|\{n_l\}\rangle \quad \text{where} \quad \boxed{E = \sum_k \omega_k n_k}$$

$$\hat{p}|\{n_l\}\rangle = \sum_k k \hat{n}_k |\{n_l\}\rangle = \sum_k k n_k |\{n_l\}\rangle$$

$$= p|\{n_l\}\rangle \quad \text{where} \quad \boxed{p = \sum_k k n_k}$$

<u>ground state $|0\rangle$</u>

$|0\rangle$ defined as state with lowest energy; we can identify $|0\rangle$ as follows:

compute

$$\langle\{n_l\}|\hat{H}|\{n_l\}\rangle = E \underbrace{\langle\{n_l\}|\{n_l\}\rangle}_{=1 \text{ normalized state}} = E$$

$$= \sum_k \omega_k \langle\{n_l\}|\hat{n}_k|\{n_l\}\rangle$$

$$= \sum_k \underbrace{\omega_k}_{\geq 0} \underbrace{\langle\{n_l\}|\hat{a}_k^+ \hat{a}_k|\{n_l\}\rangle}_{= \| \hat{a}_k |\{n_l\}\rangle \|^2 \geq 0}$$

$\Rightarrow E \geq 0$

$\Rightarrow$ lowest energy is $E = 0$ for a state $|\{n_l\}\rangle$ with $E = \sum_k \omega_k n_k = 0 \Leftrightarrow n_k = 0$ for all k

$$\Rightarrow \boxed{|0\rangle \overset{\text{def.}}{=} |\{n_l = 0\}\rangle} \quad \underline{\text{ground state}}$$

<u>what is</u> $\hat{a}_k |0\rangle$ <u>?</u>

to answer this question, we need $[\hat{H}, \hat{a}_k]$!

<u>thus:</u>

<u>compute</u> $[\hat{n}_\ell, \hat{a}_k]$

$$[\hat{n}_\ell, \hat{a}_k] = \hat{a}_\ell^+ \hat{a}_\ell \hat{a}_k - \hat{a}_k \hat{a}_\ell^+ \hat{a}_\ell$$
$$= \hat{a}_\ell^+ (\hat{a}_\ell \hat{a}_k - \hat{a}_k \hat{a}_\ell) - (\hat{a}_k \hat{a}_\ell^+ - \hat{a}_\ell^+ \hat{a}_k) \hat{a}_\ell$$
$$= \hat{a}_\ell^+ \underbrace{[\hat{a}_\ell, \hat{a}_k]}_{=0} - \underbrace{[\hat{a}_k, \hat{a}_\ell^+]}_{=\delta_{k\ell}} \hat{a}_\ell$$
$$= -\delta_{k,\ell}\, \hat{a}_\ell = -\delta_{k,\ell}\, \hat{a}_k$$

<u>and</u>

$$[\hat{n}_\ell, \hat{a}_k^+] = \hat{n}_\ell \hat{a}_k^+ - \hat{a}_k^+ \hat{n}_\ell = (\hat{a}_k \hat{n}_\ell^+ - \hat{n}_\ell^+ \hat{a}_k)^+$$
$$\overset{\hat{n}_\ell^+ = \hat{n}_\ell}{=} [\hat{a}_k, \hat{n}_\ell]^+ = -[\hat{n}_\ell, \hat{a}_k]^+$$
$$= +\delta_{k,\ell}\, \hat{a}_k^+$$

$$\Rightarrow \boxed{\begin{aligned} [\hat{n}_\ell, \hat{a}_k] &= -\delta_{k,\ell}\, \hat{a}_k \\ [\hat{n}_\ell, \hat{a}_k^+] &= +\delta_{k,\ell}\, \hat{a}_k^+ \end{aligned}} \qquad (16)$$

<u>then</u>

$$[\hat{H}, \hat{a}_k] = \sum_\ell w_\ell \underbrace{[\hat{n}_\ell, \hat{a}_k]}_{\overset{(16)}{=} -\delta_{k,\ell} \hat{a}_k} = -w_k \hat{a}_k$$

$$\Rightarrow \hat{H}(\hat{a}_k |0\rangle) = (\hat{a}_k \hat{H} - w_k \hat{a}_k)|0\rangle = (0 - w_k)\, \hat{a}_k |0\rangle$$

$\Rightarrow \hat{a}_k|0\rangle$ has energy $(-\omega_k) < 0$ contradiction to def. of $|0\rangle$

$\Rightarrow \boxed{\hat{a}_k|0\rangle = 0}$ $\hat{a}_k$ annihilation operator

for all k and $|0\rangle$ is annihilated by all $\hat{a}_k$!

<u>How to create general state $|\{n_l\}\rangle$</u>

$$\hat{n}_k\left(\hat{a}_l|\{n_q\}\rangle\right) \overset{(16)}{=} \left(\hat{a}_l\hat{n}_k - \delta_{kl}\hat{a}_l\right)|\{n_q\}\rangle$$

$$= (n_k - \delta_{kl})\,\hat{a}_l|\{n_q\}\rangle$$

$\Rightarrow \hat{a}_l|\{n_q\}\rangle \sim |n_l - 1, \{n_q\}_{q\neq l}\rangle$ one particle removed with momentum l

and

$$\hat{n}_k\left(\hat{a}_l^+|\{n_q\}\rangle\right) \overset{(16)}{=} \left(\hat{a}_l^+\hat{n}_k + \delta_{kl}\hat{a}_l^+\right)|\{n_q\}\rangle$$

$$= (n_k + \delta_{kl})\,\hat{a}_l^+|\{n_q\}\rangle$$

$\Rightarrow \hat{a}_l^+|\{n_q\}\rangle \sim |n_l + 1, \{n_q\}_{q\neq l}\rangle$ one particle created with momentum l

$\Rightarrow$ <u>multi-particle state</u>

$$|\{n_k\}\rangle = \prod_k \frac{(\hat{a}_k^+)^{n_k}}{\sqrt{n_k!}}|0\rangle$$

$(\hat{a}_k^+)^{n_k}$ creates n_k-particles with momentum k