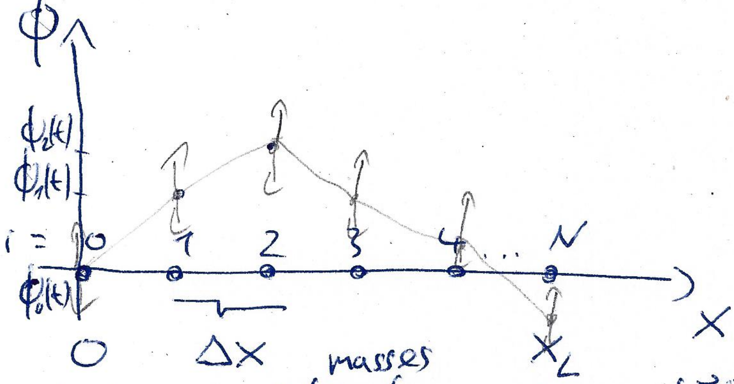
B) Classical vibrating string

Setting in two dim.



$\phi_i(t)$ vertical displacement of i-th ~~mass~~ point (mass Δm_i) mass

$\hat{=}$ generalized coordinate

- $N+1$ ~~mass~~ points of mass Δm_i at $\vec{r}_i(t) = (x_i(t), \phi_i(t))$ where $x_i(t) \approx i\Delta x_i$
- String feels force P such that its length is x_L = const.
- springs between i-th and (i+1)-th mass points
- small excitations of string: i-th mass point only moves up and down

kinetic energy of string

$$T_N(\phi_i(t)) = \sum_{i=0}^{N} \frac{1}{2} \Delta m_i \left(\frac{d\phi_i}{dt}\right)^2$$

because $\dot{x}_i(t) \approx 0$

$$\rho = \frac{\Delta m_i}{\Delta x} = \text{const.}$$

$$= \sum_{i=0}^{N} \frac{1}{2} \rho \Delta x \left(\frac{d\phi_i}{dt}\right)^2$$

$\phi_i = \phi_i(t)$

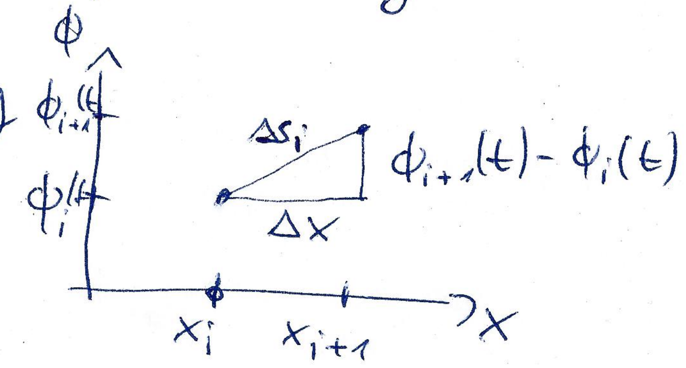
$$T(\phi(x,t)) = \lim_{N\to\infty} T_N(\phi_i(t)) = \frac{\rho}{2} \int_0^{x_L} dx \left(\frac{\partial \phi(x,t)}{\partial t}\right)^2$$

where $\phi(x_i, t) = \phi_i(t)$ and $\phi(x,t) = \phi_x(t)$ ← continuous index

↑ one gen. coord. $\phi_x(t)$ at each x

potential energy of string

excited string



$$\Delta s_i = \sqrt{\Delta x^2 + \left(\phi_{i+1}(t) - \phi_i(t)\right)^2}$$

$$= \Delta x \sqrt{1 + \left(\frac{\phi_{i+1}(t) - \phi_i(t)}{\Delta x}\right)^2} \qquad \sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon$$

$$\approx \Delta x \left[1 + \frac{1}{2}\left(\frac{\phi_{i+1}(t) - \phi_i(t)}{\Delta x}\right)^2\right]$$

for $\Delta s_i = \Delta x$ "external" force P on (x_0, ϕ_0) and (x_N, ϕ_N)

for $\Delta s_i > \Delta x$ additional potential energy $P(\Delta s_i - \Delta x)$

neglect: external force → potential energy
on x_0 and x_L

$$\Rightarrow U_N(\phi_i(t)) = \sum_{i=0}^{N} P(\Delta s_i - \Delta x)$$

$$= \sum_{i=0}^{N} \frac{P}{2} \Delta x \left(\frac{\phi_{i+1}(t) - \phi_i(t)}{\Delta x}\right)^2$$

$$\phi_i(t) = \phi(i\Delta x, t)$$

$$\frac{\phi_{i+1}(t) - \phi_i(t)}{\Delta x} = \frac{\phi((i+1)\Delta x, t) - \phi(i\Delta x, t)}{\Delta x} \approx \frac{\partial \phi(x,t)}{\partial x}$$

$$U(\phi(x,t)) = \lim_{N\to\infty} U_N(\phi_i(t))$$

$$= \frac{P}{2} \int_0^{x_L} dx \left(\frac{\partial \phi(x,t)}{\partial x}\right)^2$$

Lagrange fct. of classical string

$$L\left(\phi(x,t), \frac{\partial\phi(x,t)}{\partial x}, \frac{\partial\phi(x,t)}{\partial t}\right) = T - U$$

$$= \int_0^{x_L} dx \underbrace{\left[\frac{\rho}{2}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 - \frac{P}{2}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2\right]}_{=\mathcal{L}\left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}\right) \text{ Lagrange density}}$$

action

Lagrange fct.

$$S[\phi] = \int_{t_1}^{t_2} dt\, L[\phi]$$

$$= \int_{t_1}^{t_2} dt \int_0^{x_L} dx \left[\frac{\rho}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{P}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]$$

$$= \int_{t_1}^{t_2} dt \int_0^{x_L} dx\; \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}\right)$$

in general

Euler-Lagrange eqns for Lagrange densities

action $S[\phi]$

principle of stationary action $\delta S[\phi] \overset{!}{=} 0$

then:

$$\delta S[\phi] = S[\phi + \underbrace{\delta\phi}_{\text{small}}] - S[\phi]$$

$$= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \Big[\mathcal{L}(\phi + \delta\phi, \partial_t \phi + \partial_t \delta\phi, \partial_x \phi + \partial_x \delta\phi) - \mathcal{L}(\phi, \partial_t \phi, \partial_x \phi) \Big]$$

$$= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \Big[\cancel{\mathcal{L}(\phi, \partial_t \phi, \partial_x \phi)} + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \partial_t \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \partial_x \delta\phi - \cancel{\mathcal{L}(\phi, \partial_t \phi, \partial_x \phi)} \Big]$$

$$= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right) \delta\phi - \partial_x \left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right) \delta\phi \right]$$

$$+ \underbrace{\int_{x_1}^{x_2} dx \left[\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta\phi \right] \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left[\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta\phi \right] \Bigg|_{x_1}^{x_2}}_{=0}$$

$$\overset{!}{=} 0$$

$=0$ by assumption: $\delta\phi(x,t)$ vanishes on boundaries of $\int_{t_1}^{t_2}\int_{x_1}^{x_2}$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

~~Euler~~-Lagrange e.o.m.

e.o.m for string

$$\mathcal{L}(\partial_t \phi, \partial_x \phi) = \frac{\rho}{2}(\partial_t \phi)^2 - \frac{P}{2}(\partial_x \phi)^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \rho(\partial_t \phi) \implies \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\right) = \rho(\partial_t^2 \phi)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} = -P(\partial_x \phi) \implies \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)}\right) = -P(\partial_x^2 \phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\implies \rho(\partial_t^2 \phi) - P(\partial_x^2 \phi) = 0$$

$$\cancel{\left(\partial_t^2 - \frac{P}{\rho}\partial_x^2\right)\phi(x,t) = 0}$$

$$\left(\partial_x^2 - \frac{\rho}{P}\partial_t^2\right)\phi(x,t) = 0$$

def. $c = \sqrt{\frac{P}{\rho}}$ wave velocity

$$\boxed{\left(\partial_x^2 - \frac{1}{c^2}\partial_t^2\right)\phi(x,t) = 0}$$

wave-equation

note: conjugate momentum of $\phi(x,t)$: $\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \rho\,\dot{\phi}(x,t)$

from discrete case $\phi_i(t)$, $\pi_i(t)$ (labelled by index i):

$\{\phi_i, \phi_j\} = 0$ and $\{\phi_i, \pi_j\} = \delta_{ij}$

$\{\pi_i, \pi_j\} = 0$

now: continuous index

$\{\phi(x,t), \phi(y,t)\} = 0$ and $\{\phi(x,t), \pi(y,t)\} = \delta_{xy}$

$\{\pi(x,t), \pi(y,t)\} = 0$

redefinition: $\tilde{\phi}(x,t) = \sqrt{\rho}\,\phi(x,t)$

$$\mathcal{L}(\partial_t \tilde{\phi}, \partial_x \tilde{\phi}) = \frac{1}{2}(\partial_t \tilde{\phi})^2 - \frac{P}{2\rho}(\partial_x \tilde{\phi})^2 - V(\tilde{\phi})$$

$$\frac{P}{\rho} = c^2$$

allow for add. potential
e.g. $V(\phi) = \frac{1}{2} m^2 \tilde{\phi}^2$

drop tilde

$$\mathcal{L}(\partial_t \phi, \partial_x \phi) = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}c^2(\partial_x \phi)^2 - \frac{1}{2}m^2\phi^2$$

$$c=1: \frac{1}{2}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

e.o.m.:

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t\phi)} = \frac{\partial}{\partial t}(\partial_t\phi) = \partial_t^2\phi$$

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ $\quad g_{\mu\nu} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\partial_0 = \frac{\partial}{\partial t}$ $\quad \partial_1 = \frac{\partial}{\partial x}$

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x\phi)} = \frac{\partial}{\partial x}\left(-c^2(\partial_x\phi)\right) = -c^2\partial_x^2\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t\phi)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x\phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\boxed{\partial_t^2\phi - c^2\partial_x^2\phi + m^2\phi = 0}$$

and

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi$$