QFT Prep. Course 2018

Patrick Vaudrevange room PH 3328

dates 10. Oct. 2018 - 12. Oct. 2018

Wed.	$10^{00} - 11^{30}$	and	$12^{30} - 14^{00}$	HS3
Thu.	$9^{30} - 11^{00}$	and	$14^{00} - 15^{30}$	HS3
Fr.	$10^{00} - 11^{30}$	and	$12^{30} - 14^{00}$	LMU A348/349

topics

A) Classical mechanics

B) Classical vibrating string

C) Quantization

A) Classical mechanics

Newton's laws of motion

1) There exists a frame of reference, called inertial frame of reference, in which an object either remains at rest or continues to move with const. velocity, unless a net force acts on it:

$$\underset{\substack{\uparrow \\ \text{net force}}}{\vec{F}} = \sum_i \vec{F}_i = 0 \iff \frac{d\vec{v}}{dt} = 0$$

velocity does not change

2) In an inertial frame of reference we have

$$\vec{F} = \frac{d\vec{p}}{dt}$$

where : momentum $\vec{p} = m\vec{v}$

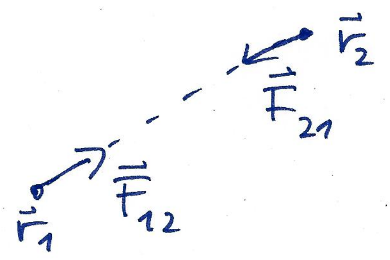
velocity $\vec{v} = \frac{d\vec{r}}{dt}$

for const. mass (i.e. $\frac{dm}{dt} = 0$):

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\,\vec{a}$$

where: $\vec{a} = \frac{d^2\vec{r}}{dt^2}$

3) actio = reactio



$$\vec{F}_{12} = -\vec{F}_{21}$$

forces are equal up to (-1), i.e. they act in opposite directions

equation of motion (e.o.m.)

consider forces $\vec{F}$ that depend only on $\vec{r}(t), \dot{\vec{r}}(t)$ and t:

$$\vec{F} = \vec{F}(\vec{r}(t), \dot{\vec{r}}(t), t)$$

Newton's second law:

$$\frac{d}{dt}\left(m\dot{\vec{r}}(t)\right) = \vec{F}(\vec{r}(t), \dot{\vec{r}}(t), t) \qquad \text{e.o.m.}$$

this is an ordinary differential equation of second order

↖ (just depends on one indep. variable t)

↑

(at most second derivatives)

examples

1) $\vec{F} = 0$ and $m = \text{const.}$

$$\Rightarrow m\frac{d^2\vec{r}}{dt} = 0 \qquad \text{e.o.m.}$$

$$\Rightarrow \vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

solution of e.o.m.

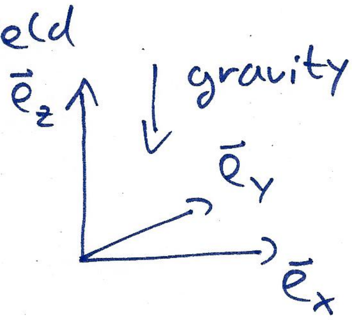
≙ trajectory of point mass m

2) $\vec{F} = -mg\vec{e}_z$ and $m = \text{const.}$

homogeneous gravitational field

↑

(indep. of position $\vec{r}(t)$)



$$\Rightarrow m\frac{d^2\vec{r}}{dt^2} = -mg\vec{e}_z \qquad \text{e.o.m.}$$

$$\Rightarrow \vec{r}(t) = \vec{r}_0 + \vec{v}_0 t - \frac{1}{2}gt^2\vec{e}_z$$

3) $\vec{F} = -k\vec{r}$ and $m = \text{const.}$

harmonic oscillator

k: spring constant $[k] = \frac{N}{m} = \frac{kg\,\frac{m}{s^2}}{m} = \frac{kg}{s^2}$

$$\Rightarrow m\frac{d^2\vec{r}}{dt^2} = -k\vec{r} \qquad \text{e.o.m.}$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \qquad \text{where } \omega = \sqrt{\frac{k}{m}}$$

$[\omega] = \frac{1}{s}$ angular fequency

$\omega = 2\pi f = \frac{2\pi}{T}$

frequency ↗ period ↖

$$\Rightarrow \vec{r}(t) = \vec{A}\sin(\omega t) + \vec{B}\cos(\omega t)$$

test: $\frac{d\vec{r}}{dt} = \vec{A}\cos(\omega t)\,\omega - \vec{B}\sin(\omega t)\,\omega$

$$\frac{d^2\vec{r}}{dt^2} = -\vec{A}\sin(\omega t)\,\omega^2 - \vec{B}\cos(\omega t)\,\omega^2$$

$$= -\omega^2\left(\vec{A}\sin(\omega t) + \vec{B}\cos(\omega t)\right) = -\omega^2\vec{r} \quad \square$$

Conservative force

a force $\vec{F}$ is called conservative if there exists a potential $U(\vec{r})$ such that

$$\vec{F} = -\vec{\nabla} U(\vec{r})$$

where

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad \text{gradient}$$

and $U(\vec{r})$ is a number

equivalently:
- $\vec{\nabla} \times \vec{F} = 0$
- work $W \equiv \oint_C \vec{F} \cdot d\vec{r} = 0$

  work done by force $\vec{F}$ along closed curve C is zero

examples

- gravity

  $U(\vec{r}) = mgz \quad \Rightarrow \quad \vec{F} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} = -mg\,\vec{e}_z$

- harmonic oscillator

  $U(\vec{r}) = \frac{1}{2} k \vec{r}^{\,2} = \frac{1}{2} k (x^2 + y^2 + z^2)$

  $\Rightarrow \vec{F} = -\frac{1}{2} k \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = -k\vec{r}$

energy for systems with conservative forces

assume $\bar{F} = -\vec{\nabla} U(\vec{r})$

and $m = \text{const.}$

$$\Rightarrow \text{e.o.m.}: \quad m\ddot{\vec{r}}(t) = -\vec{\nabla} U(\vec{r}(t)) \qquad \Big| \cdot \dot{\vec{r}}$$

$$m\left(\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial t}\right)$$

$$= -\left(\frac{\partial U}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial t}\right) \qquad \Big| \int dt$$

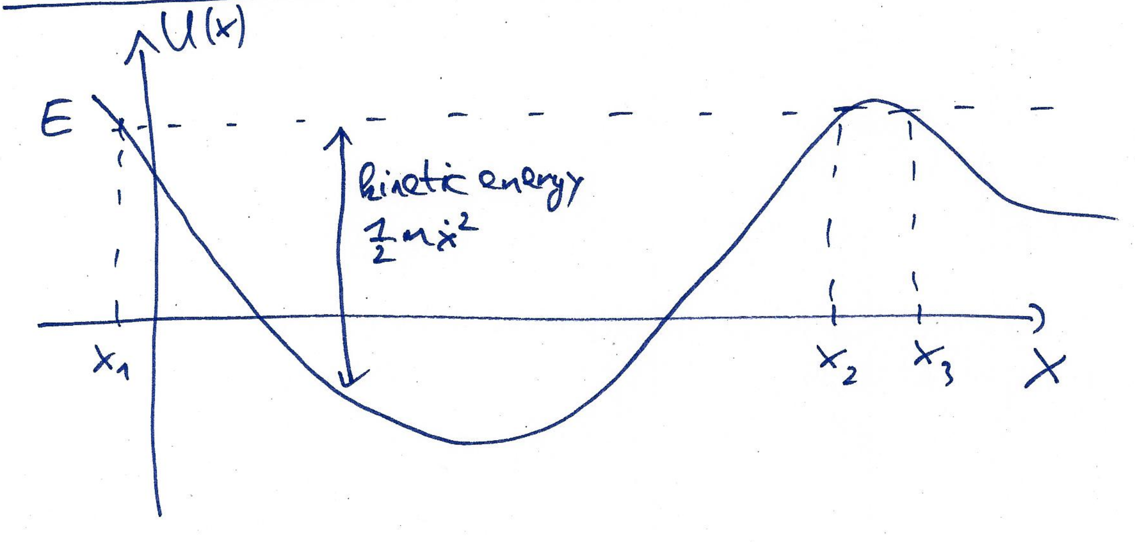
$$\frac{1}{2}m\left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2\right] = \underset{\substack{\uparrow \\ \text{integration constant}}}{E} - U(\vec{r}(t))$$

$$\Rightarrow \boxed{E = \frac{1}{2}m(\dot{\vec{r}}(t))^2 + U(\vec{r}(t)) = \text{const.}}$$

kinetic energy + potential energy = total energy E

i.e. E = const. for conservative forces

one-dimensional illustration



at x_i, $i = 1,2,3$
kinetic energy = 0

Holonomic constraints

(standard motivation to use generalized coordinates q_k instead of coordinates $\vec{r}$)

def. holonomic constraints

$$g_\alpha(\underset{\uparrow}{\vec{r}_1},\dots,\underset{\uparrow}{\vec{r}_N},t) = 0 \qquad \text{for } \alpha = 1,..,R$$

pos. of N point masses

$\Rightarrow$ # free degrees of freedom (d.o.f.): $f = \underset{\uparrow}{3}N - R$

space has 3 dimensions

Constraints induce forces onto i-th point mass:

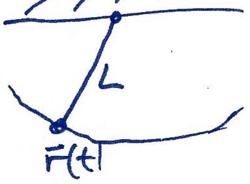
$$\vec{Z}_i(\vec{r}_i,t) = \sum_{\alpha=1}^{R} \lambda_\alpha \vec{\nabla}_i g_\alpha$$

force pushes $\vec{r}_i$ back to the "submanifold" defined by $g_\alpha(\vec{r}_1,\dots,\vec{r}_N,t) = 0$

e.g. $R=1$, $N=1$: $m\ddot{\vec{r}}(t) = \vec{F}(\vec{r}(t)) + \lambda\vec{\nabla}g(\vec{r},t)$

pendulum:

$g(\vec{r}(t)) = \vec{r}^2 - L^2$
$= 0$



and we have:

Lagrange's eqn (first kind)

$$m_i\ddot{x}_i = F_i + \sum_{\alpha=1}^{R} \lambda_\alpha \frac{\partial g_\alpha(x_1,\dots,x_{3N},t)}{\partial x_i} \qquad i=1,..,3N \qquad (1)$$

& $g_\alpha(x_1,\dots,x_{3N},t) = 0$ for $\alpha = 1,..,R$

generalized coordinates

choose $q_1, \dots q_f$ such that one can parametrize

$$x_i = x_i(q_1, \dots, q_f, t) \quad \text{for } i = 1, .., 3N$$

and for all q_i the constraints are fulfilled:

$$g_\alpha\left(x_1(q_1, \dots, q_f, t), \dots, x_{3N}(q_1, \dots, q_f, t)\right) \equiv 0$$

eliminate forces due to constraints

since $g_\alpha \equiv 0$ $\Rightarrow 0 = \dfrac{d g_\alpha}{d q_k} = \displaystyle\sum_{i=1}^{3N} \frac{\partial g_\alpha}{\partial x_i} \frac{\partial x_i}{\partial q_k}$ for $k = 1, .., f$

use eq. (1) times $\dfrac{\partial x_i}{\partial q_k}$ and sum over i:

$$\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k} = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_k} + \sum_{\alpha=1}^{R} \lambda_\alpha \underbrace{\sum_{i=1}^{3N} \frac{\partial g_\alpha}{\partial x_i} \frac{\partial x_i}{\partial q_k}}_{= 0 \text{ from above}} \quad (2)$$

where $x_i = x_i(q_1, \dots, q_f, t)$

Derivation of Lagrange's eqn (second kind)

$$x_i = x_i(q_1, \dots, q_f, t) = x_i(q, t) \qquad i = 1, \dots, 3N$$

$$\Rightarrow \dot{x}_i = \frac{d}{dt} x_i(q, t) = \sum_{k=1}^{f} \frac{\partial x_i(q,t)}{\partial q_k} \dot{q}_k + \frac{\partial x_i(q,t)}{\partial t}$$

$$\overset{\text{def.}}{=} \dot{x}_i(q, \dot{q}, t)$$

$$\Rightarrow \frac{\partial \dot{x}_i(q, \dot{q}, t)}{\partial \dot{q}_k} = \frac{\partial x_i(q,t)}{\partial q_k} \qquad (3)$$

Kinetic energy

$$T = \sum_{i=1}^{3N} \frac{m_i}{2} \dot{x}_i^2$$

$$\Rightarrow T(q,\dot{q},t) = \sum_{i=1}^{3N} \frac{m_i}{2} \left(\dot{x}_i(q,\dot{q},t)\right)^2$$

then

$$\frac{\partial T(q,\dot{q},t)}{\partial q_k} = \sum_{i=1}^{3N} m_i \dot{x}_i(q,\dot{q},t) \frac{\partial \dot{x}_i(q,\dot{q},t)}{\partial q_k} \qquad (4)$$

and

$$\frac{\partial T(q,\dot{q},t)}{\partial \dot{q}_k} = \sum_{i=1}^{3N} m_i \dot{x}_i(q,\dot{q},t) \underbrace{\frac{\partial \dot{x}_i(q,\dot{q},t)}{\partial \dot{q}_k}}_{\overset{\text{eq.}(3)}{=} \frac{\partial x_i(q,t)}{\partial q_k}} \qquad \Big| \frac{d}{dt}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k} + \underbrace{\sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k}}_{\overset{\text{eq.}(4)}{=} \frac{\partial T}{\partial q_k}}$$

use eq. (2), i.e. $\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k} = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_k}$ and obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_k} \overset{\text{def.}}{=} \underset{\text{generalized force}}{Q_k} \qquad (5)$$

assume that all forces $\vec{F}_i$ are conservative, i.e.

$$\vec{F}_i = -\vec{\nabla}_i U(\vec{r}_j) \qquad i = 1, \ldots, N$$

i.e.

$$F_i = -\frac{\partial U}{\partial x_i} \qquad i = 1, \ldots, 3N \qquad \text{and } U = U(x_1, \ldots, x_{3N})$$

def. $U(q,t) = U(x_1(q,t), \ldots, x_{3N}(q,t))$

then

$$Q_k = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_k} = -\sum_{i=1}^{3N} \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_k} = -\frac{\partial U(q,t)}{\partial q_k}$$

put this into eq. (5) and get:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k} \qquad \text{for } k = 1, \ldots, f$$

use $\frac{\partial U}{\partial \dot{q}_k} = 0 \Rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}(T-U) - \frac{\partial}{\partial q_k}(T-U) = 0$

def. $\boxed{\mathcal{L}(q, \dot{q}, t) = T(q, \dot{q}, t) - U(q,t)}$ Lagrange function

and we have found the Euler-Lagrange equs

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q_k} = 0}$$

Hamilton's principle

def. action S as $S[q] = \int_{t_1}^{t_2} dt\, \mathcal{L}(q, \dot{q}, t)$

i.e. assign a number $S[q]$ to each trajectory $q(t)$

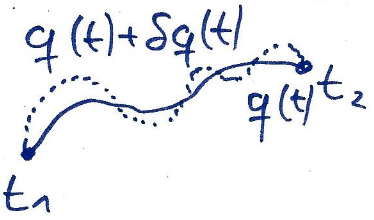
then: Hamilton's principle says that

$$\boxed{q(t) \text{ is a solution of } \delta S[q] \overset{!}{=} 0}$$

in other words:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \iff \delta S[q] = 0}$$

what is $\delta S[q]$?



$$\delta S[q] \overset{\text{def.}}{=} S[q + \underset{\uparrow\ \text{small deviation from path } q(t)}{\delta q}] - S[q]$$

$$= \int_{t_1}^{t_2} dt \left[\mathcal{L}(q+\delta q, \dot{q}+\delta\dot{q}, t) - \mathcal{L}(q, \dot{q}, t)\right]$$

$$\overset{\text{Taylor}}{=} \int_{t_1}^{t_2} dt \left[\cancel{\mathcal{L}(q, \dot{q}, t)} + \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial q_k}\delta q_k + \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\delta \dot{q}_k - \cancel{\mathcal{L}(q, \dot{q}, t)}\right]$$

$$= \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \left[\frac{\partial \mathcal{L}}{\partial q_k}\delta q_k + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\delta q_k\right)\right.$$

$$- \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \delta q_k \Bigg]$$

$$= - \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \left[\left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} \right) \delta q_k \right]$$

$$\underbrace{+ \int_{t_1}^{t_2} dt \, \frac{d}{dt} \left[\sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta q_k \right]}$$

$= 0$ because we assume

$\delta q_k(t_1) = \delta q_k(t_2) = 0$

$\overset{!}{=} 0$ for all choices of δq_k

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0} \quad \text{for } k = 1, \dots, f$$

Hamiltonian mechanics

given Lagrangian $\mathcal{L}(q, \dot{q}, t)$ perform following steps:

1) def. generalized momenta also called conjugate momenta $p_k(q, \dot{q}, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$ (6) for $k = 1, \dots f$

2) express $\dot{q}_k$ in terms of q, p and t by inverting eq. (6) $\Rightarrow$ $\dot{q}_k = \dot{q}_k(q, p, t)$

3) def. Hamiltonian H as Legendre trafo of $\mathcal{L}$:

$$H = \sum_{k=1}^{f} \dot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \mathcal{L} = \sum_{k=1}^{f} \dot{q}_k p_k - \mathcal{L}(q, \dot{q}, t)$$

insert $\dot{q} = \dot{q}(q, p, t)$

then $H = H(q,p,t)$ $\quad q,p$ indep. variables

now compute

$$\bullet\ \frac{\partial H}{\partial q_l} = \sum_{k=1}^{f} \frac{\partial \dot q_k}{\partial q_l} \underbrace{p_k}_{\overset{eq.(6)}{=} \frac{\partial \mathcal{L}}{\partial \dot q_k}} - \frac{\partial \mathcal{L}}{\partial q_l} - \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot q_k} \frac{\partial \dot q_k}{\partial q_l}$$

$$= -\frac{\partial \mathcal{L}}{\partial q_l} \overset{e.o.m.}{=} -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_l} = -\dot p_l$$

$$\bullet\ \frac{\partial H}{\partial p_l} = \sum_{k=1}^{f} \frac{\partial \dot q_k}{\partial p_l} \underbrace{p_k}_{\overset{eq.(6)}{=} \frac{\partial \mathcal{L}}{\partial \dot q_k}} + \sum_{k=1}^{f} \dot q_k \underbrace{\frac{\partial p_k}{\partial p_l}}_{= \delta_{kl}}$$

$$- \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot q_k} \frac{\partial \dot q_k}{\partial p_l} = \dot q_l$$

$$\bullet\ \frac{\partial H}{\partial t} = \sum_{k=1}^{f} \frac{\partial \dot q_k}{\partial t} \underbrace{p_k}_{\overset{eq.(6)}{=} \frac{\partial \mathcal{L}}{\partial \dot q_k}} - \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot q_k} \frac{\partial \dot q_k}{\partial t} - \frac{\partial \mathcal{L}}{\partial t}$$

$$= -\frac{\partial \mathcal{L}}{\partial t}$$

(7)

Hamilton's eqns: $\boxed{\frac{\partial H}{\partial q_k} = -\dot p_k \quad \text{and} \quad \frac{\partial H}{\partial p_k} = \dot q_k}$

$2f$ differential eqns of first order ⟺ Euler-Lagrange has f diff. eqns of second order

example for Hamilton

$$\mathcal{L}(\vec{r},\dot{\vec{r}},t) = \frac{1}{2} m (\dot{\vec{r}})^2 - U(\vec{r},t)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z,t)$$

1) generalized momenta $p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$:

$$\begin{array}{l} p_x = m\dot{x} \\ p_y = m\dot{y} \\ p_z = m\dot{z} \end{array} \Longrightarrow \vec{p} = m\dot{\vec{r}}$$

2) express $\dot{\vec{r}}$ in terms of $\vec{r}, \vec{p}, t$:

$$\dot{\vec{r}} = \frac{1}{m}\vec{p}$$

3) <u>def</u> $H = H(x,y,z,p_x,p_y,p_z)$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2} m \left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2} \right)$$

$$+ U(x,y,z,t)$$

$$= \frac{1}{2m}\vec{p}^{\,2} + U(\vec{r},t)$$

Hamilton's equs: e.g. for $q_k = x$

$$\frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} = -\dot{p}_x \Longrightarrow \dot{\vec{p}} = -\vec{\nabla} U$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x} \Longrightarrow \dot{\vec{r}} = \frac{\vec{p}}{m}$$

then $\ddot{\vec{r}} = \frac{d}{dt}\left(\frac{\vec{p}}{m}\right) \overset{m=\text{const.}}{=} \frac{1}{m}\dot{\vec{p}} = -\frac{1}{m}\vec{\nabla} U \Leftrightarrow \boxed{m\ddot{\vec{r}} = -\vec{\nabla} U}$

Conserved quantities

- energy conservation

$$\frac{dH}{dt} = \frac{d}{dt}\left(\sum_{k=1}^{f} \dot q_k \frac{\partial \mathcal{L}}{\partial \dot q_k} - \mathcal{L}\right)$$

using

$$\frac{d}{dt}\left(\sum_{k=1}^{f} \dot q_k \frac{\partial \mathcal{L}}{\partial \dot q_k}\right) = \sum_{k=1}^{f}\left(\ddot q_k \frac{\partial \mathcal{L}}{\partial \dot q_k} + \dot q_k \underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_k}}_{\overset{\text{e.o.m}}{=} \frac{\partial \mathcal{L}}{\partial q_k}}\right)$$

$$= \sum_{k=1}^{f}\left(\frac{\partial \mathcal{L}}{\partial \dot q_k}\ddot q_k + \frac{\partial \mathcal{L}}{\partial q_k}\dot q_k\right)$$

and

$$\frac{d\mathcal{L}}{dt} = \overbrace{\sum_{k=1}^{f}\left(\frac{\partial \mathcal{L}}{\partial \dot q_k}\ddot q_k + \frac{\partial \mathcal{L}}{\partial q_k}\dot q_k\right)} + \frac{\partial \mathcal{L}}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_{k=1}^{f} \dot q_k \frac{\partial \mathcal{L}}{\partial \dot q_k}\right) = \frac{d\mathcal{L}}{dt} - \frac{\partial \mathcal{L}}{\partial t} \qquad (7b)$$

$$\Rightarrow \frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \overset{\text{eq.}(7)}{=} \frac{\partial H}{\partial t}$$

$$\Rightarrow \boxed{\text{if } \frac{\partial H}{\partial t} = 0 \Longrightarrow H = \text{const.}}$$

- for cyclic coordinate q_k (with $\frac{\partial \mathcal{L}}{\partial q_k} = 0$)

$$\Rightarrow -\dot p_k \overset{\text{Ham.}}{=} \frac{\partial H}{\partial q_k} = -\frac{\partial \mathcal{L}}{\partial q_k} = 0 \qquad \text{thus } \boxed{p_k = \text{const.}}$$

Poisson bracket

$F = F(q,p,t)$ } some physical quantities
$K = K(q,p,t)$

def. $$\{F,K\} = \sum_{k=1}^{f}\left(\frac{\partial F}{\partial q_k}\frac{\partial K}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial K}{\partial q_k}\right) \quad (8)$$

properties

- $\{F,K\} = -\{K,F\}$
- $\{F,F\} \equiv 0$
- $$-\{F,q_\ell\} \overset{\text{eq.}(8)}{=} -\sum_{k=1}^{f}\left(\frac{\partial F}{\partial q_k}\underbrace{\frac{\partial q_\ell}{\partial p_k}}_{=0} - \frac{\partial F}{\partial p_k}\underbrace{\frac{\partial q_\ell}{\partial q_k}}_{=\delta_{k\ell}}\right)$$

$$= \frac{\partial F}{\partial p_\ell}$$

and
$$\{F,p_\ell\} \overset{\text{eq.}(8)}{=} \sum_{k=1}^{f}\left(\frac{\partial F}{\partial q_k}\underbrace{\frac{\partial p_\ell}{\partial p_k}}_{=\delta_{k\ell}} - \frac{\partial F}{\partial p_k}\underbrace{\frac{\partial p_\ell}{\partial q_k}}_{=0}\right)$$

$$= \frac{\partial F}{\partial q_\ell}$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial p_\ell} = -\{F,q_\ell\} \quad \text{and} \quad \frac{\partial F}{\partial q_\ell} = \{F,p_\ell\}} \quad (9)$$

- use $F = q_k$ in eq. (9) and obtain

$$\boxed{\{q_k,q_\ell\} = 0 \quad \text{and} \quad \{q_k,p_\ell\} = \delta_{k\ell}}$$

time evolution of $F(q,p,t)$

$$\frac{dF}{dt} = \sum_{k=1}^{f} \left(\frac{\partial F}{\partial q_k} \underbrace{\dot{q}_k}_{=\frac{\partial H}{\partial p_k}} + \frac{\partial F}{\partial p_k} \underbrace{\dot{p}_k}_{=-\frac{\partial H}{\partial q_k}} \right) + \frac{\partial F}{\partial t}$$

Hamilton's eqns

$$\Rightarrow \boxed{\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}} \qquad \text{for } F = F(q,p,t)$$

assume $F = F(q,p)$ i.e. $\frac{\partial F}{\partial t} = 0 \Rightarrow \frac{dF}{dt} = \{F, H\}$

for example

- $F = p_k$: $\dot{p}_k = \{p_k, H\} \overset{\text{eq.}(9)}{=} -\frac{\partial H}{\partial q_k}$
- $F = q_k$: $\dot{q}_k = \{q_k, H\} \overset{\text{eq.}(9)}{=} \frac{\partial H}{\partial p_k}$

conserved quantities

F is conserved $\Leftrightarrow \frac{dF}{dt} = 0$

assume $\frac{\partial F}{\partial t} = 0$ then

$$\boxed{F \text{ is conserved} \Leftrightarrow \{F, H\} = 0}$$

example $F = H = H(q,p)$

$$\frac{dH}{dt} = \underbrace{\{H, H\}}_{=0} + \underbrace{\frac{\partial H}{\partial t}}_{=0} = 0 \Rightarrow H \text{ is conserved}$$

Noether's theorem

consider $S[q] = \int_{t_1}^{t_2} dt \, \mathcal{L}(q, \dot{q}, t)$

perform transformation

$$q_k \longmapsto q_k' = \Psi_k(q, \dot{q}, t; \varepsilon) \approx q_k + \varepsilon \Psi_k(q, \dot{q}, t) + \mathcal{O}(\varepsilon^2)$$

$$t \longmapsto t' = \varphi(q, \dot{q}, t; \varepsilon) \approx t + \varepsilon \varphi(q, \dot{q}, t) + \mathcal{O}(\varepsilon^2)$$

where $\Psi_k(q, \dot{q}, t; 0) = q_k$ and $\varphi(q, \dot{q}, t; 0) = t$

then

$$q_k(t) \longmapsto q_k'(t') = q_k(t) + \varepsilon \Psi_k(q(t), \dot{q}(t), t) \quad ?$$

and

$$t'(t) = t + \varepsilon \varphi(q(t), \dot{q}(t), t)$$

assume $S[q]$ is invariant, i.e.

$$S[q(t)] \overset{!}{=} S'[q'(t')] \quad \Leftrightarrow \text{ symmetry trafo}$$

$$\Rightarrow \int_{t_1'}^{t_2'} dt' \, \mathcal{L}\left(q', \frac{dq'}{dt'}, t'\right) = \int_{t_1}^{t_2} dt \, \mathcal{L}\left(q, \frac{dq}{dt}, t\right)$$

then

$$\int_{t_1'}^{t_2'} dt' \, \mathcal{L}\left(q', \frac{dq'}{dt'}, t'\right) = \int_{t_1}^{t_2} dt \, \underbrace{\mathcal{L}\left(q', \frac{dq'}{dt'}, t'\right) \frac{dt'}{dt}}_{f(\varepsilon) = f(0) + \varepsilon \frac{d}{d\varepsilon} f \big|_{\varepsilon = 0}}$$

$$= \int_{t_1}^{t_2} dt \left[\mathcal{L}(q, \dot{q}, t) + \varepsilon \frac{d}{d\varepsilon} \left(\mathcal{L}\left(q', \frac{dq'}{dt'}, t'\right) \frac{dt'}{dt} \right) \bigg|_{\varepsilon = 0} \right]$$

$$\Rightarrow \text{inv.} \Leftrightarrow \frac{d}{d\varepsilon} \left(\mathcal{L}\left(q', \frac{dq'}{dt'}, t'\right) \frac{dt'}{dt} \right) \bigg|_{\varepsilon = 0} = 0$$

⑯

this corresponds to $\frac{d}{dt}\left(Q(q,\dot{q},t)\right) = 0$

i.e. $Q = \text{const}$.

as we show next:

$$\frac{dt'}{dt} = 1 + \varepsilon \frac{d}{dt}\varphi(q,\dot{q},t) = 1 + \varepsilon \frac{d\varphi}{dt}$$

$$\frac{dq_k'}{dt'} = \frac{dq_k'}{dt}\frac{dt}{dt'} = \left(\dot{q}_k + \varepsilon \frac{d\psi_k}{dt}\right)\left(1 - \varepsilon\frac{d\varphi}{dt}\right)$$

$$= \dot{q}_k + \varepsilon\frac{d\psi_k}{dt} - \varepsilon\dot{q}_k\frac{d\varphi}{dt} + O(\varepsilon^2)$$

$$\Rightarrow \frac{d}{d\varepsilon}\left(\mathcal{L}\left(q_k + \varepsilon\psi_k, \dot{q}_k + \varepsilon\frac{d\psi_k}{dt} - \varepsilon\dot{q}_k\frac{d\varphi}{dt}, t + \varepsilon\varphi\right)\left(1 + \varepsilon\frac{d\varphi}{dt}\right)\right)\Big|_{\varepsilon=}$$

$$= \underbrace{\sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial q_k}}_{= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}}\psi_k + \sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\left(\frac{d\psi_k}{dt} - \dot{q}_k\frac{d\varphi}{dt}\right) + \frac{\partial\mathcal{L}}{\partial t}\varphi + \mathcal{L}\frac{d\varphi}{dt}$$

$$= \frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\psi_k\right) + \left(\mathcal{L} - \sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right)\frac{d\varphi}{dt} + \varphi\frac{\partial\mathcal{L}}{\partial t} =$$

$\mathcal{L} = \mathcal{L}(q,\dot{q},t)$ because $\varepsilon = 0$

using eq.([redacted]), i.e. $\frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right) = \frac{d\mathcal{L}}{dt} - \frac{\partial\mathcal{L}}{\partial t}$

$$\Rightarrow \frac{d}{d\varepsilon}\left[\mathcal{L}(q',\dots)\frac{dt'}{dt}\right]\Big|_{\varepsilon=0} = \frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\psi_k\right) + \left(\mathcal{L} - \sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right)\frac{d\varphi}{dt} + \varphi\frac{d}{dt}\left(\mathcal{L} - \sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right)$$

$$= \frac{d}{dt}\left[\sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\psi_k + \left(\mathcal{L} - \sum_{k=1}^{f}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right)\varphi\right] = 0$$

$$\Rightarrow \boxed{\begin{aligned} Q &= Q(q, \dot q, t) \\ &= \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot q_k} \psi_k + \left(\mathcal{L} - \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot q_k} \dot q_k\right) \varphi = \text{const.} \end{aligned}}$$

i.e. Symmetrie $S = S' \xrightarrow{\text{Noether}} Q = Q(q, \dot q, t) = \text{const.}$

i.e. $\frac{d}{dt} Q = 0$

examples

1) $\mathcal{L} = \mathcal{L}(q_k, \dot q)$ not explicitly from time t

symmetry: $q'_k = q_k \Leftrightarrow \psi_k = 0$

$t' = t + \varepsilon \Leftrightarrow \varphi = 1$

in general:

$\psi_k = \psi_k(q, \dot q, t)$

$\varphi = \varphi(q, \dot q, t)$

$$\Rightarrow \cancel{\frac{dq'_k}{dt'}} \quad q'_k(t') = q_k(t) + \varepsilon \psi_k = q_k(t)$$

$$\frac{dt'}{dt} = \frac{d}{dt}(t + \varepsilon) = 1$$

$$\frac{dq'_k}{dt'} = \dot q_k + \varepsilon \frac{d\psi_k}{dt} - \varepsilon \dot q_k \frac{d\varphi}{dt} = \dot q_k$$

$$\Rightarrow \frac{d}{d\varepsilon}\left[\mathcal{L}\left(q'_k, \frac{dq'_k}{dt'}\right) \frac{dt'}{dt}\right]\Bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \mathcal{L}(q_k, \dot q_k)\Big|_{\varepsilon=0} = 0$$

action is invariant

$$\Rightarrow Q = \mathcal{L} - \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot q_k} \dot q_k = \text{const.} = -E$$

i.e. $\mathcal{L} = \frac{1}{2} m \dot x^2 - U(x) \Rightarrow Q = \frac{1}{2} m \dot x^2 - U(x) - m \dot x^2 = -\left(\frac{1}{2} m \dot x^2 + U(x)\right) = -E$

extended Noether's theorem

$$\mathcal{L}(q,\dot{q},t) \quad \text{and} \quad \mathcal{L}'(q,\dot{q},t) = \mathcal{L}(q,\dot{q},t) + \frac{d}{dt}\underbrace{f(q,t)}_{\text{arbitrary fct.}}$$

yield the same e.o.m.

i.e.

$$S' = \int_{t_1}^{t_2} dt\ \mathcal{L}' = S + \int_{t_1}^{t_2} dt\ \frac{d}{dt} f(q,t)$$

$$= S + f(q(t_2),t_2) - f(q(t_1),t_1)$$

then

$$\delta S' = \delta S + \sum_{k=1}^{f} \left(\frac{\partial f}{\partial q_k}\delta q_k\right)\Big|_{t_2} - \sum_{k=1}^{f}\left(\frac{\partial f}{\partial q_k}\delta q_k\right)\Big|_{t_1}$$

using $\delta q(t_1) = \delta q(t_2) = 0$

$$\Rightarrow \boxed{\delta S' = \delta S} \qquad \text{for } \boxed{\mathcal{L}' = \mathcal{L} + \frac{d}{dt} f(q,t)}$$

new invariance condition:

$$\frac{d}{d\varepsilon}\left[\mathcal{L}\left(q', \frac{dq'}{dt'}, t'\right)\frac{dt'}{dt}\right]\Big|_{\varepsilon=0} = \frac{d}{dt} f(q,t)$$

$$\Rightarrow \boxed{Q = \sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\psi_k + \left(\mathcal{L} - \sum_{k=1}^{f}\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\dot{q}_k\right)\varphi - f(q,t) = \text{const.}}$$