4 Renormalization

Main references for this section are [10, 11].

4.1 Introductory remarks

Consider a Wightman theory $(\mathcal{H}, U, \Omega, \phi_B, D)$ as in the previous lecture and suppose we want to describe a collision of several particles: The LSZ formula gives:

$$\begin{aligned}\langle 0|a(k_1)\dots a(k_\ell)\hat{S}a^\dagger(p_1)\dots a^\dagger(p_n)|0\rangle = \frac{(-i)^{n+\ell}}{(\sqrt{Z})^{n+\ell}}\prod_{i=1}^{\ell}(k_i^2-m^2)\prod_{j=1}^{n}(p_j^2-m^2)\times \\ \times\int d^4x_1\dots d^4x_\ell d^4y_1\dots d^4y_n\, e^{i\sum_{i=1}^{\ell}k_ix_i - i\sum_{j=1}^{n}p_jy_j}\times \\ \times\langle\Omega|T(\phi_B(x_\ell)\dots\phi_B(x_1)\phi_B(y_1)\dots\phi_B(y_n))\Omega\rangle,\end{aligned}$$

The 'renormalized' field $\phi := Z^{-1/2}\phi_B$ is again a Wightman field. To compute the S-matrix of $(\mathcal{H}, U, \Omega, \phi, D)$ we drop Z and the index B on the r.h.s. of the formula above.

Possibly after regularizing the time-ordered product, we can express the Green functions of ϕ above by the Wightman functions and then analytically continue to Schwinger functions $G_{E,n}$. If the theory satisfies also Osterwalder-Schrader axioms, we have

$$G_{E,n}(x_1,\dots,x_n) = \frac{\delta}{\delta J(x_1)}\dots\frac{\delta}{\delta J(x_n)}Z_E[J]|_{J=0}, \quad Z_E[J] = \int_{S'_{\mathbb{R}}} e^{\varphi(J)}d\mu(\varphi), \tag{56}$$

for some Borel measure $d\mu$ and $J \in S_{\mathbb{R}}$. (We changed $e^{i\varphi(J)}$ to $e^{\varphi(J)}$, where $J \in S_{\mathbb{R}}$, making use of the Osterwalder-Schrader axiom of analyticity).

Today we will attempt to construct this measure in $\lambda\phi^4$ theory. This endeavor will not be completely successful. In the end we will obtain $G_{E,n}$ as power series in the coupling constant λ whose convergence we will not control. Thereby we will abandon the Wightman/Osterwalder-Schrader setting and delve into perturbative QFT. Strictly speaking, we will also abandon the realm of quantum theories, as there will be no underlying Hilbert space and thus no control that transition probabilities of physical processes take values between zero and one. On the positive side, the finiteness of individual terms in this expansion will be an interesting problem in the theory of differential equations.

4.2 From action to generating functional

Consider the Euclidean action written in terms of the bare (unphysical) quantities:

$$S_E[\varphi_B] = \int d^4x\big(\frac{1}{2}\partial_\mu\varphi_B\partial^\mu\varphi_B + \frac{1}{2}m_B^2\varphi_B^2 + \frac{\lambda_B}{4!}\varphi_B^4\big). \tag{57}$$

Introduce the physical parameters via $\varphi_B = \sqrt{Z}\varphi$, $m_B = Z_m m$, $\lambda_B = Z_\lambda \lambda$. This gives

$$\begin{aligned} S_E[\varphi] &= \int d^4x \frac{1}{2}\big(\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2\big) \\ &+ \int d^4x \big\{\frac{1}{2}(Z-1)\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}(Z_m^2 Z - 1)m^2\varphi^2 + \frac{Z_\lambda Z^2}{4!}\lambda\varphi^4\big\} \\ &= S_{E,0}[\varphi] + S_{E,I}[\varphi]. \end{aligned} \tag{58}$$

Let us start with the free action $S_{E,0}$. The corresponding generating functional is

$$Z_{E,0}[J] = e^{\frac{1}{2}\int d^4p\, \overline{\widehat{J}}(p)C(p)\widehat{J}(p)} = \int_{S'_{\mathbb{R}}} e^{\varphi(J)} d\mu_C(\varphi), \tag{59}$$

$$d\mu_C(\varphi) = \text{"}\frac{1}{N_C} e^{-S_{E,0}[\varphi]}\mathcal{D}[\varphi]\text{"}, \tag{60}$$

where $C(p) = \frac{1}{p^2+m^2}$ and $d\mu_C$ is the corresponding Gaussian measure given formally by (60). The candidate generating functional for the interacting theory is

$$Z_E^{\mathrm{cand}}[J] = \frac{1}{N}\int_{S'_{\mathbb{R}}} e^{\varphi(J)} e^{-S_{E,I}[\varphi]} d\mu_C(\varphi), \tag{61}$$

where N is chosen so that $Z_E^{\mathrm{cand}}[0] = 1$. Problems:

- $d\mu_C$ is supported on distributions so $\varphi(x)^4$ appearing in $S_{E,I}[\varphi]$ is ill-defined (UV problem).
- Integral over spacetime volume in $S_{E,I}[\varphi]$ ill-defined (IR problem).

4.3 Regularized generating functional

To make sense out of (61) we need regularization: We set

$$C_\Lambda^{\Lambda_0}(p) := \frac{1}{p^2+m^2}\big(e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}}\big). \tag{62}$$

Here Λ_0 is the actual UV cut-off. $0 \le \Lambda \le \Lambda_0$ is an auxiliary cut-off which will be needed for technical reasons. Thus $C_\Lambda^{\Lambda_0}$ is essentially supported in $\Lambda \le p^2 \le \Lambda_0$.

Lemma 4.1 $d\mu_{C_\Lambda^{\Lambda_0}}$ *is supported on smooth functions.*

Thus the following regularized generating functional is meaningful

$$Z_E^{\Lambda,\Lambda_0}[J] = \frac{1}{N}\int e^{\varphi(J)} e^{-S^{\Lambda_0}_{E,I,(V)}[\varphi]} d\mu_{C_\Lambda^{\Lambda_0}}(\varphi), \tag{63}$$

where we also introduced a finite volume V. Of course, when we try to take the limit $\Lambda_0 \to \infty$, we will get back divergent expressions. The idea of renormalization is to absorb these divergencies into the parameters in $S^{\Lambda_0}_{E,I,(V)}$ (therefore we added a superscript Λ_0). For simplicity, we rename the coefficients from (58) as follows:

$$S^{\Lambda_0}_{E,I,(V)}[\varphi] = \int_V d^4x \big(a^{\Lambda_0}\partial_\mu\varphi\partial^\mu\varphi + b^{\Lambda_0}\varphi^2 + c^{\Lambda_0}\lambda\varphi^4\big). \tag{64}$$

We list the following facts:

- Z^{Λ,Λ_0}_E generates (regularized) Schwinger functions $G^{\Lambda,\Lambda_0}_{E,n}$.
- $-\log(NZ^{\Lambda,\Lambda_0}_E)$ generates (regularized) connected Schwinger functions $G^{\Lambda,\Lambda_0}_{E,\mathrm{c},n}$.
- $\Sigma^{\Lambda,\Lambda_0}[J] := -\log(NZ^{\Lambda,\Lambda_0}_E[(C^{\Lambda_0}_\Lambda)^{-1}J]) + \frac{1}{2}\langle J|(C^{\Lambda_0}_\Lambda)^{-1}J\rangle$ generates (regularized) connected amputated Schwinger functions with subtracted zero-order contribution $G^{\Lambda,\Lambda_0}_{E,\mathrm{r}\geq 1,\mathrm{a},\mathrm{c},n}$.

Given energy-momentum conservation, it is convenient to set

$$S^{\Lambda,\Lambda_0}_n(p_1,\dots,p_{n-1})(2\pi)^4\delta(p_1+\dots+p_n) := \int e^{i(p_1x_1+\dots+p_nx_n)} G^{\Lambda,\Lambda_0}_{E,\mathrm{r}\geq 1,\mathrm{a},\mathrm{c},n}(x_1,\dots,x_n)d^4x_1\dots d^4x_n. \tag{65}$$

4.4 The problem of perturbative renormalizability

The renormalization of ϕ^4 in 4 dimensions is only understood perturbatively. This means we treat S^{Λ,Λ_0}_n as a formal power series

$$S^{\Lambda,\Lambda_0}_n = \sum_{r\geq 1}\lambda^r S^{\Lambda,\Lambda_0}_{r,n}, \tag{66}$$

i.e. a series whose convergence we do not control. (Order by order the limit $V\to\mathbb{R}^4$ can be taken, so we will not discuss it anymore). Similarly, we treat the coefficients from the interaction Lagrangian as formal power series:

$$a^{\Lambda_0} = \sum_{r\geq 1}\lambda^r a^{\Lambda_0}_r, \quad b^{\Lambda_0} = \sum_{r\geq 1}\lambda^r b^{\Lambda_0}_r, \quad c^{\Lambda_0} = \sum_{r\geq 1}\lambda^r c^{\Lambda_0}_r. \tag{67}$$

Furthermore, we impose the following BPHZ renormalization conditions (RC)

$$S^{0,\Lambda_0}_{r,4}(0) = \delta_{r,1}, \quad S^{0,\Lambda_0}_{r,2}(0) = 0, \quad \partial_{p^2}S^{0,\Lambda_0}_{r,2}(0) = 0, \tag{68}$$

which fix the physical values of the parameters[2].

Theorem 4.2 *(Perturbative renormalizability) There are such* $\{a^{\Lambda_0}_r, b^{\Lambda_0}_r, c^{\Lambda_0}_r\}_{\Lambda_0\geq 0}$ *that the limits*

$$S_{r,n}(\underline{p}) := \lim_{\Lambda_0\to\infty} S^{0,\Lambda_0}_{r,n}(\underline{p}), \quad \underline{p} = (p_1,\dots,p_{n-1}) \tag{69}$$

exist and are finite, and the renormalisation conditions (68) hold.

[2] The physical meaning of our RC is not so direct, since we are in the Euclidean setting. Before substituting our n-point functions to the LSZ formula for the S-matrix, they have to be analytically continued to the real time and transferred to the on-shell renormalization scheme.

Example: Consider the leading contribution to the two-point and four-point function:

$$\begin{aligned}
S_{1,2}^{0,\Lambda_0}(p) &= \text{......O...... + ...x......} \\
&= 12c_1^{\Lambda_0}\int \frac{d^4q}{(2\pi)^4}C_0^{\Lambda_0}(q) + 2(a_1^{\Lambda_0}p^2 + b_1^{\Lambda_0}), \\
&= 12c_1^{\Lambda_0}\underbrace{\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}e^{-\frac{(q^2+m^2)}{\Lambda_0^2}}}_{I^{\Lambda_0}} + 2(a_1^{\Lambda_0}p^2 + b_1^{\Lambda_0}), \qquad (70) \\
S_{1,4}^{0,\Lambda_0}(p_1,p_2,p_3) &= 4!c_1^{\Lambda_0}. \qquad (71)
\end{aligned}$$

For any finite Λ_0 the integral above is convergent, but it diverges for $\Lambda_0 \to \infty$. We have to choose the behavior of $a_1^{\Lambda_0}, b_1^{\Lambda_0}, c_1^{\Lambda_0}$ s.t. this divergence is compensated and the RC are satisfied. We get from the three RC conditions, respectively,

$$4!c_1^{\Lambda_0} = 1, \quad 12c_1^{\Lambda_0}I^{\Lambda_0} + 2b_1^{\Lambda_0} = 0, \quad 2a_1^{\Lambda_0} = 0. \qquad (72)$$

Hence, $c_1^{\Lambda_0} = 1/4!$, $b_1^{\Lambda_0} = -(1/4)I^{\Lambda_0}$, $a_1^{\Lambda_0} = 0$ and thus $b_1^{\Lambda_0}$ absorbs the divergence of the integral I^{Λ_0}.

4.5 Flow equations

Properties of $\Sigma^{\Lambda,\Lambda_0}[J] := -\log\left(NZ_E^{\Lambda,\Lambda_0}[(C_\Lambda^{\Lambda_0})^{-1}J]\right) + \frac{1}{2}\langle J|(C_\Lambda^{\Lambda_0})^{-1}J\rangle$:

- $e^{-\Sigma^{\Lambda,\Lambda_0}[J]} = \int d\mu_{C_\Lambda^{\Lambda_0}}(\varphi)e^{-S_{E,I}^{\Lambda_0}[J+\varphi]}$.

- $\lim_{\Lambda\to\Lambda_0}\Sigma^{\Lambda,\Lambda_0}[J] = S_{E,I}^{\Lambda_0}[J]$ since $\lim_{\Lambda\to\Lambda_0} C_\Lambda^{\Lambda_0}(p) = 0$ and therefore, formally, $\lim_{\Lambda\to\Lambda_0} d\mu_{C_\Lambda^{\Lambda_0}}(\varphi) = \delta(\varphi)\mathcal{D}[\varphi]$.

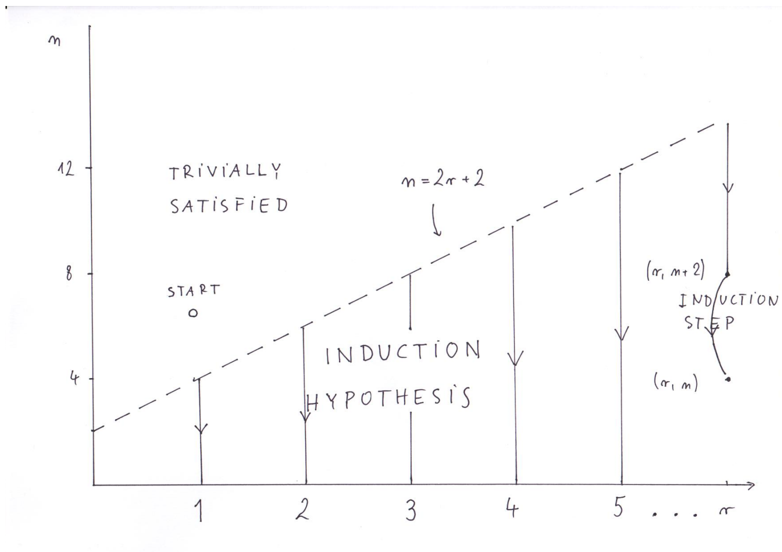
- $\partial_\Lambda(e^{-\Sigma^{\Lambda,\Lambda_0}[J]}) = \frac{1}{2}\langle\frac{\delta}{\delta J}|\partial_\Lambda C_\Lambda^{\Lambda_0}\frac{\delta}{\delta J}\rangle e^{-\Sigma^{\Lambda,\Lambda_0}[J]}$.

This gives the following equation for $S_{r,n}^{\Lambda,\Lambda_0}(\underline{p}) := S_{r,n}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1})$

$$\partial_\Lambda S_{r,n}^{\Lambda,\Lambda_0}(\underline{p}) = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}(\partial_\Lambda C_\Lambda^{\Lambda_0})(p)S_{r,n+2}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1},p,-p)$$
$$-\frac{1}{2}\sum_{\substack{r'+r''=r\\ n'+n''=n+2}}\binom{n}{n'-1}[S_{r',n'}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n'-1})(\partial_\Lambda C_\Lambda^{\Lambda_0})(q)S_{r'',n''}^{\Lambda,\Lambda_0}(-q,p_{n'},\ldots,p_{n-1})]_{\text{sym}}, \qquad (73)$$

where $q = -p_1 - \cdots - p_{n'-1}$ and sym denotes the symmetrisation in $p_1,\ldots,p_{n-1}$. Boundary conditions:

(a) $\Lambda = 0:\ S_{r,4}^{0,\Lambda_0}(0) = \delta_{r,1}, \quad S_{r,2}^{0,\Lambda_0}(0) = 0, \quad \partial_{p^2}S_{r,2}^{0,\Lambda_0}(0) = 0$ (RC).



(b) $\Lambda = \Lambda_0$: $\partial^w S^{\Lambda_0,\Lambda_0}_{r,n}(\underline{p}) = 0$ for $n + |w| \geq 5$. (By second bullet above).

With the help of the above equation one proves the following theorem:

Theorem 4.3 *The following estimate holds*

$$|\partial^w S^{\Lambda,\Lambda_0}_{r,n}(\underline{p})| \leq \begin{cases} P_1(|\underline{p}|) \text{ for } 0 \leq \Lambda \leq 1, \\ \Lambda^{4-n-w} P_2(\log \Lambda) P_3(\frac{|\underline{p}|}{\Lambda}) \text{ for } 1 \leq \Lambda \leq \Lambda_0 \end{cases},$$

where P_i are some polynomials independent of p, Λ, Λ_0, but depending on n, r, w.

In particular this shows that $S^{\Lambda,\Lambda_0}_{r,n}(\underline{p})$ stays bounded if $\Lambda = 0$ and $\Lambda_0 \to \infty$. With more effort, one also shows convergence as $\Lambda_0 \to \infty$, required by the renormalizability property (69).

4.6 Outline of the proof of Theorem 4.3

Observation[3]:

$$S^{\Lambda,\Lambda_0}_{r,n} \equiv 0 \text{ for } n > 2r + 2. \tag{74}$$

Given this, the structure of the flow equation suggests the inductive scheme as in the figure: Indeed, we can start the induction in the region where (74) holds and therefore the estimate from Theorem 4.3 is satisfied. We suppose the estimate holds for

[3]Indeed, without assuming connectedness we clearly have vanishing of these functions for $n > 4r$. Now $(r - 2)$ verticies need to use at least two lines to keep the diagram connected. 2 verticies need to spend only one line. So altogether $n > 4r - 2(r - 2) - 2$.

- (r, n_1) for $n_1 \geq n + 2$.
- (r_2, n_2) for $r_2 < r$ and any n_2.

Since only $S^{\Lambda,\Lambda_0}_{r',n'}$ as listed above appear on the r.h.s. of the flow equation (73), we can apply the estimate to this r.h.s. Then (after quite some work) the flow equation gives the required estimate on the $S^{\Lambda,\Lambda_0}_{r,n}$, which appears on the l.h.s. of the flow equation (73).

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