5 Symmetries I

Symmetries in physics are described by groups. We recall some definitions and facts from the theory of groups and their representations following [14, Chapter 1], [15, Chapter 1], [16].

5.1 Groups

1. Def. A group is a set G with an operation $\cdot : G \times G \to G$ s.t.
   - $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$,
   - There exists $e \in G$ s.t. $g \cdot e = e \cdot g = g$ for all $g \in G$,
   - For any $g \in G$ there exists $g^{-1} \in G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$.
2. Examples:
   - $\mathbb{Z}_2 = \{1, -1\}$ is the group of parity transformations.
   - Let V be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Then $GL(V)$ denotes the group of all invertible linear mappings $V \to V$.
   - For $V = \mathbb{K}^n$ we write $GL(n, \mathbb{K}) := GL(V)$. This is the group of invertible $n \times n$ matrices with entries in $\mathbb{K}$.
   - $SO(3) = \{ R \in GL(3, \mathbb{R}) \,|\, R^T R = I,\ \det R = 1\}$ - the group of rotations.
   - $SU(2) = \{U \in GL(2, \mathbb{C}) \,|\, U^\dagger U = I,\ \det U = 1 \}$ - special unitary group.
3. Let G, $\hat{G}$ be groups. Then $H : G \to \hat{G}$ is a group homomorphism if

$$H(g_1 g_2) = H(g_1) H(g_2) \text{ for any } g_1, g_2 \in G. \tag{75}$$

   If H is in addition a bijection then it is called an isomorphism.
   (For Lie groups, which we discuss below, homomorphisms are required to be smooth and isomorphisms should also have smooth inverse).

5.2 Lie groups

1. Def. G is a Lie group if it is a smooth real manifold and the group operation and taking the inverse are smooth maps. The dimension of G is the dimension of this manifold.
2. Def. A set M is an n-dimensional smooth manifold if the following hold:
   - It is a Hausdorff topological space. (Distinct points have non-overlapping neighbourhoods, unique limits).
   - There is an *open cover* i.e. a family of open sets $U_\alpha \subset \mathcal{M}$, $\alpha \in \mathcal{I}$, s.t. $\bigcup_{\alpha \in \mathcal{I}} U_\alpha = M$.

- There is an *atlas* $\mathcal{A}(M) := \{ \eta_\alpha : U_\alpha \to O_\alpha \,|\, \alpha \in I \}$ given by some family of open sets $O_\alpha \subset \mathbb{R}^n$ and charts η_α, which are homeomorphisms. (A homeomorphism is a continuous bijection whose inverse is also continuous).
- Let $O_{\alpha,\beta} := \eta_\beta(U_\alpha \cap U_\beta) \subset R^n$ and let $O_{\beta,\alpha} = \eta_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$. Then $\eta_\alpha \circ \eta_\beta^{-1} : O_{\alpha,\beta} \to O_{\beta,\alpha}$ is smooth.

3. Def. A map $F : M \to \hat{M}$ between two manifolds is smooth if all the maps $\hat{\eta}_\beta \circ F \circ \eta_\alpha^{-1} : O_\alpha \to \hat{O}_\beta$ are smooth whenever well-defined. It is called a diffeomorphism if it is a bijection and the inverse is also smooth. We denote by $C^\infty(M)$ the space of smooth maps $M \to \mathbb{R}$.

4. Def. A smooth map $\mathbb{R} \times M \ni (t, x) \mapsto \gamma_t(x) \in M$, is called a *flow* (of a vector field) if

$$\gamma_0 = \mathrm{id}_M, \quad \gamma_s \circ \gamma_t = \gamma_{s+t} \text{ for } t, s \in \mathbb{R}. \tag{76}$$

5. Def. A vector field $X : C^\infty(M) \to C^\infty(M)$ with the flow γ is given by

$$X(f) = \frac{d}{dt} f \circ \gamma_t|_{t=0}, \quad f \in C^\infty(M). \tag{77}$$

6. Fact. Given two vector fields X, Y, the commutator $[X, Y](f) := X(Y(f)) - Y(X(f))$ is again a vector field.

7. Def. A tangent vector at point $x \in M$ is a map $X_x : C^\infty(M) \to \mathbb{R}$ given by $X_x(f) = \frac{d}{dt} f \circ \gamma_t(x)|_{t=0}$. We denote by $T_x M$ the space of all tangent vectors at x (for different flows).
Example: Let $M = \mathbb{R}^n$ and $\gamma_t = (\gamma_t^1, \dots, \gamma_t^n)$ be a flow. Then

$$X_x(f) = \sum_{i=1}^n \frac{d}{dt} \gamma_t^i(x)|_{t=0} \frac{\partial f}{\partial x^i}(x). \tag{78}$$

If one 'forgets' f and thinks about $\frac{\partial}{\partial x^i}$ as basis vectors then the expression $X_x = \sum_{i=1}^n \frac{d}{dt} \gamma_t^i(x)|_{t=0} \frac{\partial}{\partial x^i}$ is clearly the tangent vector to $t \to \gamma_t(x)$ at x.

8. Def. (Transport of a vector field) Let $F : M \to \hat{M}$ be a diffeomorphism and X a vector field on M given by a flow $\{\gamma_t\}_{t\in\mathbb{R}}$. Then $F_* X$ is a vector field on $\hat{M}$ given by $F \circ \gamma_t \circ F^{-1}$. That is

$$(F_* X)(\hat{f}) = \frac{d}{dt} \hat{f} \circ F \circ \gamma_t \circ F^{-1}|_{t=0}, \quad \hat{f} \in C^\infty(\hat{M}). \tag{79}$$

9. Now we can introduce the Lie algebra G' of a Lie group G.

   - Def. Let G be a Lie group and $g \in G$. Then, the left-multiplication $L_g : G \to G$, acting by $L_g \tilde{g} = g\tilde{g}$ is a diffeomorphism.

- Def. We say that a vector field X on G is *left-invariant* if $((L_g)_* X) = X$ for any $g \in G$.
- Fact. If X, Y are left-invariant, then also $[X, Y]$ is left-invariant.
- Def. The Lie algebra G' of G is the vector space of left-invariant vector fields on G with algebraic operation given by the commutator.

10. Def. For $X \in G'$ we set $\exp(X) := \gamma_1(e)$.

11. Fact. This exponential map is a diffeomorphism of a neighbourhood of zero in G' into a neighbourhood of e in G.

5.3 From multiplication law in G to algebraic operation in G'

1. Fact. Left-invariance and (79) give $X_g(f) = X_e(f \circ L_g)$ for any $g \in G$. Thus left-invariant vector fields are determined by their values at the neutral element e. In this sense, G' can be identified with T_eG and has the same dimension as G.

2. Let us choose a basis $X^1, \ldots, X^n$ in G'. We have

$$[X^A, X^B]_e = \sum_{C=1}^{n} f^{CAB} X_e^C, \tag{80}$$

where f^{CAB} are called the structure constants. In physics one usually defines the infinitesimal generators[4] $t^A := iX^A$ and writes (80) as

$$[t^A, t^B] = \sum_{C=1}^{n} i f^{CAB} t^C, \tag{81}$$

where evaluation at e is understood. We follow the mathematics convention below, unless stated otherwise.

3. Let us determine f^{CAB} from the multiplication law of the group G. We work in some chart $\eta : U \to O \subset \mathbb{R}^n$ whose domain U is a neighbourhood of e, in which the group elements have the form $g = \eta^{-1}(\varepsilon^1, \ldots, \varepsilon^n)$, $e = \eta^{-1}(0, \ldots, 0)$. For $\varepsilon_1, \varepsilon_2 \in O$, i.e. $\varepsilon_i = (\varepsilon_i^1, \ldots, \varepsilon_i^n)$, $i = 1, 2$, we define the multiplication function

$$m(\varepsilon_1, \varepsilon_2) := \eta(\eta^{-1}(\varepsilon_1)\eta^{-1}(\varepsilon_2)) \tag{82}$$

[4]As we defined G' as a real vector space, it may be unclear what the multiplication by 'i' means. We recall that is it always possible to 'complexify' a real vector space. Furthermore, in the later part of these lectures we will represent Lie algebras on complex vector spaces. Then 'i' will be provided by the vector space.

which takes values in O i.e. $m(\varepsilon_1, \varepsilon_2) = (m^1(\varepsilon_1, \varepsilon_2), \ldots, m^n(\varepsilon_1, \varepsilon_2))$. Since $\eta^{-1}(0) = e$, we have $m(\varepsilon_1, 0) = \varepsilon_1$ and $m(0, \varepsilon_2) = \varepsilon_2$. Consequently

$$\frac{\partial m^i}{\partial \varepsilon_1^j}(0,0) = \frac{\partial m^i}{\partial \varepsilon_2^j}(0,0) = \delta_{i,j}. \tag{83}$$

Given a flow of a vector field γ_t in G we define the transported flow $\tilde{\gamma}_t(\varepsilon) = \eta \circ \gamma_t \circ \eta^{-1}(\varepsilon) = (\tilde{\gamma}_t^1(\varepsilon), \ldots, \tilde{\gamma}_t^n(\varepsilon))$.

Lemma 5.1 *Let $X^1, \ldots, X^A \ldots, X^n$ be vector fields on G whose flows satisfy*

$$\frac{d}{dt}\tilde{\gamma}^i_{A,t}(\varepsilon)|_{t=0} = \frac{\partial m^i}{\partial \varepsilon_2^A}(\varepsilon, 0). \tag{84}$$

Then these fields are left-invariant. Furthermore, they are linearly independent near e and thus span the Lie algebra.

Proof. We want to verify $X_g^A(f) = X_e^A(f \circ L_g)$ for $f \in C^\infty(M)$ and $g = \eta^{-1}(\varepsilon)$. Thus we set $\tilde{f} := f \circ \eta^{-1}$ and compute

$$\begin{aligned} X^A_{\eta^{-1}(\varepsilon)}(f) &= \frac{d}{dt} f(\gamma_{A,t} \circ \eta^{-1}(\varepsilon))|_{t=0} = \frac{d}{dt}\tilde{f}(\tilde{\gamma}_{A,t}(\varepsilon))|_{t=0} \\ &= \sum_{i=1}^n \frac{d}{dt}(\tilde{\gamma}^i_{A,t}(\varepsilon))|_{t=0} \frac{\partial \tilde{f}}{\partial \varepsilon^i}(\varepsilon) = \sum_{i=1}^n \frac{\partial m^i}{\partial \varepsilon_2^A}(\varepsilon, 0)\frac{\partial \tilde{f}}{\partial \varepsilon^i}(\varepsilon). \end{aligned} \tag{85}$$

On the other hand

$$\begin{aligned} X^A_{\eta^{-1}(0)}(f \circ L_{\eta^{-1}(\varepsilon)}) &= \frac{d}{dt} f(L_{\eta^{-1}(\varepsilon)}\gamma_{A,t} \circ \eta^{-1}(0))|_{t=0} \\ &= \frac{d}{dt} f(\eta^{-1}(\varepsilon)\eta^{-1}(\tilde{\gamma}_{A,t}(0)))|_{t=0} = \frac{d}{dt}\tilde{f}(m(\varepsilon, \tilde{\gamma}_{A,t}(0)))|_{t=0} \\ &= \sum_{i,k} \frac{\partial \tilde{f}}{\partial \varepsilon^i}(\varepsilon) \frac{\partial m^i}{\partial \varepsilon_2^k}(\varepsilon, 0) \underbrace{\frac{d}{dt}\tilde{\gamma}^k_{A,t}(0)|_{t=0}}_{\delta_{k,A} \text{ by (83),(84).}}, \end{aligned} \tag{86}$$

which concludes the proof of left-invariance. It suffices to check linear independence near e which follows from (85) and (83). □

Lemma 5.2 *Under the assumptions of the previous lemma, we have*

$$[X^A, X^B]_e = \sum_{C=1}^n f^{CAB} X_e^C, \tag{87}$$

with $f^{CAB} = \frac{\partial^2 m^C}{\partial \varepsilon_1^A \partial \varepsilon_2^B}(0,0) - \frac{\partial^2 m^C}{\partial \varepsilon_1^B \partial \varepsilon_2^A}(0,0)$. *(Homework).*

5.4 Abstract Lie algebras

1. Def. A Lie algebra is a vector space $\mathfrak{g}$ over the field $\mathbb{R}$ together with a bilinear form $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies
   - Antisymmetry: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.
   - Jacobi identity: $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ for all $X, Y, Z \in \mathfrak{g}$.
2. Examples:
   - Let V be a vector space over a field $\mathbb{K}$. Then $\mathfrak{gl}(V)$ denotes the Lie algebra of all linear mappings $V \to V$ with $[X, Y] = X \circ Y - Y \circ X$.
   - For $V = \mathbb{K}^n$ we write $\mathfrak{gl}(n, \mathbb{K}) := \mathfrak{gl}(V)$. This is the Lie algebra of all $n \times n$ matrices with entries in $\mathbb{K}$.
   - $\mathfrak{so}(3) = \{\, X \in \mathfrak{gl}(3, \mathbb{R}) \,|\, X^T = -X \,\}$ is the Lie algebra of rotations.
   - $\mathfrak{su}(2) = \{\, X \in \mathfrak{gl}(2, \mathbb{C}) \,|\, X^\dagger = -X,\ \mathrm{Tr}(X) = 0 \,\}$.
3. Def. Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie algebras. A linear map $h : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if $h([X, Y]) = [h(X), h(Y)]$. If h is a bijection, it is called an isomorphism.
4. For example, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic Lie algebras.
5. Thm. If $\mathfrak{g}$ is a finite-dimensional Lie algebra then there is a unique, up to isomorphism, **simply-connected** Lie group G s.t. $G' = \mathfrak{g}$.
   (Recall that a topological space is simply connected if any loop can be continuously contracted to a point).
6. Fact: If $H : G \mapsto \hat{G}$ is a Lie group homomorphism then
   $$h(X) = \frac{d}{dt} H(\exp(tX))|_{t=0}, \quad X \in G' \tag{88}$$
   is a Lie algebra homomorphism. Furthermore $H(\exp(tX)) = \exp(th(X))$ for all $t \in \mathbb{R}$.
7. Thm. Let $G, \hat{G}$ be Lie groups and suppose G is **simply-connected**. Then a Lie algebra homomorphism $h : G' \to \hat{G}'$ can be lifted to a Lie group homomorphism H.

5.5 Matrix Lie groups

1. Def. A closed subgroup of $GL(n, \mathbb{K})$ is called a matrix Lie group. For example $SO(3)$ and $SU(2)$.
2. Fact: Let G be a matrix Lie group. The Lie algebra G' of this Lie group is given by
   $$G' = \{\, X \in \mathfrak{gl}(n, \mathbb{K}) \,|\, e^{tX} \in G \text{ for all } t \in \mathbb{R} \,\}. \tag{89}$$

We have $GL(n,\mathbb{K})' = \mathfrak{gl}(n,\mathbb{K})$, $SO(3)' = \mathfrak{so}(3)$, $SU(2)' = \mathfrak{su}(2)$. Furthermore, abstract $\exp : G' \to G$ coincide with the exponential function of a matrix.

3. Recall that $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic Lie algebras. However $SO(3)$ and $SU(2)$ are not isomorphic Lie groups. In particular, $SU(2)$ is simply connected, $SO(3)$ is not.

6 Symmetries II

6.1 Representations

1. Def. A group homomorphism $D : G \to GL(V)$ is called a representation.

2. Let $D_1 : G \to GL(V_1)$ and $D_2 : G \to GL(V_2)$ be two reps of G.

   - Def: The direct sum of D_1, D_2, acting on $V_1 \oplus V_2$ is defined by:

   $$(D_1 \oplus D_2)(g)(v_1 \oplus v_2) = (D_1(g)v_1) \oplus (D_2(g)v_2). \tag{90}$$

   - Def: The tensor product of D_1, D_2 acting on $V_1 \otimes V_2$ is defined by

   $$(D_1 \otimes D_2)(g)(v_1 \otimes v_2) = (D_1(g)v_1) \otimes (D_2(g)v_2). \tag{91}$$

3. The property of irreducibility of a representation D can be explained as follows:

   - Def. We say that a subspace $W \subset V$ is invariant, if $D(g)w \in W$ for any $g \in G$ and $w \in W$,
   - Def. We say that a representation is irreducible if it has no invariant subspaces except for $\{0\}$ and V.
   - Def. A representation is completely reducible if it is a direct sum of irreducible representations.
   - Fact. (Schur Lemma). Let $D : G \to GL(V)$ be an irreducible representation and V complex. If $A \in GL(V)$ commutes with all $D(g)$, $g \in G$, then it has the form $A = \lambda I$, $\lambda \in \mathbb{C}$.

4. Def: A Lie algebra homomorphism $d : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a representation. Irreducibility and complete reducibility of such representations are defined analogously as for groups.

5. Thm. Let $D : G \to GL(V)$ be a representation of a Lie group. Then

   $$d(X) = \frac{d}{dt} D(\exp(tX))|_{t=0}, \text{ for } X \in G' \tag{92}$$

   defines a representation of G'.

6. Thm. Let G be a **simply-connected** Lie group and G' its Lie algebra. Let $d : G' \to \mathfrak{gl}(V)$ be a representation. Then there exists a unique representation $D : G \to GL(V)$ s.t. (92) holds.

7. Def. Let G be a matrix Lie group. We say that a bilinear form $b : G' \times G' \to \mathbb{R}$ is invariant, if $b(gXg^{-1}, gYg^{-1}) = b(X, Y)$ for all $g \in G$, $X, Y \in G'$.

8. Fact. Let $d : G' \to \mathfrak{gl}(V)$ be a representation, b an invariant bilinear form on G' and $\{X^1, \ldots, X^n\}$ a basis in G'. Then the *Casimir operator*

$$C := \sum_{A,B} b(X^A, X^B) d(X^A) d(X^B) \tag{93}$$

is basis-independent and commutes with all $d(X)$, $X \in G'$. Thus, by the Schur Lemma, it is a multiple of unity in any irreducible representation on complex V.

6.2 Projective representations

1. Let V by a complex vector space.
   - Def: $U(1) = \{e^{i\varphi} I \,|\, \varphi \in \mathbb{R}\} \subset GL(V)$.
   - Def: $GL(V)/U(1)$ is the (Lie) group generated by the equivalence classes $\underline{A} = \{e^{i\varphi} A \,|\, \varphi \in \mathbb{R}\}$.
   - Def: A homomorphism $\underline{D} : G \to GL(V)/U(1)$ is called a projective representation (a representation up to a phase).
   - For equivalence classes we have $\underline{D}(g_1)\underline{D}(g_2) = \underline{D}(g_1 g_2)$. But for a given choice of representatives $D(g) \in \underline{D}(g)$, (s.t. $D(e) = I$, $g \to D(g)$ continuous)

     $$D(g_1)D(g_2) = e^{i\varphi(g_1,g_2)} D(g_1 g_2). \tag{94}$$

     for some function $\varphi : G \times G \to \mathbb{R}$.

2. $\underline{D}$ gives rise to the homomorphism $\underline{d} : G' \to (GL(V)/U(1))'$ given by

$$\underline{d}(X) = \frac{d}{dt} \underline{D}(\exp(tX))|_{t=0}, \quad X \in G'. \tag{95}$$

3. We have $(GL(V)/U(1))' = (GL(V)'/U(1)') = \mathfrak{gl}(V)/(i\mathbb{R})$. The elements of this Lie algebra are the equivalence classes $\underline{Y} = \{Y + iz \,|z \in \mathbb{R}\}$.

4. For equivalence classes we have $[\underline{d}(X^A), \underline{d}(X^B)] = \sum_C f^{CAB} \underline{d}(X^C)$. But for any given choice of representatives $d(X^A) \in \underline{d}(X^A)$

$$[d(X^A), d(X^B)] = \sum_C f^{CAB} d(X^C) - iz^{A,B} I, \tag{96}$$

where $z^{A,B}$ are called the *central charges* and the (-) sign in the last formula is a matter of convention. In the physical notation one defines the infinitesimal generators $T^A = id(X^A)$ so that (96) reads

$$[T^A, T^B] = \sum_C if^{CAB} T^C + iz^{A,B} I. \tag{97}$$

We follow the mathematical convection below unless stated otherwise.

- In some cases it is possible to eliminate $z^{A,B}$ by passing to different representatives $\widetilde{d}(X^A) = d(X^A) + ic^A$.
- Then $\widetilde{d}$ becomes a Lie algebra representation $G' \to \mathfrak{gl}(V)$.
- Hence, by a theorem above, $\widetilde{d}$ can be lifted to a Lie group representation $\widetilde{D} : \widetilde{G} \to GL(V)$, where $\widetilde{G}$ is the unique simply-connected Lie group with the Lie algebra G'. (The universal covering group).

5. Let us explain in more detail the concept of the covering space/group:

   - Def. A topological space G is path connected, if for any $g_1, g_2 \in G$ there is a continuous map $\gamma : [0,1] \to G$ s.t. $\gamma(0) = g_1$, $\gamma(1) = g_2$.
   - Def. A topological space G is simply connected, if it is path connected and every loop in the space can be continuously contracted to a point.
   - Def. The universal cover of a connected topological space G is a simply-connected space $\widetilde{G}$ together with a covering map $H_c : \widetilde{G} \to G$. The covering map is a local homeomorphism s.t. the cardinal number of $H_c^{-1}(g)$ is independent of g. The universal cover is unique.
   - Fact: If G is a Lie group, $\widetilde{G}$ is also a Lie group and $H_c : \widetilde{G} \to G$ is a homomorphism s.t. ker H_c is a discrete subgroup.

6. The situation above occurs in particular for projective unitary representations of $SO(3)$.

   - One can choose D s.t. $e^{i\varphi(g_1,g_2)} \in \{\pm 1\}$. By continuity, $e^{i\varphi(g_1,g_2)} = 1$ for g_1, g_2 close to e, hence $z^{A,B} = 0$.
   - Thus D can be lifted to a unitary representation $\widetilde{D}$ of $\widetilde{SO}(3) = SU(2)$. More precisely, $\widetilde{D}(A) = D(H_c(A))$, where $A \in SU(2)$ and $H_c : SU(2) \to SO(3)$ is the covering homomorphism.
   - $\ker H_c := H_c^{-1}(e) = \mathbb{Z}_2$ thus $SU(2)/\mathbb{Z}_2 \simeq SO(3)$ and every element of $SO(3)$ corresponds to two elements in $SU(2)$. That is $SU(2)$ is a double covering of $SO(3)$.

6.3 Representations of rotations

1. Fact: $\mathfrak{so}(3) = \{ X \in \mathfrak{gl}(3,\mathbb{R}) \,|\, X^T = -X \}$ is the Lie algebra of $SO(3) = \{ R \in GL(3,\mathbb{R}) \,|\, R^T R = I,\ \det R = 1\}$. Indeed, let $X \in \mathfrak{so}(3)$. Then

$$(e^{tX})^T e^{tX} = e^{tX^T} e^{tX} = e^{-tX} e^{tX} = 1 \qquad (98)$$
$$\det(e^{tX}) = e^{t\mathrm{Tr}X} = 1, \qquad (99)$$

   where we used that a real anti-symmetric metric has vanishing diagonal elements and consequently is traceless.

2. We choose a basis in $\mathfrak{so}(3)$ as follows

$$L^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{100}$$

so that $e^{\theta \vec{n} \cdot \vec{L}}$ is the rotation around the axis $\vec{n}$, $\|\vec{n}\| = 1$, by angle θ.

3. One verifies the commutation relations

$$[L^i, L^j] = \varepsilon^{ijk} L^k. \tag{101}$$

These generators are related to the angular momentum operators J^i via $L^i = -iJ^i$. They satisfy accordingly

$$[J^i, J^j] = i\varepsilon^{ijk} J^k. \tag{102}$$

4. Some facts about the irreducible representations of $\mathfrak{so}(3)$:

   - From quantum mechanics we know that there is only one Casimir operator $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, whose eigenvalues are $j(j+1)$, for $j = 0, 1/2, 1, \ldots$.
   - The irreducible representations $d^{(j)}$ are labelled by j and are $2j + 1$ dimensional.
   - The basis vectors are denoted $|j, m\rangle$, $m = -j, -j + 1, \ldots j$, where $d^{(j)}(J_3)|j, m\rangle = m|j, m\rangle$.

5. Recall that $H_c : SU(2) \to SO(3)$ is the covering homomorphism. It gives rise to the isomorphism $h_c : \mathfrak{su}(2) \to \mathfrak{so}(3)$ which can be described as follows:

   - let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices. Then $Y^j = \frac{1}{2i}\sigma_j$ is a basis of $\mathfrak{su}(2)$, since $[Y^i, Y^j] = \varepsilon^{ijk} Y^k$.
   - Then h_c is given in this basis by $h_c(Y^j) = L^j$.
   - $h_c^{-1} = d^{(1/2)}$.

6. Since $SU(2)$ is simply-connected, any representation $d^{(j)}$ gives rise to a representation $D^{(j)}$ of $SU(2)$ according to $D^{(j)}(e^{t(h_c)^{-1}(X)}) = e^{td^{(j)}(X)}$. In particular $D^{(1/2)}$, is the defining representation of $SU(2)$ as one can show using that the Pauli matrices are traceless and hermitian.

7. Fact: Only for integer j the representation $D^{(j)}$ of $SU(2)$ can be lifted to a representation of $SO(3)$.
   But, as we know from the previous subsection, for any j it can be lifted to a *projective* representation of $SO(3)$. In order to accommodate half-integer spins, we need to justify that the projective representations of $SO(3)$ correspond to rotation symmetry of physical quantum systems. This will be done later today, using that quantum states are determined up to a phase.

8. For any $j = 0, 1/2, 1, \ldots$ and a rotation R given by the axis $\vec{n}$ and angle θ we define the *Wigner functions*

$$D^{(j)}_{mm'}(R) := \langle j, m | e^{-i\theta \vec{n}\cdot \vec{J}^{(j)}} | j, m' \rangle, \tag{103}$$

where $\vec{J}^{(j)}$ denotes here the angular momentum in the representation $d^{(j)}$.

9. Let us illustrate how the projective character of $D^{(j)}(R)$ for half-integer j comes about: Consider a rotation $R_{2\pi}$ by 2π around the $\vec{e}_3$-axis in the $j = 1/2$ representation. It can be computed in two ways: First, since rotation by 2π is equal to identity, we obtain $D^{(1/2)}(R) = I$. On the other hand, formula (103) gives

$$D^{(1/2)}(R_{2\pi}) = e^{-i2\pi\frac{\sigma_3}{2}} = \exp\left(- i\pi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{bmatrix} = -I. \tag{104}$$

Since we got two different results, $D^{(1/2)}$ is not a well defined homomorphism $SO(3) \to GL(V)$ (i.e. representation). But it can still be a well defined homomorphism $SO(3) \to GL(V)/U(1)$ (i.e. projective representation) as the two results differ only by a sign and thus belong to the same equivalence class.

10. Any finite dimensional representation of $SU(2)$ is completely reducible i.e. can be represented as a direct sum of the irreducible representations $D^{(j)}$. In particular, we have

$$D^{(j_1)} \otimes D^{(j_2)} = \oplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)}. \tag{105}$$

This is what is called "addition of angular momenta".

6.4 Symmetries of quantum theories

When studying symmetries of a quantum theory, one has to take it seriously that physical states are defined up to a phase. Thus we consider the following setting:

1. $\mathcal{H}$ - Hilbert space of physical states.

2. For $\Psi \in \mathcal{H}$, $\|\Psi\| = 1$ define the ray $\hat{\Psi} := \{ \mathrm{e}^{i\theta}\Psi \,|\, \theta \in \mathbb{R} \}$.

3. $\hat{\mathcal{H}}$ - set of rays with the ray product $[\hat{\Phi}|\hat{\Psi}] := |\langle \Phi|\Psi\rangle|^2$.

Definition 6.1 *A symmetry transformation of a quantum system is an invertible map $\hat{U} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ s.t. $[\hat{U}\hat{\Phi}|\hat{U}\hat{\Psi}] = [\hat{\Phi}|\hat{\Psi}]$. Such transformations form a group.*

Theorem 6.2 *(Wigner) For any symmetry transformation $\hat{U} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ we can find a unitary or anti-unitary operator $U : \mathcal{H} \to \mathcal{H}$ s.t. $\hat{U}\hat{\Psi} = \widehat{U\Psi}$. U is unique up to phase.*

Anti-unitary operators are defined as follows:

- A map $U : \mathcal{H} \to \mathcal{H}$ is anti-linear if $U(c_1\Psi_1 + c_2\Psi_2) = \bar{c}_1 U\Psi_1 + \bar{c}_2 U\Psi_2$.
- The adjoint of an anti-linear map is given by $\langle\Phi|U^\dagger\Psi\rangle = \overline{\langle U\Phi|\Psi\rangle}$.
- An anti-linear map is called anti-unitary if $U^\dagger U = UU^\dagger = I$ in which case $\langle U\Phi|U\Psi\rangle = \overline{\langle\Phi|U^\dagger U\Psi\rangle} = \langle\Psi|\Phi\rangle$.

Anti-unitary operators will be needed to implement discrete symmetries, e.g. time reversal. For symmetries described by connected Lie groups anti-unitary operators can be excluded, as we indicate below.

Application of the Wigner theorem:

1. Suppose a connected Lie group G is a symmetry of our theory i.e. there is a group homomorphism $G \ni g \mapsto \hat{U}(g)$ into symmetry transformations.

2. The Wigner theorem gives corresponding unitary operators $U(g)$. Since they are determined up to a phase, they form only a *projective* representation:

$$U(g_1)U(g_2) = \mathrm{e}^{i\theta_{1,2}} U(g_1 g_2). \tag{106}$$

   (Since G is connected, we can exclude that some $U(g)$ are anti-unitary. Indeed for a connected group we have $g = g_0^2$ for some $g_0 \in G$. Now $U(g) = \mathrm{e}^{-i\theta} U(g_0)U(g_0)$ which is unitary no matter if $U(g_0)$ is unitary or anti-unitary).

3. As discussed above[5], for a large class of connected Lie groups G (including $SO(3)$ and $\mathcal{P}_+^\uparrow$) a projective unitary representation of G corresponds to an *ordinary* unitary representation of the covering group $\widetilde{G}$

$$\widetilde{G} \ni \tilde{g} \mapsto \tilde{U}(\tilde{g}) \in B(\mathcal{H}). \tag{107}$$

   In particular, projective unitary representations of $SO(3)$ correspond to ordinary unitary representations of $SU(2)$ and thus there is room for half-integer spin!

[5]Strictly speaking, in previous sections we tacitly assumed that the representations act on finite-dimensional vector spaces, while here $\mathcal{H}$ can be infinite dimensional. Fortunately, the relevant results generalize.

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