3 Scattering theory

The main reference for this section is [3, Chapter 16].

3.1 Setting

We consider a Wightman theory $(\mathcal{H}, U, \Omega, \phi, D)$. Recall the key properties

1. Covariance: $U(\Lambda, a)\phi(x)U(\Lambda, a)^{-1} = \phi(\Lambda x + a)$,
2. Locality: $[\phi(x), \phi(y)] = 0$ for $(x - y)^2 < 0$,
3. Cyclicity of Ω: Vectors of the form $\phi(f_1)\dots\phi(f_n)\Omega$ span a dense subspace in $\mathcal{H}$,

where smearing with test-functions from S in variables x, y is understood in properties 1. and 2. Furthermore $U(a) = e^{iP_\mu a^\mu}$ and $\mathrm{Sp}\, P \subset \bar{V}_+$. Today we will impose stronger assumptions on the spectrum:

A.1. The spectrum contains an isolated mass hyperboloid H_m i.e.

$$H_m \subset \mathrm{Sp}\, P \subset \{0\} \cup H_m \cup G_{\tilde{m}}, \tag{28}$$

where $H_m = \{\, p \in \mathbb{R}^4 \,|\, p^0 = \sqrt{\vec{p}^2 + m^2}\,\}$, $G_{\tilde{m}} = \{\, p \in \mathbb{R}^4 \,|\, p^0 \geq \sqrt{\vec{p}^2 + \tilde{m}^2}\,\}$ for $\tilde{m} > m$. (In other words, the mass-operator $\sqrt{P_\mu P^\mu}$ has an isolated eigenvalue m. Embedded eigenvalues can also be treated [13], but then scattering theory is more difficult).

A.2. Define the single-particle subspace $\mathcal{H}(H_m)$ as the spectral subspace of H_m. That is, $\mathcal{H}(H_m) = \chi(P)\mathcal{H}$, where $\chi(P)$ is the characteristic function of H_m evaluated at $P = (P_0, P_1, P_2, P_3)$. We assume that U restricted to $\mathcal{H}(H_m)$ is an irreducible representation of $\mathcal{P}_+^\uparrow$. (One type of particles).

Theorem 3.1 *(Källen-Lehmann representation). For a Wightman field ϕ with $\langle \Omega, \phi(x)\Omega\rangle = 0$ we have*

$$\langle \Omega|\phi(x)\phi(y)\Omega\rangle = \int d\rho(M^2)\Delta_+(x - y; M^2), \tag{29}$$

$$\Delta_+(x - y; M^2) := \langle 0|\phi_0^{(M)}(x)\phi_0^{(M)}(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(y-x)}, \tag{30}$$

where $d\rho(M^2)$ is a measure on $\mathbb{R}_+$, $p_0 = \sqrt{\vec{p}^2 + M^2}$, $\phi_0^{(M)}$ is the free scalar field of mass M and $|0\rangle$ is the vacuum vector in the Fock space of this free field theory, whereas Ω is the vacuum of the (possibly interacting) Wightman theory. Furthermore, given the structure of the spectrum (28), we have

$$d\rho(M^2) = Z\delta(M^2 - m^2)d(M^2) + d\tilde{\rho}(M^2) \tag{31}$$

where $Z \geq 0$ and $d\tilde{\rho}$ is supported in $[\tilde{m}^2, \infty)$

We assume in the following that:

A.3. $\langle\Omega, \phi(x)\Omega\rangle = 0$. This is not a restriction, since a shift by a constant $\phi(x) \mapsto \phi(x) + c$ gives a new Wightman field.

A.4. $Z \neq 0$ to ensure that $\langle\Psi_1|\phi(x)\Omega\rangle \neq 0$ for some single-particle vector Ψ_1 (i.e. a vector living on H_m). This means that the particle is 'elementary' (as opposed to composite) and we do not need polynomials in the field to create it from the vacuum. This assumption can be avoided at a cost of complications.

3.2 Problem and strategy

Take two single-particle states $\Psi_1, \Psi_2 \in \mathcal{H}(H_m)$. We would like to construct vectors $\Psi^{\mathrm{out}}, \Psi^{\mathrm{in}}$ describing outgoing/ incoming configuration of these two single-particle states Ψ_1, Ψ_2. Mathematically this problem consists in finding two 'multiplications'

$$\Psi^{\mathrm{out}} = \Psi_1 \overset{\mathrm{out}}{\times} \Psi_2, \tag{32}$$

$$\Psi^{\mathrm{in}} = \Psi_1 \overset{\mathrm{in}}{\times} \Psi_2, \tag{33}$$

which have all the properties of the (symmetrised) tensor product but take values in $\mathcal{H}$ (and not in $\mathcal{H} \otimes \mathcal{H}$). After all, we know from quantum mechanics, that symmetrised tensor products describe configurations of two undistinguishable bosons.

The strategy is suggested by the standard Fock space theory: With the help of the field ϕ we will construct certain 'time-dependent creation operators' $t \mapsto A^{\dagger}_{1,t}$, $t \mapsto A^{\dagger}_{2,t}$ s.t.

$$\Psi_1 = \lim_{t\to\pm\infty} A^{\dagger}_{1,t}\Omega, \quad \Psi_2 = \lim_{t\to\pm\infty} A^{\dagger}_{2,t}\Omega. \tag{34}$$

Then we can try to construct

$$\Psi^{\mathrm{out}} = \lim_{t\to\infty} A^{\dagger}_{1,t}A^{\dagger}_{2,t}\Omega, \quad \Psi^{\mathrm{in}} = \lim_{t\to-\infty} A^{\dagger}_{1,t}A^{\dagger}_{2,t}\Omega. \tag{35}$$

Of course analogous consideration applies to n-particle scattering states.

Plan of the remaining part of the lecture:

- Construction of $A^{\dagger}_t$.
- Existence of limits in definitions of Ψ^{out}, Ψ^{in}.
- Wave-operators, S-matrix and the LSZ reduction formula.

3.3 Definition of $A_t^\dagger$

The operators $A_t^\dagger$ are defined in (41) below. In order to motivate this definition, we state several facts about the free field. It should be kept in mind that we are interested in the interacting field, and the following discussion of the free field is merely a motivating digression.

Recall the definition of the free scalar field:

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3 2p^0}(e^{ipx}a^\dagger(p) + e^{-ipx}a(p)). \tag{36}$$

(Here and in the following we reserve the letter p for momenta restricted to the mass-shell i.e. $p = (p^0, \vec{p}) = (\sqrt{\vec{p}^2 + m^2}, \vec{p})$. For other momenta I will use q). There are two ways to extract $a^\dagger$ out of ϕ_0:

1. Use the formula from the lecture:

$$a^\dagger(p) = i \int d^3x\, \phi_0(x) \overleftrightarrow{\partial}_0 e^{-ipx} \tag{37}$$

   Since $a^\dagger(p)$ is not a well-defined operator (only an operator valued distribution) we will smear both sides of this equality with a test-function. For this purpose we define for any $f \in C_0^\infty(\mathbb{R}^4)$

$$a^\dagger(f) := \int \frac{d^3p}{(2\pi)^3 2p^0} a^\dagger(p) f(p), \quad f_m(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ipx} f(p), \tag{38}$$

   where the latter is a positive-energy solution of the KG equation, that is $(\Box + m^2) f_m(x) = 0$. We get

$$a^\dagger(f) = i \int d^3x\, \phi_0(x) \overleftrightarrow{\partial}_0 f_m(x). \tag{39}$$

2. Pick a function $h \in S$ s.t. $\operatorname{supp} \widehat{h}$ is compact and $\operatorname{supp} \widehat{h} \cap \operatorname{Sp} P \subset H_m$. Then

$$\phi_0(h) = (2\pi)^2 a^\dagger(\widehat{h}), \text{ where } \widehat{h}(q) = \frac{1}{(2\pi)^2} \int e^{iqx} h(x) d^4x. \tag{40}$$

Now we come back to our (possibly) interacting Wightman field ϕ and perform both operations discussed above to obtain the 'time dependent creation operator'

$$A_t^\dagger := i \int d^3x\, \phi(h)(t, \vec{x}) \overleftrightarrow{\partial}_0 f_m(t, \vec{x}), \tag{41}$$

where $\phi(h)(t, \vec{x}) := U(t, \vec{x}) \phi(h) U(t, \vec{x})^\dagger = \phi(h_{(t,\vec{x})})$.

3.4 Construction of scattering states

Theorem 3.2 *(Haag-Ruelle) For $f_1, \dots, f_n$ with disjoint supports, the following limits exist*

$$\Psi_n^{\mathrm{out}} = \lim_{t\to\infty} A_{1,t}^\dagger \dots A_{n,t}^\dagger \Omega, \tag{42}$$

$$\Psi_n^{\mathrm{in}} = \lim_{t\to-\infty} A_{1,t}^\dagger \dots A_{n,t}^\dagger \Omega \tag{43}$$

and define outgoing/incoming scattering states.

Proof. For $n = 1$ the expression

$$A_{1,t}^\dagger \Omega = i \int d^3x \, \big(\phi(h)(t,\vec{x})\Omega\big) \overleftrightarrow{\partial}_0 f_m(t,\vec{x}) \tag{44}$$

is independent of t and thus $\lim_{t\pm\infty} A_{1,t}^\dagger(f_1)\Omega$ (trivially) exist. Moreover, it is a single-particle state. Justification:

- $x \mapsto \phi(h)(x)\Omega$ is a solution of the KG equation. This can be shown using the Källen-Lehmann representation and the support property of $\widehat{h}$ to eliminate the contribution from $d\tilde{\rho}$. Assumption A.1. enters here. (Howework).

- For any two solutions g_1, g_2 of the KG equation $\int d^3x \, g_1(t,\vec{x})\overleftrightarrow{\partial}_0 g_2(t,\vec{x})$ is independent of t.

- We have $i[P_\mu, \phi(h)(x)] = (\frac{\partial}{\partial x^\mu})\phi(h)(x)$. Since $P_\mu\Omega = 0$, we can write

$$\begin{aligned} P^2\phi(h)(x)\Omega &= P_\mu P^\mu \phi(h)(x)\Omega = -i[P_\mu, i[P^\mu, \phi(h)(x)]]\Omega \\ &= -\Box_x \phi(h)(x)\Omega = m^2\phi(h)(x)\Omega, \end{aligned} \tag{45}$$

  where in the last step we used the first item above. Hence $\phi(h)(x)\Omega$ are single-particle states of mass m.

For $n = 2$ we set $\Psi_t := A_{1,t}^\dagger A_{2,t}^\dagger \Omega$ and try to verify the Cauchy criterion:

$$\|\Psi_{t_2} - \Psi_{t_1}\| = \| \int_{t_1}^{t_2} \partial_\tau \Psi_\tau d\tau \| \le \int_{t_1}^{t_2} \|\partial_\tau \Psi_\tau\| d\tau. \tag{46}$$

If we manage to show that $\|\partial_\tau\Psi_\tau\| \le c/\tau^{1+\eta}$ for some $\eta > 0$ then the Cauchy criterion will be satisfied as we will have

$$\|\Psi_{t_2} - \Psi_{t_1}\| \le c \left| \frac{1}{t_1^\eta} - \frac{1}{t_2^\eta} \right|. \tag{47}$$

(Note that we use the completeness of $\mathcal{H}$ here i.e. the property that any Cauchy sequence converges).

Thus we study $\partial_\tau \Psi_\tau$. The Leibniz rule gives

$$\begin{aligned} \partial_\tau \Psi_\tau &= (\partial_\tau A^\dagger_{1,\tau}) A^\dagger_{2,\tau}\Omega + A^\dagger_{1,\tau}(\partial_\tau A^\dagger_{2,\tau})\Omega \\ &= [(\partial_\tau A^\dagger_{1,\tau}), A^\dagger_{2,\tau}]\Omega + A^\dagger_{2,\tau}(\partial_\tau A^\dagger_{1,\tau})\Omega + A^\dagger_{1,\tau}(\partial_\tau A^\dagger_{2,\tau})\Omega. \end{aligned} \tag{48}$$

Since $(\partial_\tau A^\dagger_{i,\tau})\Omega = 0$ by the first part of the proof, only the term with the commutator above is non-zero. To analyze it, we need some information about KG wave-packets:

- Def. For the KG wave-packet $f_{i,m}$ we define the velocity support as

$$V_i = \left\{ \frac{\vec{p}}{p^0} \,|\, p \in \mathrm{supp} f_i \right\} \tag{49}$$

  and let V_i^δ be slightly larger sets.

- Fact. For any $N \in \mathbb{N}$ we can find a c_N s.t.

$$|f_{i,m}(\tau, \vec{x})| \leq \frac{c_N}{\tau^N} \text{ for } \frac{\vec{x}}{\tau} \notin V_i^\delta. \tag{50}$$

Due to (50), the contributions to $\|[(\partial_\tau A^\dagger_{1,\tau}), A^\dagger_{2,\tau}]\Omega\|$ coming from the part of the integration region in (41) where $\frac{\vec{x}}{\tau} \notin V_i^\delta$, are rapidly vanishing with τ. So we only have to worry about the dominant parts:

$$A^{\dagger(D)}_{i,t} := i \int_{\frac{\vec{x}}{t} \in V_i^\delta} d^3x\, \phi(h)(t, \vec{x}) \overleftrightarrow{\partial}_0 f_{i,m}(t, \vec{x}). \tag{51}$$

Since V_1^δ, V_2^δ are disjoint, the Wightman axiom of locality gives for sufficiently large τ.

$$\|[(\partial_\tau A^{\dagger(D)}_{1,\tau}), A^{\dagger(D)}_{2,\tau}]\Omega\| \leq \frac{c_N}{\tau^N}. \tag{52}$$

This concludes the proof. □

3.5 Wave operators, scattering matrix, LSZ reduction

In the following we choose h s.t. $\widehat{h}(p) f(p) = (2\pi)^{-2} Z^{-1/2} f(p)$. This can be done, since f has compact support. After this fine-tuning, exploiting assumptions A.2, A.3, A.4 one obtains the following simple formula for scalar products of scattering states:

Theorem 3.3 *(Haag-Ruelle) Let $\Psi_n^{\mathrm{out}}, (\Psi'_{n'})^{\mathrm{out}}$ be as in the previous theorem. Then their scalar products can be computed as if these were vectors on the Fock space:*

$$\langle \Psi_n^{\mathrm{out}} | (\Psi'_{n'})^{\mathrm{out}} \rangle = \langle 0 | a(f_n) \ldots a(f_1) a^\dagger(f'_1) \ldots a^\dagger(f'_{n'}) | 0 \rangle \tag{53}$$

and analogously for incoming states.

Let $\mathcal{F}$ be the symmetric Fock space. (This is not the Hilbert space of our Wightman theory, but merely an auxiliary object needed to define the wave-operators). We define the outgoing wave-operator $W^{\rm out}:\mathcal{F}\to\mathcal{H}$ as

$$W^{\rm out}(a^\dagger(f_1)\dots a^\dagger(f_n)|0\rangle)=\lim_{t\to\infty}A^\dagger_{1,t}\dots A^\dagger_{n,t}\Omega. \tag{54}$$

By Theorem 3.3 it is an isometry i.e. $(W^{\rm out})^\dagger W^{\rm out}=I$. If it is also a unitary i.e. $\mathrm{Ran}\,W^{\rm out}=\mathcal{H}$ then we say that the theory is *asymptotically complete* that is every vector in $\mathcal{H}$ can be interpreted as a collection of particles from $\mathcal{H}(H_m)$. This property does not follow from Wightman axioms (there are counterexamples) and it is actually not always expected on physical grounds. For a more thorough discussion of asymptotic completeness we refer to [12].

The incoming wave-operator $W^{\rm in}:\mathcal{F}\to\mathcal{H}$ is defined by taking the limit $t\to-\infty$ in (54). The scattering matrix $\hat{S}:\mathcal{F}\to\mathcal{F}$ is given by[1]

$$\hat{S}=(W^{\rm out})^\dagger W^{\rm in}. \tag{55}$$

If $\hat{S}\neq I$ we say that a theory is interacting. If $\mathrm{Ran}\,W^{\rm out}=\mathrm{Ran}\,W^{\rm in}$, then $\hat{S}$ is a unitary (even without asymptotic completeness).

Corollary 3.4 *(LSZ reduction) [8] For $f_1,\dots,f_\ell,g_1,\dots g_n\in S$ with mutually disjoint supports, we have*

$$\begin{aligned}\langle 0|a(f_1)\dots a(f_\ell)\hat{S}a^\dagger(g_1)\dots a^\dagger(g_n)|0\rangle=\int\frac{d^3k_1}{(2\pi)^3 2k_1^0}\cdots\frac{d^3p_n}{(2\pi)^3 2p_n^0}f_1(k_1)\dots g_n(p_n)\times\\ \times\frac{(-i)^{n+\ell}}{(\sqrt{Z})^{n+\ell}}\prod_{i=1}^{\ell}(k_i^2-m^2)\prod_{j=1}^{n}(p_j^2-m^2)\times\\ \times\int d^4x_1\dots d^4x_\ell d^4y_1\dots d^4y_n\,e^{i\sum_{i=1}^{\ell}k_ix_i-i\sum_{j=1}^{n}p_jy_j}\times\\ \times\langle\Omega|T(\phi(x_\ell)\dots\phi(x_1)\phi(y_1)\dots\phi(y_n))\Omega\rangle,\end{aligned}$$

where T is the time-ordered product (which needs to be regularized in the Wightman setting).

By analytic continuation one can relate the Green functions to Schwinger functions. The latter can be studied using path integrals as explained in the previous lecture. This led to a proof that for ϕ^4 in 2-dimensional spacetime $\hat{S}\neq I$ [9]. It is a big open problem if there is a Wightman theory in 4-dimensional spacetime with $\hat{S}\neq I$.

[1] The notation $\hat{S}$ is used to avoid confusion with the Schwartz class S.

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