2 Path integrals

The main references for this section are [5, Chapter 6] [6, Chapter 1].

2.1 Wightman and Schwinger functions

Consider a theory $(\mathcal{H}, U, \Omega, \phi, D)$ of one scalar Wightman field.

- Wightman functions are defined as

$$W_n(x_1, \ldots, x_n) = \langle \Omega | \phi(x_1) \ldots \phi(x_n) \Omega \rangle. \tag{13}$$

  They are tempered distributions.

- Green functions are defined as

$$G_n(x_1, \ldots, x_n) = \langle \Omega | T(\phi(x_1) \ldots \phi(x_n)) \Omega \rangle \tag{14}$$

  Recall that $T\phi(x_1)\phi(x_2) = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1)$. This multiplication of distributions by a discontinuous function may be ill-defined in the Wightman setting. Approximation of θ by smooth functions may be necessary. Then we obtain tempered distributions.

- Euclidean Green functions (Schwinger functions) are defined as

$$G_{E,n}(x_1, \ldots, x_n) = W_n((ix_1^0, \vec{x}_1), \ldots, (ix_n^0, \vec{x}_n)). \tag{15}$$

  The analytic continuation is justified in the Wightman setting. We obtain real-analytic functions on $\mathbb{R}^{4n}_{\neq} = \{(x_1, \ldots, x_n) \,|\, x_i \neq x_j \ \ \forall i \neq j\,\}$, symmetric under the exchange of variables.

The Schwinger functions are central objects of mathematical QFT based on path-integrals. The idea is to express $G_{E,n}$ as moment functions of a measure μ on the space $S'_{\mathbb{R}}$ of real-valued tempered distributions

$$G_{E,n}(x_1, \ldots, x_n) = \int_{S'_{\mathbb{R}}} \varphi(x_1) \ldots \varphi(x_n) d\mu(\varphi). \tag{16}$$

Today's lecture:

- Measure theory on topological spaces.
- Conditions on $d\mu$ which guarantee that formula (16) really gives Schwinger functions of some Wightman QFT. (Osterwalder-Schrader axioms).
- Remarks on construction of interacting functional measures $d\mu$

2.2 Elements of measure theory

1. Def. We say that X is a topological space, if it comes with a family of subsets $\mathcal{T} = \{O_i\}_{i\in\mathbb{I}}$ of X satisfying the following axioms:
   - $\emptyset, X \in \mathcal{T}$,
   - $\bigcup_{j\in\mathbb{J}} O_j \in \mathcal{T}$,
   - $\bigcap_{j=1}^{N} O_j \in \mathcal{T}$.

   O_i are called the open sets.

2. **Example:** $S'_{\mathbb{R}}$ is a topological space. In fact, given $\varphi_0 \in S'_{\mathbb{R}}$, a finite family $J_1, \dots J_N \in S_{\mathbb{R}}$ and $\varepsilon_1, \dots, \varepsilon_N > 0$ we can define a *neighbourhood* of φ_0 as follows:

$$B(\varphi_0; J_1, \dots, J_N; \varepsilon_1, \dots \varepsilon_N)$$
$$:= \{ \varphi \in S'_{\mathbb{R}} \,|\, |\varphi(J_1) - \varphi_0(J_1)| < \varepsilon_1, \dots, |\varphi(J_N) - \varphi_0(J_N)| < \varepsilon_N \}. \quad (17)$$

   All open sets in $S'_{\mathbb{R}}$ can be obtained as unions of such neighbourhoods.

3. Def. Let X be a topological space. A family $\mathcal{M}$ of subsets of X is a σ-algebra in X if it has the following properties:
   - $X \in \mathcal{M}$,
   - $A \in \mathcal{M} \Rightarrow X\backslash A \in \mathcal{M}$,
   - $A_n \in \mathcal{M}$, $n \in \mathbb{N}$, $\Rightarrow A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

   If $\mathcal{M}$ is a σ-algebra in X then X is called a *measurable space* and elements of $\mathcal{M}$ are called *measurable sets*.

4. Def. The Borel σ-algebra is the smallest σ-algebra containing all open sets of X. Its elements are called Borel sets.

5. Def. Let X be a measurable space and Y a topological space. Then a map $f : X \to Y$ is called *measurable* if for any open $V \subset Y$ the inverse image $f^{-1}(V)$ is a measurable set.

6. Def. A measure is a function $\mu : \mathcal{M} \to [0, \infty]$ s.t. for any countable family of disjoint sets $A_i \in \mathcal{M}$ we have

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i). \quad (18)$$

   Also, we assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{M}$.
   - If $\mu(X) = 1$, we say that μ is a probability measure.
   - If μ is defined on the Borel σ-algebra, we call it a Borel measure.

7. We denote by $\mathcal{L}^p(X, d\mu)$, $1 \leq p < \infty$ the space of measurable functions $f : X \to \mathbb{C}$ s.t.

$$\|f\|_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} < \infty. \tag{19}$$

We denote by $L^p(X, d\mu)$ the space of equivalence classes of functions from $\mathcal{L}^p(X, d\mu)$ which are equal except at sets of measure zero. The following statements are known as the Riesz-Fisher theorem:

- $L^p(X, d\mu)$ is a Banach space with the norm (19).
- $L^2(X, d\mu)$ is even a Hilbert space w.r.t. $\langle f_1|f_2\rangle = \int \bar{f}_1(x) f_2(x) d\mu(x)$.

8. The following theorem allows us to construct measures on $S'_{\mathbb{R}}$:

Theorem 2.1 *(Bochner-Minlos) Let $Z_E : S_{\mathbb{R}} \to \mathbb{C}$ be a map satisfying*

(a) (Continuity) $Z_E[J_n] \to Z_E[J]$ if $J_n \to J$ in $S_{\mathbb{R}}$

(b) (Positive definiteness) For any $J_1, \ldots, J_N \in S_{\mathbb{R}}$, the matrix $A_{i,j} := Z_E[J_i - J_j]$ is positive. This means $z^\dagger A z := \sum_{i,j} \bar{z}_i A_{i,j} z_j \geq 0$ for any $z \in \mathbb{C}^N$.

(c) (Normalisation) $Z_E[0] = 1$.

Then there exists a unique Borel probability measure μ on $S'_{\mathbb{R}}$ s.t.

$$Z_E[J] = \int_{S'_{\mathbb{R}}} e^{i\varphi(J)} d\mu(\varphi) \tag{20}$$

$Z_E[J]$ is called the characteristic function of μ or the (Euclidean) generating functional of the moments of μ. Indeed, formally we have:

$$(-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z_E[J]|_{J=0} = \int_{S'_{\mathbb{R}}} \varphi(x_1) \ldots \varphi(x_n) d\mu(\varphi), \tag{21}$$

so the generating functional carries information about all the moments of the measure (cf. (16) above).

9. **Example:** Let $C = \frac{1}{-\Delta + m^2}$, where $\Delta = \frac{\partial^2}{(\partial x^0)^2} + \cdots + \frac{\partial^2}{(\partial x^3)^2}$ is the Laplace operator on $\mathbb{R}^4$. We consider the expectation value of C on $J \in S_{\mathbb{R}}$:

$$\langle J|CJ\rangle := \int d^4p\, \bar{\hat{J}}(p) \frac{1}{p^2 + m^2} \hat{J}(p). \tag{22}$$

and set $Z_{E,C}[J] := e^{-\frac{1}{2}\langle J|CJ\rangle}$. This map satisfies the assumptions of the Bochner-Minlos theorem and gives a measure $d\mu_C$ on $S'_{\mathbb{R}}$ called the Gaussian measure with covariance (propagator) C. In the physics notation:

$$\begin{aligned}\int F(\varphi) d\mu_C(\varphi) &= \int F(\varphi) \frac{1}{N_C} e^{-\frac{1}{2}\int d^4x\, \varphi(x)(-\Delta+m^2)\varphi(x)} \mathcal{D}[\varphi] \\ &= \int F(\varphi) \frac{1}{N_C} e^{-\frac{1}{2}\int d^4x\, (\partial_\mu\varphi(x)\partial^\mu\varphi(x) + m^2\varphi^2(x))} \mathcal{D}[\varphi],\end{aligned} \tag{23}$$

for any $F \in L^1(S'_{\mathbb{R}}, d\mu_C)$. Since we chose imaginary time, we have a Gaussian damping factor and not an oscillating factor above. This is the main reason to work in the Euclidean setting.

2.3 Osterwalder-Schrader axioms

Now we formulate conditions, which guarantee that a given measure μ on $S'_{\mathbb{R}}$ gives rise to a Wightman theory:

Definition 2.2 *We say that a Borel probability measure μ on $S'_{\mathbb{R}}$ defines an Osterwalder-Schrader QFT if this measure, resp. its generating functional $Z_E : S_{\mathbb{R}} \to \mathbb{C}$, satisfies:*

1. *(Analyticity) The function $\mathbb{C}^N \ni (z_1, \ldots, z_N) \to Z_E[\sum_{i=1}^N z_j J_j] \in \mathbb{C}$ is entire analytic for any $J_1, \ldots J_N \in S_{\mathbb{R}}$.*

   Gives existence of Schwinger functions.

2. *(Regularity) For some $1 \leq p \leq 2$, a constant c and all $J \in S_{\mathbb{R}}$, we have*
$$|Z_E[J]| \leq e^{c(\|J\|_1 + \|J\|_p^p)}. \tag{24}$$

   Gives temperedness of the Wightman field.

3. *(Euclidean invariance) $Z_E[J] = Z_E[J_{(R,a)}]$ for all $J \in S_{\mathbb{R}}$, where $J_{(R,a)}(x) = J(R^{-1}(x-a))$, $R \in SO(4)$, $a \in \mathbb{R}^4$.*

   Gives Poincaré covariance of the Wightman theory.

4. *(Reflection positivity) Define:*

   - $\theta(x^0, \vec{x}) = (-x^0, \vec{x})$ *the Euclidian time reflection.*
   - $J_\theta(x) := J(\theta^{-1}x) = J(\theta x)$ *for* $J \in S_{\mathbb{R}}$.
   - $\mathbb{R}^4_+ = \{(x^0, \vec{x}) \,|x^0 > 0\}$

   Reflection positivity requires that for functions $J_1, \ldots, J_N \in S_{\mathbb{R}}$, supported in $\mathbb{R}^4_+$, the matrix $M_{i,j} := Z_E[J_i - (J_j)_\theta]$ is positive.

   Gives positivity of the scalar product in the Hilbert space $\mathcal{H}$ (i.e. $\langle\Psi|\Psi\rangle \geq 0$ for all $\Psi \geq 0$). Also locality and spectrum condition.

5. *(Ergodicity) Define:*

   - $J_s(x) = J(x^0 - s, \vec{x})$ *for* $J \in S_{\mathbb{R}}$.
   - $(T(s)\varphi)(J) = \varphi(J_s)$ *for* $\varphi \in S'_{\mathbb{R}}$.

Ergodicity requires that for any function $A \in L^1(S'_\mathbb{R}, d\mu)$ and $\varphi_1 \in S'_\mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t A(T_s\varphi_1)ds = \int_{S'_\mathbb{R}} A(\varphi)d\mu(\varphi). \tag{25}$$

Gives the uniqueness of the vacuum.

Theorem 2.3 *Let μ be a measure on $S'_\mathbb{R}$ satisfying the Osterwalder-Schrader axioms. Then the moment functions*

$$G_{E,n}(x_1, \dots, x_n) = \int_{S'_\mathbb{R}} \varphi(x_1)\dots\varphi(x_n)d\mu(\varphi) \tag{26}$$

exist and are Schwinger functions of a Wightman QFT.

Remark 2.4 *The Gaussian measure $d\mu_C$ from the example above satisfies the Osterwalder-Schrader axioms and gives the (scalar, Hermitian) free field.*

Some ideas of the proof: The Hilbert space and the Hamiltonian of the Wightman theory is constructed as follows:

- Def: $\mathcal{E} := L^2(S'_\mathbb{R}, d\mu)$.
- Def: $A_J(\varphi) := e^{i\varphi(J)}$ for any $J \in S_\mathbb{R}$ and $(\theta A_J)(\varphi) := e^{i\varphi(J_\theta)}$.
- Fact: $\mathcal{E} = \overline{\text{Span}\{A_J \,|\, J \in S_\mathbb{R}\}}$
- Def: $\mathcal{E}_+ = \overline{\text{Span}\{A_J \,|\, J \in S(\mathbb{R}^4_+)_\mathbb{R}\}}$, where $S(\mathbb{R}^4_+)_\mathbb{R}$ are real Schwartz-class functions supported in $\mathbb{R}^4_+$.
- Fact: $\langle A_1|A_2\rangle := \int \overline{(\theta A_1)}(\varphi)A_2(\varphi)d\mu(\varphi)$ is a bilinear form on $\mathcal{E}_+$, which is positive (i.e. $\langle A|A\rangle \geq 0$) by reflection positivity. Due to the presence of θ it differs from the the scalar product in $\mathcal{E}$.
- Def: $\mathcal{N} = \{\, A \in \mathcal{E}_+ \,|\, \langle A|A\rangle = 0\}$ and set $\mathcal{H} = (\mathcal{E}_+/\mathcal{N})^{\text{cpl}}$, where cpl denotes completion. This $\mathcal{H}$ is the Hilbert space of the Wightman theory.
- $T(t) : \mathcal{E}_+ \to \mathcal{E}_+$ for $t \geq 0$. It gives rise to a semigroup $e^{-tP_0} : \mathcal{H} \to \mathcal{H}$ with a self-adjoint, positive generator P_0 - the Hamiltonian. Thus $e^{itP_0} : \mathcal{H} \to \mathcal{H}$ gives unitary time-evolution.

2.4 Interacting measure

Interacting measures are usually constructed by perturbing the Gaussian measure $d\mu_C$. Reflection positivity severely restricts possible perturbations. Essentially, one has to write:

$$d\mu_I(\varphi) = \frac{1}{N} e^{-\int L_{\text{E,I}}(\varphi(x))d^4x} d\mu_C(\varphi), \tag{27}$$

where N is the normalisation constant and $L_{\text{E,I}} : \mathbb{R} \to \mathbb{R}$ some function (the Euclidean interaction Lagrangian). For example $L_{\text{E,I}}(\varphi(x)) = \frac{\lambda}{4!}\varphi(x)^4$. But this leads to problems:

- φ is a distribution so $\varphi(x)^4$ in general does not makes sense.
  This *ultraviolet problem* can sometimes be solved by *renormalization*.

- Integral over whole spacetime ill-defined. (But enforced by the translation symmetry).

For ϕ^4 theory in two-dimensional spacetime these problems were overcome and $d\mu_I$ satisfying the Osterwalder-Schrader axioms was constructed. It was also shown that the resulting theory is interacting, i.e. has non-trivial S-matrix. In the next lecture we will discuss the S-matrix is in the Wightman setting.

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