Lectures on mathematical foundations of QFT

Wojciech Dybalski

October 27, 2018

Introduction

The central question of these mathematical lectures is the following:

- Is QFT logically consistent?

Although it may not seem so, this question is quite relevant for physics. For example, if QFT contained a contradiction and, say, the magnetic moment of the electron could be computed in two different ways giving two completely different results, which of them should be compared with experiments? It turns out that such a situation is not completely ruled out in QFT, since we don't have enough control over the convergence of the perturbative series. If we take first few terms of this series, we often obtain excellent agreement with experiments. But if we managed to compute all of them and sum them up, most likely the result would be infinity.

For this and other reasons, the problem of logical consistency of QFT fascinated generations of mathematical physicists. They managed to solve it only in toy models, but built impressive mathematical structures some of which I will try to explain in these lectures.

The strategy to study the logical consistency of QFT can be summarized as follows: Take QFT as presented in the physics part of this course. Take the whole mathematics with its various sub-disciplines. Now try to 'embed' QFT into mathematics, where the problem of logical consistency is under good control. The 'image' of this embedding will be some subset of mathematics which can be called Mathematical QFT. It intersects with many different sub-disciplines including algebra, analysis, group theory, measure theory and many others. It differs from the original QFT in several respects: First, some familiar concepts from the physical theory will not reappear on the mathematics side, as tractable mathematical counterparts are missing. Second, many concepts from mathematics will enter the game, some of them without direct physical meaning (e.g. different notions of continuity and convergence). Their role is to control logical consistency within mathematics.

It should also be said that the efforts to 'embed' QFT into mathematics triggered a lot of new mathematical developments. Thus advancing mathematics is another important source of motivation to study Mathematical QFT.

1 Wightman quantum field theory

The main references for this section are [1, Section VIII], [2, Section IX.8,X.7].

1.1 Relativistic Quantum Mechanics

We consider a quantum theory given by a Hilbert space $\mathcal{H}$ (a space with a scalar product $\langle\ |\ \rangle$, which is complete in the norm $\|\cdot\| = \sqrt{\langle\cdot|\cdot\rangle}$) and:

(a) Observables $\{O_i\}_{i\in\mathbb{I}}$. Hermitian / self-adjoint operators.

(b) Symmetry transformations $\{U_j\}_{j\in\mathbb{J}}$. Unitary/anti-unitary operators.

1.1.1 Observables

1. Consider an operator $O : D(O) \to \mathcal{H}$ i.e. a linear map from a dense domain $D(O) \subset \mathcal{H}$ to $\mathcal{H}$. $D(O) = \mathcal{H}$ only possible for bounded operators O (i.e. with bounded spectrum). In other words, $O \in \mathcal{L}(D(O), \mathcal{H})$, which is the space of linear maps between the two spaces.

2. $D(O^\dagger) := \{\, \Psi \in \mathcal{H}\,|\ |\langle\Psi|O\Psi'\rangle| \leq c_\Psi \|\Psi'\| \text{ for all } \Psi' \in D(O)\,\}$. Thereby, $O^\dagger\Psi$ is well defined for any $\Psi \in D(O^\dagger)$ via the Riesz theorem.

3. We say that O is Hermitian, if $D(O) \subset D(O^\dagger)$ and $O^\dagger\Psi = O\Psi$ for all $\Psi \in D(O)$.

4. We say that O is self-adjoint, if it is Hermitian and $D(O) = D(O^\dagger)$. Advantage: we can define e^{itO} and then also a large class of other functions via the Fourier transform. E.g. $f(O) = (2\pi)^{-1/2} \int dt\, e^{itO} \check{f}(t)$ for $f \in C_0^\infty(\mathbb{R})$ (smooth, compactly supported, complex-valued).

5. We say that operators O_1, O_2 weakly commute on some dense domain $D \subset D(O_1) \cap D(O_2) \cap D(O_1^\dagger) \cap D(O_2^\dagger)$ if for all $\Psi_1, \Psi_2 \in D$

$$
\begin{aligned}
0 = \langle\Psi_1|[O_1, O_2]\Psi_2\rangle &= \langle\Psi_1|O_1O_2\Psi_2\rangle - \langle\Psi_1|O_2O_1\Psi_2\rangle \\
&= \langle O_1^\dagger\Psi_1|O_2\Psi_2\rangle - \langle O_2^\dagger\Psi_1|O_1\Psi_2\rangle. \qquad (1)
\end{aligned}
$$

6. We say that two self-adjoint operators O_1, O_2 strongly commute, if

$$
[e^{it_1O_1}, e^{it_2O_2}] = 0 \text{ for all } t_1, t_2 \in \mathbb{R}. \qquad (2)
$$

   No domain problems here, since e^{itO} is always bounded, hence $D(e^{itO}) = \mathcal{H}$.

7. Let $O_1, \dots, O_n$ be a family of self-adjoint operators which mutually strongly commute. For any $f \in C_0^\infty(\mathbb{R}^n)$ we define

$$
f(O_1, \dots, O_n) = (2\pi)^{-n/2} \int dt_1 \dots dt_n\, e^{it_1O_1} \dots e^{it_nO_n} \check{f}(t_1, \dots, t_n). \qquad (3)
$$

Definition 1.1 *[4] The joint spectrum* $\mathrm{Sp}(O_1, \ldots, O_n)$ *of such a family of operators is defined as follows:* $p \in \mathrm{Sp}(O_1, \ldots, O_n)$ *if for any open neighbourhood* V_p *of this point there is a function* $f \in C_0^\infty(\mathbb{R}^n)$ *s.t. suppf* $\subset V_p$ *and* $f(O_1, \ldots, O_n) \neq 0$.

It is easy to see that for one operator O with purely point spectrum (e.g. the Hamiltonian of the harmonic oscillator) $\mathrm{Sp}(O)$ is the set of all the eigenvalues. But the above definition captures also the continuous spectrum without using 'generalized eigenvectors'.

1.1.2 Symmetry transformations

We treat today only symmetries implemented by unitaries.

1. A linear bijection $U : \mathcal{H} \to \mathcal{H}$ is called a unitary if $\langle U\Psi_1|U\Psi_2\rangle = \langle \Psi_1|\Psi_2\rangle$ for all $\Psi_1, \Psi_2 \in \mathcal{H}$. We denote by $\mathcal{U}(\mathcal{H})$ the group of all unitaries on $\mathcal{H}$. Unitaries are suitable to describe symmetries as they preserve transition amplitudes of physical processes.

2. The Minkowski metric is invariant under Poincaré transformations $x \mapsto \Lambda x + a$, where $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ (the proper ortochronous Poincaré group). We consider a unitary representation of this group on $\mathcal{H}$, i.e. a map $\mathcal{P}_+^\uparrow \ni (\Lambda, a) \mapsto U(\Lambda, a) \in \mathcal{U}(\mathcal{H})$ with the property
$$U(\Lambda_1, a_1)U(\Lambda_2, a_2) = U((\Lambda_1, a_1)(\Lambda_2, a_2)), \tag{4}$$
i.e. a group homomorphism.

3. We say that such a representation is continuous, if $(\Lambda, a) \mapsto \langle \Psi_1|U(\Lambda, a)\Psi_2\rangle \in \mathbb{C}$ is a continuous function for any $\Psi_1, \Psi_2 \in \mathcal{H}$.

1.1.3 Energy-momentum operators and the spectrum condition

The following fact is an immediate consequence of the Stone theorem and continuity is crucial here:

Theorem 1.2 *Given a continuous unitary representation of translations* $\mathbb{R}^4 \ni a \mapsto U(a) := U(I, a) \in \mathcal{U}(\mathcal{H})$*, there exist four strongly commuting self-adjoint operators* P_μ*,* $\mu = 0, 1, 2, 3$*, s.t.*
$$U(a) = e^{iP_\mu a^\mu}. \tag{5}$$
We call $P = \{P_0, P_1, P_2, P_3\}$ *the energy-momentum operators.*

In physical theories P_μ are unbounded operators (since values of the energy-momentum can be arbitrarily large), defined on some domains $D(P_\mu) \subset \mathcal{H}$. However, to guarantee stability of physical systems, the energy should be bounded from below in any inertial system. The mathematical formulation is the spectrum condition:

Definition 1.3 *We say that a Poincaré covariant quantum theory satisfies the spectrum condition if*

$$\mathrm{Sp}\, P := \mathrm{Sp}(P_0, P_1, P_2, P_3) \subset \overline{V}_+, \tag{6}$$

where $\overline{V}_+ := \{ (p_0, \vec{p}) \in \mathbb{R}^4 \,|\, p_0 \geq |\vec{p}| \}$ *is the closed future lightcone.*

1.1.4 Vacuum state

1. A unit vector $\Omega \in \mathcal{H}$ is called the vacuum state if $U(\Lambda, a)\Omega = \Omega$ for all $(\Lambda, a) \in \mathbb{R}^4$. This implies $P_\mu \Omega = 0$ for $\mu = 0, 1, 2, 3$.

2. By the spectrum condition, Ω is the ground state of the theory.

3. We say that the vacuum is unique, if Ω is the only such vector in $\mathcal{H}$ up to multiplication by a phase.

1.1.5 Relativistic Quantum Mechanics: Summary

Definition 1.4 *A relativistic quantum mechanics is given by*

1. *Hilbert space* $\mathcal{H}$.

2. *A continuous unitary representation* $\mathcal{P}_+^\uparrow \ni (\Lambda, a) \mapsto U(\Lambda, a) \in \mathcal{U}(\mathcal{H})$ *satisfying the spectrum condition.*

3. *Observables* $\{O_i\}_{i\in\mathbb{I}}$, *including* P_μ.

Furthermore, $\mathcal{H}$ *may contain a vacuum vector* Ω *(unique or not).*

So far in our collection of observables $\{O_i\}_{i\in\mathbb{I}}$ we have identified only global quantities like P_μ. (For example, to measure P_0, we would have to add up the energies of all the particles in the universe of our theory). But actual measurements are performed locally, i.e. in bounded regions of spacetime and we would like to include the corresponding observables. We have to do it in a way which is consistent with Poincaré symmetry, spectrum condition and locality (Einstein causality). This is the role of quantum fields.

1.2 Quantum fields as operator-valued distributions

1.2.1 Tempered distributions

We recall some definitions:

1. The Schwartz-class functions:

$$S = \{ f \in C^\infty(\mathbb{R}^4) \,|\, \sup_{x\in\mathbb{R}^4} |x^\alpha \partial^\beta f(x)| < \infty, \quad \alpha, \beta \in \mathbb{N}_0^4 \}, \tag{7}$$

where $x^\alpha := x_0^{\alpha_0} \dots x_3^{\alpha_3}$ and $\partial^\beta = \frac{\partial^{|\beta|}}{(\partial x^0)^{\beta_0} \dots (\partial x^3)^{\beta_3}}$, $|\beta| = \beta_0 + \dots + \beta_3$.

2. The semi-norms $\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^4} |x^\alpha \partial^\beta f(x)|$ give a notion of convergence in S: $f_n \to f$ in S if $\|f_n - f\|_{\alpha,\beta} \to 0$ for all α, β.

3. We say that a linear functional $\varphi : S \to \mathbb{C}$ is continuous, if for any finite set F of multiindices there is a constant c_F s.t.

$$|\varphi(f)| \leq c_F \sum_{\alpha,\beta\in F} \|f\|_{\alpha,\beta}. \tag{8}$$

(Note that if $f_n \to f$ in S then $\varphi(f_n) \to \varphi(f)$). Such continuous functionals are called tempered distributions and form the space S' which is the topological dual of S.

Any measurable, polynomially growing function $x \mapsto \varphi(x)$ defines a tempered distribution via

$$\varphi(f) = \int d^4x\, \varphi(x) f(x). \tag{9}$$

The notation (9) is often used also if there is no underlying function, e.g. $\delta(f) =: \int \delta(x) f(x) d^4x = f(0)$.

Definition 1.5 *We consider:*

1. *A map $S \ni f \mapsto \phi(f) \in \mathcal{L}(D(\phi(f)), \mathcal{H})$.*
2. *A dense domain $D \subset \mathcal{H}$ s.t. for all $f \in S$*
   - *$D \subset D(\phi(f)) \cap D(\phi(f)^\dagger)$,*
   - *$\phi(f) : D \to D$,*
   - *$\phi(f)^\dagger : D \to D$.*

We say that (ϕ, D) is an operator valued distribution if for all $\Psi_1, \Psi_2 \in D$ the map

$$S \ni f \mapsto \langle \Psi_1 | \phi(f) \Psi_2 \rangle \in \mathbb{C} \tag{10}$$

is a tempered distribution. We say that (ϕ, D) is Hermitian, if $\phi(f)$ is a Hermitian operator for any real valued *$f \in S$.*

Note that a posteriori $S \ni f \mapsto \phi(f) \in \mathcal{L}(D, D)$.

1.2.2 Wightman QFT

Definition 1.6 *A theory of one scalar Hermitian Wightman field is given by:*

1. *A relativistic quantum theory $(\mathcal{H}, U)$ with a unique vacuum state $\Omega \in \mathcal{H}$.*
2. *A Hermitian operator-valued distribution (ϕ, D) s.t. $\Omega \in D$ and $U(\Lambda, a)D \subset D$ for all $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ satisfying:*

(a) *(Locality)* $[\phi(f_1), \phi(f_2)] = 0$ *if* $\mathrm{supp}\, f_1$ *and* $\mathrm{supp}\, f_2$ *spacelike separated. (In the sense of weak commutativity on* D*).*

(b) *(Covariance)* $U(\Lambda, a)\phi(f)U(\Lambda, a)^\dagger = \phi(f_{(\Lambda,a)})$, *for all* $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ *and* $f \in S$. *Here* $f_{(\Lambda,a)}(x) = f(\Lambda^{-1}(x - a))$.

(c) *(Cyclicity of the vacuum)* $\mathcal{D} = \mathrm{Span}\{\, \phi(f_1) \ldots \phi(f_m)\Omega \,|\, f_1, \ldots f_m \in S, m \in \mathbb{N}_0 \,\}$ *is a dense subspace of* $\mathcal{H}$.

The distribution (ϕ, D) *is called the Wightman quantum field.*

Remarks:

1. Operator valued *functions* satisfying the Wightman axioms do not exist (we really need distributions). The physical reason is the uncertainty relation: Measuring ϕ strictly at a point x causes very large fluctuations of energy and momentum, which prevent $\phi(x)$ from being a well defined operator. Such observations were made already in [7], before the theory of distributions was developed.

2. It is possible to choose $D = \mathcal{D}$.

Example: Let $\mathcal{H}$ be the symmetric Fock space, then the energy-momentum operators

$$P^0 = \int \frac{d^3p}{(2\pi)^3 2p^0} p^0 a^\dagger(p)a(p), \quad \vec{P} = \int \frac{d^3p}{(2\pi)^3 2p^0} \vec{p}\, a^\dagger(p)a(p), \tag{11}$$

where $p_0 = \sqrt{p^2 + m^2}$, satisfy the spectrum condition and generate a unitary representation of translations $U(a) = e^{iP_\mu a^\mu}$. Clearly, $\Omega = |0\rangle$ is the unique vacuum state of this relativistic QM. The Hermitian operator-valued distribution, given in the function notation by

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} (e^{ipx} a^\dagger(p) + e^{-ipx} a(p)). \tag{12}$$

is a scalar Hermitian Wightman field.