BASICS ON LIE GROUPS

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Definition of group: a set G equipped with an operation $\bullet$ that combines any two elements a, b to form another element $a \bullet b$. $(G, \bullet)$ must satisfy:

1) closure $\quad \forall a, b \in G, \; a \bullet b \in G$
2) associativity $\quad \forall a, b, c \in G, \; (a \bullet b) \bullet c = a \bullet (b \bullet c)$
3) identity $\quad$ there exists an element $e \in G$: $\forall a \in G, \; e \bullet a = a \bullet e = a$
4) inverse $\quad \forall a \in G$ there exists $b \in G$ such that $a \bullet b = b \bullet a = e$ (b is denoted a^{-1})

BOOKS: e.g.
H. Georgi
"Lie Algebras in Particle Physics"
Frontiers in Physics

A Lie group is a group with ∞ number of elements that is also a differentiable manifold. Any group element continuously connected with $\mathbb{1}$ can be written as

$$U = e^{i \alpha^a T^a} \mathbb{1}$$

where α^a are numbers and T^a are the group generators.
If we know the explicit form of the group elements U, we can deduce the form of the T^a by expanding around $\mathbb{1}$.
For example: orthochronous Lorentz, $SO(1,3)$

boost along x^1 axis is

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \begin{cases} t' = \gamma(t + \beta x) \\ x' = \gamma(\beta t + \gamma x) \\ y' = y \\ z' = z \end{cases}$$

expand for small β

at $O(\beta)$,

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} 1 & \beta & & \\ \beta & 1 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) = \delta^\mu{}_\nu + \beta \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \end{array}\right)$$

now

$$U \simeq 1 + i\alpha^a T^a \qquad \text{so} \qquad \underline{\alpha^a \to \beta}$$

and $\quad i T^a \to \omega^\mu{}_\nu = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \end{array}\right)$ $\quad$ generator of boosts along x^1

The generators T^a form a Lie algebra, defined through the commutation relations

$$[T^a, T^b] = i \underbrace{f^{abc}}_{\uparrow \text{structure constants}} T^c$$

the group is
- Abelian if $f^{abc} = 0$
- Non-Abelian, otherwise

e.g. $su(2)$ has $f^{abc} = \underset{\substack{\uparrow \\ \text{totally antisym} \\ \text{with } \varepsilon^{123}=1}}{\varepsilon^{abc}}$

note that $[A,B] = AB - BA$

$$[A,[B,C]] = [A, BC - CB]$$

$$= ABC - ACB - BCA + CBA \quad \text{and so}$$

$$\circledast \quad [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \quad \text{Jacobi identity}$$

Now in terms of structure constants, $\begin{cases} A \to a \\ B \to b \\ C \to c \end{cases}$

$$[T^a, [T^b, T^c]] = [T^a, i f^{bcd} T^d]$$

$$= i f^{bcd} [T^a, T^d] = i f^{bcd} \, i f^{ade} T^e$$

$$= - f^{bcd} f^{ade} T^e$$

then $\circledast$ is

$$(- f^{bcd} f^{ade} - f^{cad} f^{bde} - f^{abd} f^{cde}) T^e = 0$$

$$\underline{f^{bcd} f^{ade} + f^{cad} f^{bde} + f^{abd} f^{cde} = 0}$$

Jacobi id. for structure constants

The comm. relations completely determine structure of the group sufficiently close to $\mathbb{1}$; if we go far away then global aspects matter (e.g. $SU(2)$ and $O(3)$, which have same comm. relations but differ globally).

But this is NOT relevant for introductory description of non-Abelian gauge theories.

Note also that f^{abc} are antisym in first 2 indices:

$$[T^a, T^b] = i f^{abc} T^c$$

$$\parallel$$

$$-[T^b, T^a] = -i f^{bac} T^c \quad \Rightarrow \quad f^{abc} = - f^{bac}.$$

An IDEAL is a sub-algebra $I \subset \mathfrak{g}$ such that

$$[g, n] \in I \quad \text{for any} \begin{cases} g \in \mathfrak{g} \\ n \in I \end{cases}$$ (invariant subalgebra)

$\{0\}$ and the whole $\mathfrak{g}$ are trivial ideals; an algebra that does not admit a non-trivial ideal is a SIMPLE ALGEBRA

e.g. $su(N)$
$so(N)$

A semi-simple algebra is the direct sum of simple algebras

e.g. SM $\quad su(3) \oplus su(2) \oplus u(1)$

<u>THEOREM</u>: all finite-dimensional reprs. of <u>semisimple algebras</u> are <u>Hermitian</u>

↘ can construct theories where the symm is a <u>unitary</u> transform on fields

[unitary theories conserve probabilities]

We are also interested in the case where the number of generators is FINITE

→ <u>compact</u> algebras (because the corresponding Lie group is a finite-dim compact manifold)

The requirement of being SIMPLE and COMPACT is very stringent: <u>classification</u>

- UNITARY GROUPS $U(N)$

defining repr. acts on space of N-dim complex vectors

$$U = e^{i\alpha T} \quad \text{with } T \text{ Hermitian, } T^\dagger = T$$

then
$$U^\dagger U = \left(e^{i\alpha T}\right)^\dagger \left(e^{i\alpha T}\right) = e^{-i\alpha T^\dagger} e^{i\alpha T}$$
$$= e^{-i\alpha T + i\alpha T} = \mathbb{1}$$

then a complex inner product is preserved: take ψ, χ states then

$$[U\psi]^\dagger U\chi = \psi^\dagger \underbrace{U^\dagger U}_{\mathbb{1}} \chi = \psi^\dagger \chi$$

now $U(N)$ contains the pure phase transformations

$U = e^{i\alpha}$, which form a $U(1)$ subgroup

these are removed by requiring $\det U = 1$ (and not a complex phase with $|\cdot| = 1$)

which gives the simple group $SU(N)$

whose dimension is $d(SU(N)) = N^2 - 1$

because
count real params:

$N \times N$ complex matrix	$2N^2$
$U^\dagger U = \mathbb{1}$	N^2 conditions
$\det U = 1$	1 condition

$$\Rightarrow \quad 2N^2 - N^2 - 1 = N^2 - 1 .$$

- ORTHOGONAL GROUPS preserve a real inner product

ψ, χ vectors $\qquad O\psi \cdot O\chi = \psi^T \underbrace{O^T O}_{\mathbb{1}} \chi = \psi^T \chi$

$$O^T O = \mathbb{1}$$

how many generators? $\quad$ real $N \times N$ matrix satisfying $\underbrace{O^T O = \mathbb{1}}_{}$ symm, so $N + \frac{N^2 - N}{2}$ independent conditions

$$d(O(N)) = N^2 - \left[N + \frac{N^2 - N}{2}\right]$$

$$= \frac{N(N-1)}{2}$$

now $\det O = \pm 1 \quad \rightarrow$ take $+1$ to get $SO(N)$ rotation group in N dimensions

then clear that $d(SO(N)) = \frac{N(N-1)}{2}$: number of planes in N dim

- SYMPLECTIC GROUPS $\quad Sp(N) \quad N$ even

preserve an antisymm inner product

$$\psi^T \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}}_{} \chi$$

$\equiv J$ (in $\frac{N}{2} \times \frac{N}{2}$ block form)

$$S^T J S = J \qquad d(Sp(N)) = N^2 - \left[\frac{N^2 - N}{2}\right]$$

$$= \frac{N(N+1)}{2}$$

number of conditions from $S^T J S = J$ antisymm since $(S^T J S)^T = S^T J^T S = -S^T J S$

- EXCEPTIONAL GROUPS (5 of them): G_2, F_4, E_6, E_7, E_8

$\underbrace{G_2}$ ↑ RANK of the group = dimension of Cartan subgroup, which is the max commuting subgroup; equivalently, # of diag generators

[SU(N) has rank $N-1$]

| This completes classification of compact simple Lie algebras.

Note also that if there is any Hermitian repr. (as for semisimple algebras) then the structure constants are REAL:

$$[T^a, T^b] = i f^{abc} T^c \quad \Rightarrow \quad [T^a,T^b]^\dagger = -i f^{abc*} T^c$$

$$\begin{aligned} [T^a,T^b]^\dagger &= (T^a T^b - T^b T^a)^\dagger \\ &= [T^b, T^a] = i f^{bac} T^c \\ &= -i f^{abc} T^c \end{aligned}$$

f^{abc} is antisym in first 2 indices

so

$$\underline{f^{abc} = f^{abc*}} .$$

GENERAL PROPERTIES OF REPRS

Reprs. of algebra are constructed by embedding generators into matrices. One can show that for compact semi-simple Lie algebras, for any repr. R we can choose a basis for the generators such that

$$\mathrm{Tr}\,(T_R^a T_R^b) = \underbrace{T(R)}_{\text{constant for each } R} \delta^{ab}$$

now combined with $[T_R^a, T_R^b] = i f^{abc} T_R^c$

this implies

$$[T_R^a, T_R^b]\, T_R^c = i f^{abd} T_R^d T_R^c \qquad \text{take trace}$$

$$\begin{aligned} \mathrm{Tr}\left[[T_R^a, T_R^b]\, T_R^c\right] &= i f^{abd}\, \mathrm{Tr}\left[T_R^d T_R^c\right] \\ &= i f^{abd}\, T(R)\, \delta^{dc} = i\, T(R)\, f^{abc} \end{aligned}$$

$$\Rightarrow \quad f^{abc} = -\frac{i}{T(R)}\, \mathrm{Tr}\left[[T_R^a, T_R^b]\, T_R^c\right]$$

Now this implies that $f^{abc} = f^{bca}$ because

$$\mathrm{Tr}\left[[T_R^a, T_R^b] T_R^c\right] = \mathrm{Tr}\left[T_R^a T_R^b T_R^c - T_R^b T_R^a T_R^c\right]$$

cyclicity of trace

$$\stackrel{?}{=} \mathrm{Tr}\left[T_R^b T_R^c T_R^a - T_R^c T_R^b T_R^a\right]$$

$$= \mathrm{Tr}\left[[T_R^b, T_R^c] T_R^a\right].$$

Now recall that $f^{abc} = -f^{bac}$ => combined, they tell us that f^{abc} are completely antisymm.

Now for a given R, infinitesimal transform under the group is

$$\phi \to (1 + i\alpha^a T_R^a)\, \phi$$ hence complex conj

$$\phi^* \to (1 - i\alpha^a T_R^{a*})\, \phi^*$$ but also we can

define the CONJUGATE REPR. $\phi^* \to (1 + i\alpha^a T_{\bar{R}}^a)\, \phi^*$

hence

$$T_{\bar{R}}^a = -(T_R^a)^* = -(T_R^a)^T.$$

Now $\bar{R}$ is equivalent to R, if there exists a unitary transform U such that

$$T_{\bar{R}}^a = U T_R^a U^\dagger.$$

In this case, R is a REAL repr.

Then we can always find a matrix E_{ab} such that if $\varphi, \chi \in R$ then $\varphi_a E_{ab} \chi_b$ is invariant

If E_{ab} → SYM STRICTLY REAL

↘ ANTISYM PSEUDOREAL

REPRS. OF SU(N) GROUPS

Free theory of N complex massless fields is automatically invariant under $U(N) \simeq SU(N) \times U(1)$ → hence the importance of SU(N) groups

The two most important representations are: FUNDAMENTAL and ADJOINT.

- FUNDAMENTAL (or DEFINING): acts on space of N-dim complex vectors. Smallest non-trivial repr. of the algebra. Dimension is N. For SU(N), set of $N \times N$ Hermitian, <u>traceless</u> matrices.

[Why traceless? Consider

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{T}} \quad \text{and diagonalize} \quad V\, \vec{\alpha}\cdot\vec{T}\, V^{-1} = \underset{\text{diagonal}}{D}$$

(Hermitian matrices are diagonalizable)

then

$$\det U \underset{\substack{\uparrow \\ \text{det is basis-indep}}}{=} \det e^{iD} = \prod_j e^{i[D]_{jj}} = e^{i\sum_j [D]_{jj}} = e^{i\,\mathrm{Tr}\,D} \underset{\text{Tr is basis-indep}}{=} e^{i\,\mathrm{Tr}(\vec{\alpha}\cdot\vec{T})}$$

hence if $\mathrm{Tr}\, T^a = 0$

$\Rightarrow \det U = 1$. Note that traceless argument applies for <u>any</u> semisimple algebra.]

- For $SU(2)$, $T^a = \dfrac{\sigma^a}{2}$ $\quad \sigma^a$ are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

they satisfy

$$[T^a, T^b] = \frac{1}{4}[\sigma^a, \sigma^b] = \frac{1}{4} 2i\, \varepsilon^{abc} \sigma^c = i \underbrace{\varepsilon^{abc}}_{\text{structure constants}} T^c$$

- For $SU(3)$, $T^a = \dfrac{\lambda^a}{2}$ $\quad \lambda^a$ are Gell-Mann matrices

Now for the 2 of SU(2) we have

$$T_R^a = \frac{\sigma^a}{2} \qquad T_{\bar{R}}^a = -\frac{\sigma^{a*}}{2}$$ but we know that

$$\sigma^2 \sigma^a \sigma^2 = -\sigma^{a*}$$ hence

$$T_{\bar{R}}^a = -\frac{\sigma^{a*}}{2} = \sigma^2 \left(\frac{\sigma^a}{2}\right) \sigma^2 = \sigma^2 T_R^a \sigma^2$$

i.e. REAL repr. with $U = \sigma^2$

also, $E_{ab} = (i\sigma^2)_{ab} = \varepsilon_{ab}$ [EXERCISE]

so it is <u>pseudoreal</u>

For $N>2$, the N of SU(N) is instead complex
(so in part. $3 \neq \bar{3}$ for SU(3))

- ADJOINT: acts on the space spanned by the generators themselves

$$(T_{adj}^a)_{bc} = q f^{abc}$$

(q purely Im)

dimension is N^2-1
(number of generators)

e.g. for SU(2):

$$(T_{adj}^a)_{bc} = q \varepsilon^{abc} \quad \rightarrow T_{adj}^1 = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ etc.}$$

EXERCISE: find q such that T_{adj}^a satisfy comm. relations of algebra.

Adjoint repr. is important because it is the repr. where non-Abelian gauge fields transform.

Note that the conjugate repr of the adjoint is

$$T_{\bar{adj}}^a = -(T_{adj}^a)^* = -(q f^{abc})^* \underset{\substack{\uparrow \\ q \in Im \\ f^{abc} \text{ real}}}{=} +q f^{abc} = T_{adj}^a$$

hence the adjoint is always <u>real</u>

$(U = \mathbb{1})$.

CASIMIR OPERATOR

For any repr. T_R^a, we know that $T^2 \equiv T_R^a T_R^a$ (sum over a) commutes with all the generators:

$$[T^2, T_R^b] = [T_R^a T_R^a, T_R^b] \qquad [AB, C] = A[B,C] + [A,C]B$$

$$= T_R^a [T_R^a, T_R^b] + [T_R^a, T_R^b] T_R^a$$

$$= T_R^a \, i f^{abc} T_R^c + i f^{abc} T_R^c T_R^a$$

$$= \underbrace{i f^{abc}}_{\text{antisym}} \underbrace{\{T_R^a, T_R^c\}}_{\text{sym}} = 0 .$$

Hence T^2 must be proportional to the identity, by Schur's lemma. So

$$T_R^a T_R^a = C_2(R)\, \mathbb{1} \qquad (\ast)$$

$\uparrow$ QUADRATIC CASIMIR of R

[Familiar example: $SU(2)$, $J^2 = T_R^a T_R^a$ is Casimir with eigenvalues $j(j+1)$ (total spin)]

As we discussed we also have for an appropriate choice of basis

$$\mathrm{Tr}\,(T_R^a T_R^b) = \underbrace{T(R)}\, \delta^{ab}$$

$\leftarrow$ index of R (sometimes called $C(R)$)

now contract this with δ^{ab}

$$\mathrm{Tr}\,(T_R^a T_R^a) = T(R)\, d(G)$$

From $(\ast)$ $\quad \parallel$

$$C_2(R)\, d(R) \qquad \Rightarrow \qquad d(R)\, C_2(R) = T(R)\, d(G)$$

which is useful to compute $C_2(R)$

- FUNDAMENTAL of $SU(N)$,

typical to choose $\quad T(\text{fund}) = \frac{1}{2} \equiv T_F$ as in the case of $SU(2)$

then $\quad C_2(\text{fund}) = \dfrac{T(\text{fund})\, d(G)}{d(\text{fund})} = \dfrac{1}{2}\dfrac{N^2-1}{N} \equiv \underline{C_F}$

$\uparrow$ often used

- ADJOINT of SU(N)

first note that $d(R) = d(G)$, so $C_2(\mathrm{adj}) = T(\mathrm{adj})$.

How do we compute $C_2(\mathrm{adj})$? Instructive to do it by building adjoint as product of fund and antifund: given 2 reprs. r_1 and r_2, their <u>direct product</u> is a repr of dim. $d(r_1)\cdot d(r_2)$. Objects transforming under it are <u>tensors</u> ξ_{pq}, $p \in r_1$, $q \in r_2$. The product repr. can in general be decomposed as direct sum of irreprs: $r_1 \times r_2 = \sum_i r_i$ (✱)

The matrices are $$t^a_{r_1\times r_2} = t^a_{r_1} \otimes 1\!\!1 + 1\!\!1 \otimes t^a_{r_2}$$

Now the quadr. Casimir of the product is

$$t^a_{r_1\times r_2}\, t^a_{r_1\times r_2} = (t^a_{r_1})^2 \otimes 1\!\!1 + 1\!\!1 \otimes (t^a_{r_2})^2 + 2\, t^a_{r_1} \otimes t^a_{r_2}$$

take the trace

$$\mathrm{Tr}\,(t^{a\,2}_{r_1\times r_2}) = \underbrace{\mathrm{Tr}\,(t^{a\,2}_{r_1})}\, d(r_2) + d(r_1)\, \underbrace{\mathrm{Tr}\,(t^{a\,2}_{r_2})} + \underset{\substack{\uparrow \\ t^a_{r_i} \text{ are} \\ \text{traceless}}}{0}$$

$$= \overbrace{C_2(r_1)\, d(r_1)}\, d(r_2) + d(r_1)\, \overbrace{C_2(r_2)\, d(r_2)}$$

$$= [C_2(r_1) + C_2(r_2)]\, d(r_1) d(r_2) \qquad (A)$$

But also, from (✱) we get

$$\mathrm{Tr}\,(t^{a\,2}_{r_1\times r_2}) \underset{\substack{\uparrow \\ \text{block-diag form}}}{=} \sum_i \mathrm{Tr}\,(t^2_{r_i}) = \sum_i C_2(r_i)\, d(r_i) \qquad (B)$$

Now apply to case of $r_1 = N$, $r_2 = \bar{N}$ in SU(N):

$$N \times \bar{N} = \underset{\substack{\uparrow \\ \xi_{pq} = \delta_{pq}}}{1} + (N^2-1)$$

(N^2-1) ← remaining pieces: $N\times N$ $\underset{\text{tensor}}{\text{traceless}}$ → ADJOINT repr.

Then from (A) = (B) get

$$[C_2(N) + C_2(\bar{N})]\, N^2 = C_2(\mathrm{adj})\, d(\mathrm{adj})$$ ← for the singlet rep. we have $T^a_1 = 0$ so $C_2(1) = 0$

see above

$$2\, \widehat{C_2(N)}\, N^2 = C_2(\mathrm{adj})\,(N^2-1)$$

$$\Rightarrow C_2(\mathrm{adj}) = \cancel{2}N^{\cancel{2}}\, \frac{\widehat{\cancel{N^2-1}}}{\cancel{2N}}\, \frac{1}{\cancel{N^2-1}} = N \equiv C_A$$ Hence $C_2(\mathrm{adj}) = C_A = N$, $T(\mathrm{adj}) = T_A = N$.

So we have seen for the <u>fundamental</u> of SU(N) that

$$\mathrm{Tr}\,(t^a t^b) = T_F\,\delta^{ab} \qquad T_F = \frac{1}{2}$$

$$\sum_a (t^a t^a)_{ij} = C_F\,\delta_{ij} \qquad C_F = \frac{N^2-1}{2N}$$

Another useful relation is

$$\underline{\sum_a (t^a)_{ij}\,(t^a)_{kl} = T_F\left(\delta_{il}\,\delta_{jk} - \frac{1}{N}\,\delta_{ij}\,\delta_{kl}\right)}$$

"completeness relation"

To prove it, begin with the fact that for any NxN complex matrix M we can write

$$(*) \qquad M = \underset{\substack{\uparrow\\ \text{complex}}}{M_0}\,\mathbb{1} + \underset{\substack{\uparrow\\ \text{coeffs}}}{M^a}\,\overset{\text{traceless}}{t^a}$$

then

$$\mathrm{Tr}\,(M) = M_0 N$$

$$\mathrm{Tr}\,(M t^b) = \mathrm{Tr}\,\left((M_0\mathbb{1} + M^a t^a)\,t^b\right) \overset{t^b \text{ is traceless}}{=} M^a\,\mathrm{Tr}\,\underbrace{(t^a t^b)}_{} = T_F\,M^b$$

($\mathrm{Tr}\,(t^a t^b) = T_F\delta^{ab}$)

$$\Rightarrow \begin{cases} M_0 = \dfrac{1}{N}\,\mathrm{Tr}\,(M) \\[2mm] M^b = \dfrac{1}{T_F}\,\mathrm{Tr}\,(M t^b) \end{cases}$$

now plug back here (*)

$$M = \frac{1}{N}\,\mathrm{Tr}\,(M)\,\mathbb{1} + \frac{1}{T_F}\,\mathrm{Tr}\,(M t^a)\,t^a$$

$$M_{ij} = \frac{1}{N}\,M_{kk}\,\delta_{ij} + \frac{1}{T_F}\,M_{lk}\,(t^a)_{kl}\,(t^a)_{ij}$$

$$\downarrow$$

$$\delta_{il}\,\delta_{jk}\,M_{lk} = \frac{1}{N}\,M_{lk}\,\delta_{lk}\,\delta_{ij} + \frac{1}{T_F}\,M_{lk}\,(t^a)_{kl}\,(t^a)_{ij}$$

now since M is an arbitrary matrix, the coeff. of M_{lk} must vanish

$$\Rightarrow \delta_{il}\,\delta_{jk} = \frac{1}{N}\,\delta_{ij}\,\delta_{kl} + \frac{1}{T_F}\,(t^a)_{ij}\,(t^a)_{kl}$$

which is what we wanted.

Another useful result is

$$\underline{\sum_{a,b} f^{abc}\,f^{abd} = C_A\,\delta^{cd}}$$

which is just the def. of the quadr. Casimir for the adjoint:

$$(T^a T^a)_{cd} = C_2(\text{adj})\,\delta_{cd}$$

but now we saw that $(T^a)_{bc} = q\, f^{abc}$ $(q \in \mathrm{Im})$, then

$$= (T^a)_{cb}(T^a)_{bd} = q f^{acb} q f^{abd} = -f^{acb} f^{abd} = +f^{abc} f^{abd}$$ as desired.

These results will be useful for calculations in non-abelian gauge theories.