BASICS ON LIE GROUPS

Definition of group: a set G equipped with an operation $\circ$ that combines any two elements a, b to form another element $a \cdot b$.
$(G, \circ)$ must satisfy:

1) closure $\forall a, b \in G$, $a \cdot b \in G$
2) associativity $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3) identity there exists an element $e \in G$: $\forall a \in G$, $e \cdot a = a \cdot e = a$
4) inverse $\forall a \in G$ there exists $b \in G$ such that $a \cdot b = b \cdot a = e$ (b is denoted a^{-1})

Books: e.g.
H. Georgi
"Lie Algebras in Particle Physics"
Frontiers in Physics

A Lie group is a group with ∞ number of elements that is also a differentiable manifold. Any group element continuously connected with $\mathbb{1}$ can be written as

$$U = e^{i \alpha^a T^a} \mathbb{1}$$

where α^a are numbers and T^a are the group generators.
If we know the explicit form of the group elements U, we can deduce the form of the T^a by expanding around $\mathbb{1}$.
For example: orthochronous Lorentz, $SO(1,3)$

boost along x^1 axis is

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\begin{cases} t' = \gamma(t + \beta x) \\ x' = \gamma(\beta t + \gamma x) \\ y' = y \\ z' = z \end{cases}$$

expand for small β

at $O(\beta)$,

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} 1 & \beta & & \\ \beta & 1 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) = \delta^\mu{}_\nu + \beta \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & & \\ & & & \end{array}\right)$$

now

$$U \simeq 1 + i\alpha^a T^a \qquad \text{so} \qquad \underline{\alpha^a \to \beta}$$

and

$$iT^a \to \omega^\mu{}_\nu = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & & \\ & & & \end{array}\right)$$

generator of boosts along x^1

The generators T^a form a Lie algebra, defined through the commutation relations

$$[T^a, T^b] = i \underbrace{f^{abc}}_{\uparrow \text{structure constants}} T^c$$

the group is

- Abelian if $f^{abc} = 0$
- Non-Abelian otherwise

e.g. $su(2)$ has $f^{abc} = \underset{\uparrow}{\varepsilon^{abc}}$ totally antisym with $\varepsilon^{123} = 1$

note that $[A,B] = AB - BA$

$$[A,[B,C]] = [A, BC - CB]$$

$$= ABC - ACB - BCA + CBA \quad \text{and so}$$

(*) $$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \qquad \text{Jacobi identity}$$

Now in terms of structure constants, $\left\{ \begin{array}{l} A \to a \\ B \to b \\ C \to c \end{array} \right.$

$$[T^a,[T^b,T^c]] = [T^a, i f^{bcd} T^d]$$

$$= i f^{bcd} [T^a, T^d] = i f^{bcd}\, i f^{ade} T^e$$

$$= - f^{bcd} f^{ade} T^e$$

then (*) is

$$(-f^{bcd} f^{ade} - f^{cad} f^{bde} - f^{abd} f^{cde}) T^e = 0$$

$$\underline{f^{bcd} f^{ade} + f^{cad} f^{bde} + f^{abd} f^{cde} = 0}$$

Jacobi id. for structure constants

The comm. relations completely determine structure of the group sufficiently close to $\mathbb{1}$; if we go far away then global aspects matter (e.g. $SU(2)$ and $O(3)$, which have same comm. relations but differ globally).

But this is NOT relevant for introductory description of non-Abelian gauge theories.

Note also that f^{abc} are antisym in first 2 indices:

$$[T^a, T^b] = i f^{abc} T^c$$

$$\quad \|$$

$$-[T^b, T^a] = -i f^{bac} T^c \quad \Rightarrow \quad f^{abc} = - f^{bac}.$$

An IDEAL is a sub-algebra $I \subset \mathcal{G}$ such that

$$[g, i] \in I \quad \text{for any} \begin{cases} g \in \mathcal{G} \\ i \in I \end{cases}$$ (invariant subalgebra)

0 and the whole $\mathcal{G}$ are trivial ideals; an algebra that does not admit a non-trivial ideal is a SIMPLE ALGEBRA

e.g. $su(N)$
$so(N)$

A semi-simple algebra is the direct sum of simple algebras

e.g. SM $su(3) \oplus su(2) \oplus u(1)$

THEOREM: all finite-dimensional reprs. of semisimple algebras are Hermitian

↘ can construct theories where the symm is a unitary transform on fields

[unitary theories conserve probabilities]

We are also interested in the case where the number of generators is FINITE

→ compact algebras (because the corresponding Lie group is a finite-dim compact manifold)

The requirement of being SIMPLE and COMPACT is very stringent: classification

- UNITARY GROUPS $U(N)$

defining repr. acts on space of N-dim complex vectors

$$U = e^{i\alpha T} \quad \text{with } T \text{ Hermitian} \quad T^{\dagger} = T$$

then

$$U^{\dagger} U = \left(e^{i\alpha T}\right)^{\dagger} \left(e^{i\alpha T}\right) = e^{-i\alpha T^{\dagger}} e^{i\alpha T}$$

$$= e^{-i\alpha T + i\alpha T} = \mathbb{1}$$

then a complex inner product is preserved: take ψ, χ states then

$$[U\psi]^{\dagger} U\chi = \psi^{\dagger} \underbrace{U^{\dagger} U}_{\mathbb{1}} \chi = \psi^{\dagger} \chi$$

now $U(N)$ contains the pure phase transformations

$$U = e^{i\alpha} \quad \text{, which form a } U(1) \text{ subgroup}$$

these are removed by requiring $\det U = 1$ (and not a complex phase with $|\cdot| = 1$)

which gives the simple group $SU(N)$

whose dimension is $d(SU(N)) = N^2 - 1$

because
count real params:

$N \times N$ complex matrix	$2N^2$
$U^\dagger U = \mathbb{1}$	N^2 conditions
$\det U = 1$	1 condition

$$\Rightarrow \quad 2N^2 - N^2 - 1 = N^2 - 1 .$$

- ORTHOGONAL GROUPS preserve a real inner product

φ, χ vectors $\qquad O\varphi \cdot O\chi = \varphi^T \underbrace{O^T O}_{\mathbb{1}} \chi = \varphi^T \chi$

$$O^T O = \mathbb{1}$$

how many generators? real $N \times N$ matrix satisfying $\underbrace{O^T O}_{} = \mathbb{1}$

symm, so $N + \frac{N^2 - N}{2}$
indep conditions

$$d(O(N)) = N^2 - \left[N + \frac{N^2 - N}{2} \right]$$
$$= \frac{N(N-1)}{2}$$

now $\det O = \pm 1 \quad \rightarrow$ take $+1$ to get $SO(N)$

rotation group in N dimensions

then clear that
$d(SO(N)) = \frac{N(N-1)}{2}$:
number of planes in N dim

- SYMPLECTIC GROUPS $Sp(N)$ N even

preserve an antisymm inner product

$$\varphi^T \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}}_{} \chi$$

$\equiv J$ in $\frac{N}{2} \times \frac{N}{2}$ block form

$$S^T J S = J \qquad d(Sp(N)) = N^2 - \left[\frac{N^2 - N}{2} \right]$$
$$= \frac{N(N+1)}{2}$$

number of conditions from $\underline{S^T J S} = J$
antisymm since
$(S^T J S)^T = S^T J^T S$
$= - S^T J S$

* EXCEPTIONAL GROUPS (5 of them): G_2, F_4, E_6, E_7, E_8

[SU(N) has rank N−1]

RANK of the group = dimension of Cartan subgroup, which is the max commuting subgroup; equivalently, # of diag generators

This completes classification of compact simple Lie algebras.

Note also that if there is any Hermitian repr. (as for semisimple algebras) then the structure constants are REAL:

$$[T^a, T^b] = i f^{abc} T^c \quad \Rightarrow \quad [T^a,T^b]^\dagger = -i f^{abc*} T^c$$

$$\begin{aligned} &\| \\ &(T^aT^b - T^bT^a)^\dagger \\ &\| \\ &[T^b, T^a] = i f^{bac} T^c \\ &\qquad\quad = -i f^{abc} T^c \end{aligned}$$

f^{abc} is antisym in first 2 indices

so

$$\underline{f^{abc} = f^{abc*}}.$$

GENERAL PROPERTIES OF REPRS.

Reprs. of algebra are constructed by embedding generators into matrices. One can show that for compact semi-simple Lie algebras, for any repr. R we can choose a basis for the generators such that

$$\mathrm{Tr}\,(T_R^a T_R^b) = \underbrace{T(R)}_{\text{constant for each } R}\,\delta^{ab}$$

now combined with $[T_R^a, T_R^b] = i f^{abc} T_R^c$

this implies

$$[T_R^a, T_R^b]\, T_R^c = i f^{abd} T_R^d T_R^c \qquad \text{take trace}$$

$$\mathrm{Tr}\left[[T_R^a, T_R^b]\, T_R^c\right] = i f^{abd}\, \mathrm{Tr}\,[T_R^d T_R^c]$$

$$= i f^{abd}\, T(R)\, \delta^{dc} = i\, T(R)\, f^{abc}$$

$$\Rightarrow \quad f^{abc} = -\frac{i}{T(R)}\, \mathrm{Tr}\left[[T_R^a, T_R^b]\, T_R^c\right]$$

Now this implies that $f^{abc} = f^{bca}$ because

$$\mathrm{Tr}\left[[T_R^a, T_R^b]\, T_R^c\right] = \mathrm{Tr}\left[T_R^a T_R^b T_R^c - T_R^b T_R^a T_R^c\right]$$

cyclicity of trace

$$\overset{\downarrow}{=} \mathrm{Tr}\left[T_R^b T_R^c T_R^a - T_R^c T_R^b T_R^a\right]$$

$$= \mathrm{Tr}\left[[T_R^b, T_R^c]\, T_R^a\right].$$

Now recall that $f^{abc} = -f^{bac}$ $\Rightarrow$ combined, they tell us that f^{abc} are completely antisym.

Now for a given R, infinitesimal transform under the group is

$$\phi \to (1 + i\alpha^a T_R^a)\,\phi$$ hence complex conj

$$\phi^* \to (1 - i\alpha^a T_R^{a*})\,\phi^*$$ but also we can define the CONJUGATE REPR.

$$\phi^* \to (1 + i\alpha^a T_{\bar{R}}^a)\,\phi$$

hence

$$T_{\bar{R}}^a = -(T_R^a)^* = -(T_R^a)^T.$$

Now $\bar{R}$ is equivalent to R, if there exists a unitary transform U such that

$$T_{\bar{R}}^a = U T_R^a U^\dagger.$$

In this case, R is a REAL repr.

Then we can always find a matrix E_{ab} such that if $\varphi, \chi \in R$ then $\varphi_a E_{ab} \chi_b$ is invariant

If E_{ab} $\to$ SYM STRICTLY REAL

$\searrow$ ANTISYM PSEUDOREAL

REPRS. OF SU(N) GROUPS

Free theory of N complex massless fields is automatically invariant under $U(N) \simeq SU(N) \times U(1)$ → hence the importance of SU(N) groups

The two most important representations are : FUNDAMENTAL and ADJOINT.

- FUNDAMENTAL : acts on space of N-dim complex vectors. Smallest non-trivial repr. of the algebra. Dimension is N.
  For SU(N), set of N×N Hermitian, traceless matrices.

[Why traceless ? Consider

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{T}}$$

and diagonalize $V\,\vec{\alpha}\cdot\vec{T}\,V^{-1} = \underset{\substack{\uparrow \\ \text{diagonal}}}{D}$

(Hermitian matrices are diagonalizable)

then

$$\det U \underset{\substack{\uparrow \\ \det \text{ is basis-indep}}}{=} \det e^{iD} = \prod_j e^{i[D]_{jj}} = e^{i\sum_j [D]_{jj}} = e^{i\,\mathrm{Tr}\,D} \overset{\substack{\mathrm{Tr}\text{ is} \\ \text{basis-indep}}}{=} e^{i\,\mathrm{Tr}\,(\vec{\alpha}\cdot\vec{T})}$$

hence if $\mathrm{Tr}\, T^a = 0$

$\Rightarrow$ $\det U = 1$. Note that tracelessness argument applies for any semisimple algebra.]

- For SU(2), $T^a = \frac{\sigma^a}{2}$ $\quad$ σ^a are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

they satisfy

$$[T^a, T^b] = \frac{1}{4}[\sigma^a, \sigma^b]$$

$$= \frac{1}{4} 2i\, \varepsilon^{abc} \sigma^c = i \underbrace{\varepsilon^{abc}}_{\text{structure constants}} T^c$$

- For SU(3), $T^a = \frac{\lambda^a}{2}$ $\quad$ λ^a are Gell-Mann matrices

Now for the 2 of SU(2) we have

$$T_R^a = \frac{\sigma^a}{2} \qquad T_{\bar{R}}^a = -\frac{\sigma^{a*}}{2}$$ but we know that

$$\sigma^2 \sigma^a \sigma^2 = -\sigma^{a*}$$ hence

$$T_{\bar{R}}^a = -\frac{\sigma^{a*}}{2} = \sigma^2 \left(\frac{\sigma^a}{2}\right) \sigma^2 = \sigma^2 T_R^a \sigma^2$$

i.e. REAL repr. with $U = \sigma^2$,
$U^\dagger = U$

also, $E_{ab} = (i\sigma^2)_{ab} = \varepsilon_{ab}$ [EXERCISE]

so it is <u>pseudoreal</u>

For $N>2$, the N of $SU(N)$ is instead complex
(so in part. $3 \neq \bar{3}$ for $SU(3)$)

- ADJOINT : acts on the space spanned by the generators themselves

$$(T_{adj}^a)_{bc} = q f^{abc}$$
(q purely Im)

dimension is N^2-1
(number of generators)

e.g. for $SU(2)$:

$$(T_{adj}^a)_{bc} = q\,\varepsilon^{abc} \quad \rightarrow \quad T_{adj}^1 = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{etc.}$$

EXERCISE: find q such that T_{adj}^a satisfy comm. relations of algebra.

Adjoint repr. is important because it is the repr. where non-Abelian gauge fields transform.

Note that the conjugate repr of the adjoint is

$$T_{\overline{adj}}^a = -(T_{adj}^a)^* = -(q f^{abc})^* \underset{\substack{\uparrow \\ q \in \mathrm{Im} \\ f^{abc}\ \text{real}}}{=} +q f^{abc} = T_{adj}^a$$

hence the adjoint is always <u>real</u>

$(U = \mathbb{1})$.

CASIMIR OPERATOR

For any repr. T_R^a, we know that $T^2 \equiv T_R^a T_R^a$ (sum over a) commutes with all the generators:

$$[T^2, T_R^b] = [T_R^a T_R^a, T_R^b]$$

$[AB, C] = A[B,C] + [A,C]B$

$$= T_R^a [T_R^a, T_R^b] + [T_R^a, T_R^b] T_R^a$$

$$= T_R^a \, i f^{abc} T_R^c + i f^{abc} T_R^c T_R^a$$

$$= \underbrace{i f^{abc}}_{\text{antisym}} \underbrace{\{T_R^a, T_R^c\}}_{\text{sym}} = 0 .$$

Hence T^2 must be proportional to the identity, by Schur's lemma. So

$$T_R^a T_R^a = C_2(R)\, \mathbb{1} \quad (*)$$

↑ QUADRATIC CASIMIR of R

[Familiar example: $SU(2)$, $J^2 = T_R^a T_R^a$ is Casimir with eigenvalues $j(j+1)$ (total spin)]

As we discussed we also have for an appropriate choice of basis

$$\mathrm{Tr}\,(T_R^a T_R^b) = \underbrace{T(R)}\, \delta^{ab}$$

← index of R (sometimes called $C(R)$)

now contract this with δ^{ab}

$$\mathrm{Tr}\,(T_R^a T_R^a) = T(R)\, d(G)$$

From $(*)$: $\quad \mathrm{Tr}\,(T_R^a T_R^a) = C_2(R)\, d(R)$

$\Rightarrow$ $\underline{d(R)\, C_2(R) = T(R)\, d(G)}$

which is useful to compute $C_2(R)$

- FUNDAMENTAL of $SU(N)$, typical to choose $T(\text{fund}) = \frac{1}{2}$ as in the case of $SU(2)$

then $C_2(\text{fund}) = \dfrac{T(\text{fund})\, d(G)}{d(\text{fund})} = \dfrac{1}{2}\dfrac{N^2-1}{N} \equiv \underline{C_F}$

↑ often used

- ADJOINT of SU(N)

first note that $d(R) = d(G)$, so $C_2(\text{adj}) = T(\text{adj})$.

How do we compute $C_2(\text{adj})$? Instructive to do it by building adjoint as product of fund and antifund: given 2 reprs. r_1 and r_2, their <u>direct product</u> is a repr of dim. $d(r_1)\cdot d(r_2)$. Objects transforming under it are <u>tensors</u> ξ_{pq}, $p \in r_1$, $q \in r_2$. The product repr. can in general be decomposed as direct sum of irreprs: $r_1 \times r_2 = \sum_{\hat{r}} r_{\hat{r}}$ (*)

The matrices are $\quad t^a_{r_1 \times r_2} = t^a_{r_1} \otimes 1\!\!1 + 1\!\!1 \otimes t^a_{r_2}$

Now the quadr. casimir of the product is

$$t^a_{r_1\times r_2}\, t^a_{r_1\times r_2} = (t^a_{r_1})^2 \otimes 1\!\!1 + 1\!\!1 \otimes (t^a_{r_2})^2 + 2\, t^a_{r_1} \otimes t^a_{r_2}$$

take the trace

$$\mathrm{Tr}\,(t^{a\,2}_{r_1\times r_2}) = \underbrace{\mathrm{Tr}\,(t^{a\,2}_{r_1})}\, d(r_2) + d(r_1)\,\underbrace{\mathrm{Tr}\,(t^{a\,2}_{r_2})} + \underset{\substack{\uparrow \\ t^a_{\hat{r}} \text{ are} \\ \text{traceless}}}{0}$$

$$= \overbrace{C_2(r_1)\, d(r_1)}\; d(r_2) + d(r_1)\; \overbrace{C_2(r_2)\, d(r_2)}$$

$$= \left[C_2(r_1) + C_2(r_2)\right] d(r_1)\, d(r_2) \qquad \text{(A)}$$

But also, from (*) we get

$$\mathrm{Tr}\,(t^{a\,2}_{r_1\times r_2}) \underset{\substack{\uparrow \\ \text{block-diag form}}}{=} \sum_{\hat{r}} \mathrm{Tr}\,(t^2_{r_{\hat{r}}}) = \sum_{\hat{r}} C_2(r_{\hat{r}})\, d(r_{\hat{r}}) \qquad \text{(B)}$$

Now apply to case of $r_1 = N$, $r_2 = \bar{N}$ in SU(N):

$$N \times \bar{N} = \underset{\substack{\uparrow \\ \xi_{pq} = \delta_{pq}}}{1} + (N^2-1)$$

(N^2-1) ← remaining pieces: $N\times N$ <u>traceless</u> tensor → ADJOINT repr.

Then from (A) = (B) get

$$\left[C_2(N) + C_2(\bar{N})\right] N^2 = C_2(\text{adj})\, d(\text{adj})$$

see above

$$2\, C_2(N)\, N^2 = C_2(\text{adj})\,(N^2-1)$$

$$\Rightarrow C_2(\text{adj}) = \cancel{2}N^{\cancel{2}}\, \frac{\cancel{N^2-1}}{\cancel{2N}}\, \frac{1}{\cancel{N^2-1}} = N \equiv C_A \qquad \text{Hence} \quad \begin{array}{l} C_2(\text{adj}) = C_A = N \\ T(\text{adj}) = T_A = N \end{array}.$$