# TRANSVERSITY OF GAUGE BOSON SELF-ENERGY FROM BRST SYMMETRY

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The discussion largely follows Prof. Beneke's notes for QFT II. However, any mistakes are mine.

Our starting point is the gauge-fixed Lagrangian, which for the purpose of exploiting BRST is best written using the auxiliary bosonic field $B^a$: in a covariant gauge, we have

$$\tilde{\mathcal{L}} = \underset{\text{gauge + matter}}{\underset{\uparrow}{\mathcal{L}}} + B^a \partial_\mu A^{\mu a} + \frac{\xi}{2}(B^a)^2 + (\partial_\mu \bar{c})^a (D^\mu c)^a$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - g f^{abc} A_\mu^c$$

(convention is $D_\mu = \partial_\mu - igA_\mu$)

Let us see explicitly how this $\tilde{\mathcal{L}}$ arises: start from naive functional integral, $\boxed{**}$

$$\langle \Omega | T\{O_i(x_i) O_j(x_j) \ldots\} | \Omega \rangle = |N|^2 \int DA \; e^{iS[A]} \; O_i O_j \ldots$$

now we must eliminate the redundancy through a gauge fixing (pick one representative for each gauge orbit). Insert identity as

$$1 = \int D\varepsilon \; \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right) \delta\left(f[A_\varepsilon] - F\right) \qquad (*1)$$

where $f$ will be our gauge-fixing function, and $A_\varepsilon$ is the gauge field transformed by a small $\varepsilon$:

$$A^a_{\mu\varepsilon} = A^a_\mu + \frac{1}{g} D_\mu^{ac} \varepsilon^c$$

"$\boxed{**}$ throughout the discussion, "$T$" actually means "$T^*$", as usual when using the path integral. This allows us, in particular, to pull derivatives in front of the $T$.

then

$$= |N|^2 \int DA \; D\varepsilon \; e^{iS[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right) \delta\left(f[A_\varepsilon] - F\right) O_i O_j \ldots$$

now $S$ and $O_{i,j,\ldots}$ are gauge invariant, and also the measure satisfies $DA = DA_\varepsilon$, so

$$= |N|^2 \int DA_\varepsilon \; D\varepsilon \; e^{iS[A_\varepsilon]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right) \delta\left(f[A_\varepsilon] - F\right) O_i O_j \ldots$$

redefining $A_\varepsilon \to A$ as int. variable we obtain $\triangle$

$$= |N|^2 \left(\int D\varepsilon\right) \int DA \; e^{iS[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right)_{\varepsilon=0} \delta\left(f[A] - F\right) O_i O_j \ldots$$

$$= |N'|^2 \int DA \, e^{iS[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right)_{\varepsilon=0} \delta\left(f[A] - F\right) O_i O_j \ldots \qquad (*2)$$

where in the last step we used the fact that the integral over $D\varepsilon$ just gives an (infinite) overall constant.

Now the current expression does not depend on F (recall *1), so let us multiply times the constant $\int DF\, \tilde{G}[F]$ with a functional $\tilde{G}[F]$ of our choice. Pick then

$$\tilde{G}[F] = \int DB^a \, e^{i\int d^4x\, B^a \left(F^a + \frac{\xi}{2} B^a\right)}$$

which in fact is equivalent to the choice we made when deriving $\tilde{\mathcal{L}}$ in the formulation without auxiliary field: complete the square

$$\tilde{G}[F] = \underbrace{\int DB^a \, e^{i\int d^4x\, \frac{\xi}{2}\left(B^a + \frac{F^a}{\xi}\right)^2}}_{\text{constant}} e^{i\int d^4x \left(-\frac{F^{a\,2}}{2\xi}\right)}$$

$$= \text{CONST} \cdot e^{i\int d^4x\left(-\frac{F^{a\,2}}{2\xi}\right)} = \text{CONST} \cdot G[F] \quad \text{where}$$

$G[F]$ leads to the formulation without $B^a$. Then *2 becomes

$$|N'|^2 \int DA\, e^{i\int d^4x\, \mathcal{L}} \det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0} \delta\left(f[A]-F\right) \int DF\, DB^a\, e^{i\int d^4x\, B^a \left(F^a + \frac{\xi}{2}B^a\right)} O_i O_j \ldots$$

$$= |N'|^2 \int DA\, DB^a\, e^{i\int d^4x\left\{\mathcal{L} + B^a f^a + \frac{\xi}{2}(B^a)^2\right\}} \det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0} O_i O_j \ldots$$

Now write the determinant as

$$\det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0} = \int D[c^a, \bar{c}^a]\, e^{i\int d^4x\, d^4y\, \bar{c}^a(x)(-i)\left.\frac{\delta f^a[A_\varepsilon(x)]}{\delta\varepsilon^b(y)}\right|_{\varepsilon=0} c^b(y)}$$

and for the covariant gauges, $f^a = \partial_\mu A^{\mu a}$, obtain

$$\left.\frac{\delta f^a[A_\varepsilon(x)]}{\delta\varepsilon^b(y)}\right|_{\varepsilon=0} = \frac{1}{g}\partial_\mu D^{\mu ab}\delta^{(4)}(x-y)$$

so

$$i\int d^4x\, d^4y\, \bar{c}^a(x)(-i)\frac{\delta f^a[A_\varepsilon(x)]}{\delta\varepsilon^b(y)} c^b(y) = i\int d^4x\, \bar{c}^a(x)(-i)\frac{1}{g}\partial_\mu D^{\mu ab} c^b(x)$$

int. by parts and redefine $\frac{i c^a}{g} \to c^a$

$$= i\int d^4x\, (\partial_\mu \bar{c}^a)(D^\mu c)^a \qquad \text{hence the path int. becomes}$$

$$= |N'|^2 \int DA\, DB^a\, D[c,\bar{c}]\, e^{i\int d^4x \underbrace{\left\{\mathcal{L} + B^a\partial_\mu A^{\mu a} + \frac{\xi}{2}(B^a)^2 + \partial_\mu\bar{c}^a (D^\mu c)^a\right\}}_{\tilde{\mathcal{L}}}} O_i O_j \ldots$$

gauge-fixed Lagrangian

So far we did not include matter fields. Their effect simply goes into $\mathcal{L}$, provided the integration measure for fermions is invariant under $\psi \to \psi_\varepsilon$. Proceed under this assumption (non-anomalous gauge sym).

The gauge-fixed Lagrangian is invariant under BRST transformations, $\delta_\vartheta \tilde{\mathcal{L}} = 0$, where the action of BRST is $\phi \to \phi + \underbrace{\vartheta \Delta \phi}_{\delta_\vartheta \phi}$ for any $\phi$ ($\vartheta$ = Grassmann number, $\Delta$ = Slavnov operator).
The action on individual fields is

$$\begin{cases} \delta_\vartheta \psi = i g \vartheta c^a T^a \psi \\ \delta_\vartheta A^a_\mu = \vartheta (D_\mu c)^a \\ \delta_\vartheta c^a = -\frac{1}{2} g \vartheta f^{abc} c^b c^c \\ \delta_\vartheta \bar{c}^a = \vartheta B^a \\ \delta_\vartheta B^a = 0 \end{cases}$$

$\Delta$ is nilpotent,
$\delta_\vartheta (\Delta \phi) = 0$.
[see Exercise Sheet 2]

Now invariance under BRST gives rise to Ward identities obeyed by the Green's functions of the non-Abelian gauge theory:

$$(*3) \quad \sum_{k=1}^{m} \langle \Omega | T\{ \phi_{n_1}(x_1) \dots \phi_{n_{k-1}}(x_{k-1})\, \vartheta \Delta(\phi_{n_k}(x_k))\, \phi_{n_{k+1}}(x_{k+1}) \dots \phi_{n_m}(x_m)\} | \Omega \rangle = 0$$

where $\phi_n$ are any of the fields in the theory:

$$\psi, A_\mu, c, \bar{c}, B, \dots$$

This identity is obtained assuming BRST is non-anomalous, which happens if the gauge sym. is non-anomalous.

We will also exploit the following EOM identities,

$$\langle \Omega | T\left\{ \frac{\delta S[\phi_n]}{\delta \phi_n(x)} \phi_{n_1}(x_1) \dots \phi_{n_m}(x_m) \right\} | \Omega \rangle$$

$$(*4) \quad = i \sum_{i=1}^{m} \delta^{(4)}(x - x_i)\, \delta_{n n_i} \langle \Omega | T\{ \phi_{n_1}(x_1) \dots \underbrace{\hat{\phi}_{n_i}(x_i)}_{\text{means "omitted from list"}} \dots \phi_{n_m}(x_m)\} | \Omega \rangle$$

vanishes unless $n = n_i$ and $x = x_i$ for one of the $\phi_{n_1} \dots \phi_{n_m}$

These identities can be obtained by appropriate manipulations of the generating functional $Z[\phi_n]$. I reviewed this in class, but will not repeat here. Please refer to Prof. Beneke's notes for QFT I, where they are derived very clearly.

Finally, preparations are finished and we can move on to our main goal. We will apply the Ward id's of BRST (and EOM id's) to prove that <u>the gauge boson self-energy is transverse</u>.

Take the two-point function

$$\langle \Omega | T\{A^a_\mu(x) A^b_\nu(y)\} | \Omega \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} G^{ab}_{\mu\nu}(k)$$

where the most general form of the Fourier transform is

$$G^{ab}_{\mu\nu}(k) = \delta^{ab} \frac{i}{k^2+i\varepsilon} \left[ \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right) A(k^2) - \frac{k_\mu k_\nu}{k^2} B(k^2) \right]$$

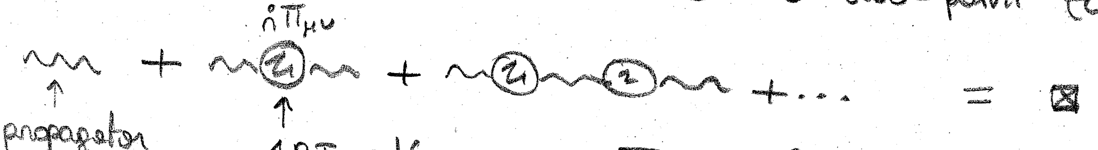
At lowest order in coupling constant $g$ this is the propagator, so we know

$$\begin{cases} A = 1 + O(g^2) \\ B = \xi + O(g^2) \end{cases}$$

[the propagator is $\dfrac{-i}{k^2+i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\dfrac{k_\mu k_\nu}{k^2}\right)$ ].

We will now show that <u>$B = \xi$</u> at <u>all orders in $g$</u>.

If this is true, then we can write the two-point function as




1PI self-energy $\quad \Pi_{\mu\nu} = (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) + k_\mu k_\nu \Pi_2(k^2)$

It's easy to convince oneself that the fact that $B$ does not get corrected at any order in $g$, is equivalent to <u>$\Pi_2 = 0$</u> (*): the <u>self-energy is transverse</u>, $k^\mu \Pi_{\mu\nu} = 0$.

Then it is also easy to resum the series,

$$\boxtimes = \frac{i}{k^2}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)(1 + \Pi + \Pi^2 + \ldots) + \frac{i}{k^2}\left(-\xi \frac{k_\mu k_\nu}{k^2}\right)$$

from first term only

$$= \frac{i}{k^2}\left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi} - \xi\frac{k_\mu k_\nu}{k^2}\right]$$

hence

$$A(k^2) = \frac{1}{1-\Pi(k^2)}.$$

Now let us prove that <u>$B = \xi$</u> at all orders.

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(*) because the first term ~ already gives $B = \xi$!

Use EOM identity in the form

$$0 = \langle\Omega|T\Big\{ \frac{\delta S}{\delta B^a(x)}\, \partial^\mu A^b_\mu(y)\Big\}|\Omega\rangle = \qquad \text{now} \quad \frac{\delta S}{\delta B^a(x)} = \partial_\nu A^{\nu a}(x) + \xi B^a(x)$$

( fields do not match, RHS of *4 is zero )

$$= \langle\Omega|T\{(\partial_\nu A^{\nu a}(x)+\xi B^a(x))\,\partial^\mu A^b_\mu(y)\}|\Omega\rangle$$

which we rewrite as $\underline{\underline{\partial^x_\nu \partial^y_\mu \langle\Omega|T\{A^{\nu a}(x)\, A^{\mu b}(y)\}|\Omega\rangle}} =$

$$= -\xi\,\langle\Omega|T\{\underbrace{B^a(x)}_{\displaystyle \overset{\shortparallel}{\Delta\bar c^a(x)}}\,\partial^\mu A^b_\mu(y)\}|\Omega\rangle = \underline{\underline{-\xi\langle\Omega|T\{\Delta\bar c^a(x)\,\partial^\mu A^b_\mu(y)\}|\Omega\rangle}}.$$

Now look at $=\!=$ : the LHS is already $\partial_\mu\partial_\nu$ (two-point function), but we need to work on the RHS. Let us use the Ward identity for 2 fields: *3 reads

$$0 = \langle\Omega|T\{\vartheta\Delta\bar c^a(x)\,\partial^\mu A^b_\mu(y)\}|\Omega\rangle + \langle\Omega|T\{\bar c^a(x)\,\vartheta\Delta(\partial^\mu A^b_\mu(y))\}|\Omega\rangle$$

↙ Grassmann vars.

$$= \langle\Omega|T\{\vartheta\Delta\bar c^a(x)\,\partial^\mu A^b_\mu(y)\}|\Omega\rangle - \langle\Omega|T\{\vartheta\,\bar c^a(x)\,\Delta(\partial^\mu A^b_\mu(y))\}|\Omega\rangle$$

now $\quad \Delta(\partial^\mu A^b_\mu(y)) = \partial^\mu(\Delta A^b_\mu(y)) = \partial^\mu(D_\mu c(y))^b$ but also,

$$\frac{\delta S}{\delta\bar c^b(y)} = -\partial^\mu D^{bc}_\mu c^c(y)$$

↑ integrate by parts before taking derivative

hence

$$0 = \langle\Omega|T\{\Delta\bar c^a(x)\,\partial^\mu A^b_\mu(y)\}|\Omega\rangle + \langle\Omega|T\Big\{\bar c^a(x)\,\frac{\delta S}{\delta\bar c^b(y)}\Big\}|\Omega\rangle$$

and $=\!=$ becomes

$$\partial^x_\nu\partial^y_\mu\langle\Omega|T\{A^{\nu a}(x)\,A^{\mu b}(y)\}|\Omega\rangle = +\xi\,\langle\Omega|T\Big\{\bar c^a(x)\,\frac{\delta S}{\delta\bar c^b(y)}\Big\}|\Omega\rangle$$

$$= -\xi\,\underbrace{\langle\Omega|T\Big\{\frac{\delta S}{\delta\bar c^b(y)}\,\bar c^a(x)\Big\}|\Omega\rangle}$$

this is precisely the case where the RHS of the EOM id. of *4 does not vanish!

We arrive at

$$\partial^x_\nu\partial^y_\mu\langle\Omega|T\,A^{\nu a}(x)\,A^{\mu b}(y)|\Omega\rangle = -\xi\, i\,\delta^{(4)}(x-y)\,\delta^{ab}\underbrace{\langle\Omega|\Omega\rangle}_{=1}.$$

The rest is easy: write

$$\partial_\nu^x \partial_\mu^y \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} G^{ab}_{\mu\nu}(k) = -i\xi\delta^{ab} \underbrace{\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}}_{\text{integral repr. of } \delta^{(4)}}$$

$$(-ik^\nu)(+ik^\mu)\, G^{ab}_{\mu\nu}(k) = -i\xi\delta^{ab}$$

$$k^\mu k^\nu G^{ab}_{\mu\nu}(k) = -i\xi\delta^{ab}$$ then recall form of $G^{ab}_{\mu\nu}$:

$$k^\mu k^\nu G^{ab}_{\mu\nu}(k) = \frac{i}{\cancel{k^2}}\delta^{ab}\left(-\cancel{k^2}B\right) = -i\xi\delta^{ab} \quad \Rightarrow \quad B = \xi$$

which is an <u>exact</u> result.

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Some more justification for $\triangle$:

$$\int DA_\varepsilon\, D\varepsilon\, e^{iS[A_\varepsilon]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)\delta(f[A_\varepsilon]-F)\,O_i O_j \ldots =$$

now
$$\frac{\delta f^a[A_\varepsilon]}{\delta\varepsilon^b} = \frac{\delta f^a[A_\varepsilon]}{\delta A^c_\varepsilon}\frac{\delta A^c_\varepsilon}{\delta\varepsilon^b} = f'^{ac}[A_\varepsilon]\,\frac{1}{g}D^{cb}_\mu$$

det is gauge invariant

now $\det\left(\frac{1}{g}D_\mu\right) = \det\left(\frac{1}{g}U_\varepsilon D_\mu U_\varepsilon^\dagger\right)$
$= \det\left(\frac{1}{g}D_\mu^{(\varepsilon)}\right)$

then

$$= \int DA_\varepsilon\, D\varepsilon\, e^{iS[A_\varepsilon]} \det\left(f'^{ac}[A_\varepsilon]\right) \det\left(\frac{1}{g}D^{cb}_\mu\right)\delta(f[A_\varepsilon]-F)\,O_i O_j \ldots$$

hence renaming $A_\varepsilon \to A$ gives

$$= \int D\varepsilon\, DA\, e^{iS[A]} \det\left(f'^{ac}[A]\right)\det\left(\frac{1}{g}D^{cb}_\mu\right)\delta(f[A]-F)\,O_i O_j \ldots$$

$$= \left(\int D\varepsilon\right)\int DA\, e^{iS[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0} \delta(f[A]-F)\,O_i O_j \ldots$$