

TRANSUERSTY OF GAUGE BOSON SELF-ENERGY FROM BRST SYMMETRY

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The discussion largely follows prof. Beneke's notes for QFT II. However, any mistakes are mine.

Our starting point is the gauge-fixed Lagrangian, which for the purpose of exploiting BRST is best written using the auxiliary bosonic field B^a : in a covariant gauge, we have

$$\tilde{L} = L + B^a \partial_\mu A^{a\mu} + \frac{g}{2} (B^a)^2 + (\partial_\mu \bar{c})^a (D^\mu c)^a$$

\uparrow
gauge + matter

$$D_\mu^{ab} = g^{ab} \partial_\mu - g f^{abc} A_\mu^c \quad (\text{convention is } D_\mu = \partial_\mu - igf_\mu^a)$$

Let us see explicitly how this \tilde{L} arises: start from naive functional integral,

$$\langle \Omega | T \{ O_i(x_i) O_j(x_j) \dots \} | \Omega \rangle = \int N^2 \int DA e^{-S[A]} O_i O_j \dots$$

now we must eliminate the redundancy through a gauge fixing (pick one representative for each gauge orbit). Insert identity as

$$1 = \int DE \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right) \delta (f[A_\epsilon] - F) \quad (*1)$$

where f will be our gauge-fixing function, and A_ϵ is the gauge field transformed by a small ϵ :

$$A_{ME}^a = A_M^a + \frac{1}{g} D_M^{ac} \epsilon^c$$

then

$$= \int N^2 \int DA \int DE e^{-S[A]} \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right) \delta (f[A_\epsilon] - F) O_i O_j \dots$$

now S and $O_i, O_j \dots$ are gauge invariant, and also the measure satisfies $DA = DA_\epsilon$, so

$$= \int N^2 \int DA_\epsilon \int DE e^{-S[A_\epsilon]} \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right) \delta (f[A_\epsilon] - F) O_i O_j \dots$$

redefining $A_\epsilon \rightarrow A$ as int. variable we obtain Δ

$$= \int N^2 \left(\int DE \right) \int DA e^{-S[A]} \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right)_{\epsilon=0} \delta (f[A] - F) O_i O_j \dots$$

$$= \int N^2 \int DA e^{-S[A]} \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right)_{\epsilon=0} \delta (f[A] - F) O_i O_j \dots \quad (*2)$$

where in the last step we used the fact that the integral over DE just gives an (infinite) overall constant.

Now the current expression does not depend on F (recall *1), (2)

so let us multiply times the constant $\int DF \tilde{G}[F]$

with a functional $\tilde{G}[F]$ of our choice. Pick then

$$\tilde{G}[F] = \int DB^a e^{\int d^4x B^a (F^a + \frac{S}{2} B^a)}$$

which in fact is equivalent to the choice we made when deriving \tilde{L} in the formulation without auxiliary field: complete the square

$$\tilde{G}[F] = \underbrace{\int DB^a e^{\int d^4x \frac{S}{2} (B^a + \frac{F^a}{S})^2}}_{\text{constant}} e^{\int d^4x (-\frac{F^a}{2S})}$$

$$= \text{CONST. } e^{\int d^4x (-\frac{F^a}{2S})} = \text{CONST. } G[F] \quad \text{where}$$

$G[F]$ leads to the formulation without B^a . Then *2 becomes

$$\begin{aligned} & \text{IN}^{1/2} \int DA e^{\int d^4x \mathcal{L}} \det \left(\frac{\delta \mathcal{L}[A_\epsilon]}{\delta \epsilon} \right)_{\epsilon=0} \delta(\mathcal{L}[A] - F) \int DF DB^a e^{\int d^4x B^a (F^a + \frac{S}{2} B^a)} \\ &= \text{IN}^{1/2} \int DA DB^a e^{\int d^4x \left\{ \mathcal{L} + B^a F^a + \frac{S}{2} (B^a)^2 \right\}} \det \left(\frac{\delta \mathcal{L}[A_\epsilon]}{\delta \epsilon} \right)_{\epsilon=0} \end{aligned} \quad \text{O:Oj...}$$

Now write the determinant as

$$\det \left(\frac{\delta \mathcal{L}[A_\epsilon]}{\delta \epsilon} \right)_{\epsilon=0} = \int D[c^a, \bar{c}^a] e^{\int d^4x d^4y \bar{c}^a(x) (-i) \frac{\delta \mathcal{L}[A_\epsilon(x)]}{\delta \epsilon^b(y)} c^b(y)}$$

and for the covariant gauges, $\mathcal{L}^a = \partial_\mu A^{\mu a}$, obtain

$$\frac{\delta \mathcal{L}^a[A_\epsilon(x)]}{\delta \epsilon^b(y)} \Big|_{\epsilon=0} = \frac{1}{8} \partial_\mu D^{\mu ab} \delta^{(a)}(x-y)$$

so

$$\int d^4x d^4y \bar{c}^a(x) (-i) \frac{\delta \mathcal{L}^a[A_\epsilon(x)]}{\delta \epsilon^b(y)} c^b(y) = \int d^4x \bar{c}^a(x) (-i) \frac{1}{8} \partial_\mu D^{\mu ab} c^b(x)$$

int. by parts and redefine $\frac{i c^a}{8} \rightarrow c^a$

$$= \int d^4x (\partial_\mu \bar{c}^a) (D^\mu c)^a \quad \text{hence the path int. becomes}$$

$$\begin{aligned} & \int d^4x \left\{ \mathcal{L} + B^a \partial_\mu A^{\mu a} + \frac{S}{2} (B^a)^2 + \partial_\mu \bar{c}^a (D^\mu c)^a \right\} \\ &= \text{IN}^{1/2} \int DA DB^a D[c, \bar{c}] e^{\int d^4x \mathcal{L}} \end{aligned} \quad \text{O:Oj...}$$

\tilde{L} gauge-fixed Lagrangian

So far we did not include matter fields. Their effect simply goes into \tilde{L} , provided the integration measure for fermions is invariant under $\Psi \rightarrow \Psi + \delta \phi$. Proceed under this assumption (non-anomalous gauge sym.). (3)

The gauge-fixed Lagrangian is invariant under BRST transformations, $\delta_\theta \tilde{L} = 0$, where the action of BRST is $\phi \rightarrow \phi + \vartheta \Delta \phi$, $\delta_\theta \phi$ for any ϕ (ϑ = Grassmann number, Δ = Slavnov operator). The action on individual fields is

$$\left\{ \begin{array}{l} \delta_\theta \psi = ig \vartheta c^a T^a \psi \\ \delta_\theta A_\mu^a = \vartheta (D_\mu c)^a \\ \delta_\theta c^a = -\frac{1}{2} g \vartheta f_{abc} c^b c^c \\ \delta_\theta \bar{c}^a = \vartheta B^a \\ \delta_\theta B^a = 0 \end{array} \right. \quad \begin{array}{l} \Delta \text{ is nilpotent,} \\ \delta_\theta (\Delta \phi) = 0. \end{array}$$

[see Exercise Sheet 3]

Now invariance under BRST gives rise to Ward identities obeyed by the Green's functions of the non-Abelian gauge theory:

$$(*) 3 \quad \sum_{n=1}^m \langle \Omega | T \{ \phi_{n_1}(x_1) \dots \phi_{n_{n-1}}(x_{n-1}) \vartheta \Delta(\phi_{n_n}(x_n)) \phi_{n_{n+1}}(x_{n+1}) \dots \phi_{n_m}(x_m) \} | \Omega \rangle = 0$$

where ϕ_n are any of the fields in the theory:

$$\psi, A_\mu, c, \bar{c}, B, \dots$$

This identity is obtained assuming BRST is non-anomalous, which happens if the gauge sym. is non-anomalous.

We will also exploit the following EOM identities,

$$(*) 4 \quad \begin{aligned} & \langle \Omega | T \left\{ \frac{\delta S[\phi_n]}{\delta \phi_n(x)} \phi_{n_1}(x_1) \dots \phi_{n_m}(x_m) \right\} | \Omega \rangle \\ &= i \sum_{i=1}^m \delta^{(4)}(x - x_i) \delta_{nn_i} \langle \Omega | T \left\{ \phi_{n_1}(x_1) \dots \overset{\uparrow}{\phi_{n_i}(x_i)} \dots \phi_{n_m}(x_m) \right\} | \Omega \rangle \end{aligned}$$

means
"omitted from last"
vanishes unless $n = n_i$
and $x = x_i$ for one of the $\phi_{n_1} \dots \phi_{n_m}$

These identities can be obtained by appropriate manipulations of the generating functional $Z[\phi_n]$. I reviewed this in class, but will not repeat here. Please refer to Prof. Beneke's notes for QFT I, where they are derived very clearly.

Finally, preparations are finished and we can move on to our main goal. We will apply the Ward id's of BRST (and EOM id's) to prove that the gauge boson self-energy is transverse.

Take the two-point function

$$\langle \Sigma | T\{A_\mu^a(x) A_\nu^b(y)\} | \Sigma \rangle = \int \frac{d^4 u}{(2\pi)^4} e^{-iu(x-y)} G_{\mu\nu}^{ab}(u)$$

where the most general form of the Fourier transform is

$$G_{\mu\nu}^{ab}(u) = \delta^{ab} \frac{i}{u^2 + i\epsilon} \left[\left(-g_{\mu\nu} + \frac{\kappa_{\mu\nu}}{u^2} \right) A(u^2) - \frac{\kappa_{\mu\nu}}{u^2} B(u^2) \right]$$

At lowest order in coupling constant g this is the propagator, so we know

$$\begin{cases} A = 1 + O(g^2) \\ B = \xi + O(g^2) \end{cases}$$

$$[\text{the propagator is } \frac{-i}{u^2 + i\epsilon} \left(g_{\mu\nu} - (1-\xi) \frac{\kappa_{\mu\nu}}{u^2} \right)].$$

We will now show that $B = \xi$ at all orders in g .

If this is true, then we can write the two-point function as

$$\underset{\substack{\uparrow \\ \text{propagator}}}{m} + \underset{\substack{\uparrow \\ \text{1PI self-energy}}}{m} + \underset{\substack{\uparrow \\ \text{1PI self-energy}}}{m} + \dots = \boxtimes$$

$$1\text{PI self-energy: } \Pi_{\mu\nu} = (u^2 g_{\mu\nu} - \kappa_{\mu\nu}) \Pi(u^2) + \kappa_{\mu\nu} \Pi_2(u^2)$$

It's easy to convince oneself that the fact that B does not get corrected at any order in g , is equivalent to $\Pi_2 = 0$: the self-energy is transverse, $\kappa^\mu \Pi_{\mu\nu} = 0$.

Then it is also easy to resum the series,

$$\begin{aligned} \boxtimes &= \frac{i}{u^2} \left(-g_{\mu\nu} + \frac{\kappa_{\mu\nu}}{u^2} \right) (1 + \Pi + \Pi^2 + \dots) + \frac{i}{u^2} \left(-\xi \frac{\kappa_{\mu\nu}}{u^2} \right) \\ &= \frac{i}{u^2} \left[\left(-g_{\mu\nu} + \frac{\kappa_{\mu\nu}}{u^2} \right) \frac{1}{1-\Pi} - \xi \frac{\kappa_{\mu\nu}}{u^2} \right] \end{aligned}$$

from first term only
hence

$$A(u^2) = \frac{1}{1-\Pi(u^2)}.$$

Now let us prove that $B = \xi$ at all orders.

Use EOM identity in the form

$$0 = \langle \mathcal{R} | T \left\{ \frac{\delta S}{\delta B^a(x)} \partial^\mu A_\mu^b(y) \right\} | \mathcal{R} \rangle = \text{now } \frac{\delta S}{\delta B^a(x)} = \partial_\mu A^{\mu a}(x) + \xi B^a(x)$$

(fields do
not match,
RHS of *4
is zero)

which we rewrite as $\partial_\nu^* \partial_\mu^y \langle \mathcal{R} | T \{ A^{\mu a}(x) A^{\nu b}(y) \} | \mathcal{R} \rangle =$

$$= -\xi \langle \mathcal{R} | T \left\{ \underbrace{B^a(x) \partial^\mu A_\mu^b(y)}_{\Delta \bar{c}^a(x)} \right\} | \mathcal{R} \rangle = -\xi \langle \mathcal{R} | T \{ \Delta \bar{c}^a(x) \partial^\mu A_\mu^b(y) \} | \mathcal{R} \rangle.$$

Now look at $=$: the LHS is already $\partial_\mu \partial_\nu$ (two-point function),
but we need to work on the RHS. Let us use the Ward identity
for 2 fields: *3 reads

$$0 = \langle \mathcal{R} | T \{ \partial \Delta \bar{c}^a(x) \partial^\mu A_\mu^b(y) \} | \mathcal{R} \rangle + \langle \mathcal{R} | T \{ \bar{c}^a(x) \partial \Delta (\partial^\mu A_\mu^b(y)) \} | \mathcal{R} \rangle$$

↙ Grammatical error.

$$= \langle \mathcal{R} | T \{ \partial \Delta \bar{c}^a(x) \partial^\mu A_\mu^b(y) \} | \mathcal{R} \rangle - \langle \mathcal{R} | T \{ \partial \bar{c}^a(x) \Delta (\partial^\mu A_\mu^b(y)) \} | \mathcal{R} \rangle$$

now $\Delta (\partial^\mu A_\mu^b(y)) = \partial^\mu (\Delta A_\mu^b(y)) = \partial^\mu (D_\mu c^b(y))$ but also,

$$\frac{\delta S}{\delta \bar{c}^b(y)} = -\partial^\mu D_\mu^{bc} c^c(y)$$

integrate by parts before
taking derivative

hence

$$0 = \langle \mathcal{R} | T \{ \Delta \bar{c}^a(x) \partial^\mu A_\mu^b(y) \} | \mathcal{R} \rangle + \langle \mathcal{R} | T \{ \bar{c}^a(x) \frac{\delta S}{\delta \bar{c}^b(y)} \} | \mathcal{R} \rangle$$

and $=$ becomes

$$\partial_\nu^* \partial_\mu^y \langle \mathcal{R} | T \{ A^{\mu a}(x) A^{\nu b}(y) \} | \mathcal{R} \rangle = +\xi \langle \mathcal{R} | T \{ \bar{c}^a(x) \frac{\delta S}{\delta \bar{c}^b(y)} \} | \mathcal{R} \rangle$$

$$= -\xi \langle \mathcal{R} | T \{ \underbrace{\frac{\delta S}{\delta \bar{c}^b(y)} \bar{c}^a(x)}_{\text{this is precisely the case where}} \} | \mathcal{R} \rangle$$

this is precisely the case where
the RHS of the EOM id. of *4
does not vanish!

We arrive at

$$\partial_\nu^* \partial_\mu^y \langle \mathcal{R} | T \{ A^{\mu a}(x) A^{\nu b}(y) \} | \mathcal{R} \rangle = -\xi \stackrel{\text{?}}{\delta} \delta^{(4)}(x-y) \delta^{ab} \langle \mathcal{R} | \mathcal{R} \rangle.$$

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The rest is easy: write

(6)

$$d_\nu^x d_\mu^y \int \frac{d^4 u}{(2\pi)^4} e^{-iu \cdot (x-y)} G_{\mu\nu}^{ab}(u) = -i \int \delta^{ab} \int \frac{d^4 u}{(2\pi)^4} e^{-iu \cdot (x-y)}$$

integral repr. of
 $\delta^{(4)}$

$$(-iu^\nu)(+iu^\mu) G_{\mu\nu}^{ab}(u) = -i \int \delta^{ab}$$

$$u^\mu u^\nu G_{\mu\nu}^{ab}(u) = -i \int \delta^{ab} \quad \text{then recall form of } G_{\mu\nu}^{ab}:$$

$$u^\mu u^\nu G_{\mu\nu}^{ab}(u) = \frac{i}{u^2} \delta^{ab} (-u^2 B) = -i \int \delta^{ab} \Rightarrow B = f$$

which is an exact
result.

Some more justification for Δ :

$$\int D A_\epsilon DE e^{iS[A_\epsilon]} \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right) \delta(f[A_\epsilon] - F) O_i O_j \dots =$$

$$\text{now } \frac{\delta f[A_\epsilon]}{\delta \epsilon^b} = \frac{\delta f[A_\epsilon]}{\delta A_\epsilon^c} \frac{\delta A_\epsilon^c}{\delta \epsilon^b} = \underbrace{\delta^{1ac}[A_\epsilon]}_{\delta f[A_\epsilon]} \frac{1}{g} D_\mu^{cb}$$

then

dependence on ϵ is
all contained here

$$= \int D A_\epsilon DE e^{iS[A_\epsilon]} \det \left(\delta^{1ac}[A_\epsilon] \right) \det \left(\frac{1}{g} D_\mu^{cb} \right) \delta(f[A_\epsilon] - F) O_i O_j \dots$$

now remove $A_\epsilon \rightarrow A$

$$= \int DE DA e^{iS[A]} \det(\delta^{1ac}[A]) \det \left(\frac{1}{g} D_\mu^{cb} \right) \delta(f[A] - F) O_i O_j \dots$$

$$= (\int DE) \int DA e^{iS[A]} \det \left(\frac{\delta f[A_\epsilon]}{\delta \epsilon} \right)_{\epsilon=0} \delta(f[A] - F) O_i O_j \dots$$