EXACT RENORMALIZATION GROUP AND ASYMPTOTIC SAFETY I

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1 Renormalization group (RG)

As Wilson puts it, "the renormalization group approach is a strategy for dealing with problems involving many length scales" [12]. The difficulty arises when complex *microscopic* behaviour underlies *macroscopic* effects, and the former fluctuations do not average out when only the macroscopic scales are considered. Thus, the core idea is that "the original, very many degrees of freedom are replaced by a smaller set of effective degrees of freedom" that represent an average over the former, microscopic degrees of freedom [13]. Wilson's strategy was to tackle the problem in steps, one step for each length scale in momentum space [13]. We will later take the limit to infinitesimal momentum shells as was initially proposed in [10]. "The resulting change of the effective action can be expressed in terms of a formally exact functional differential equation describing the evolution of the effective action as a function of the cutoff" [3], which, combined with smooth cutoffs [6],will lead us to the exact renormalization group equation [9, 3].

2 Traditional vs functional approach to RG

We can now take a look at the transformation of our action when taking infinitesimal steps to change the scale of our theory. We find that this transformation is continuous and corresponds to a trajectory in the space of possible theories. These transformations are collected in the Renormalization Group. It is not really a group but a semi-group because we cannot invert integrating out degrees of freedom [5]. The non invertible character is physically clear. Many different microscopic models can have the same long-wavelength properties. "The elimination of the short-wavelength fluctuations is a *projection of the coupling space of the microscopic theory onto a reduced coupling space associated with an effective model with the same long-wavelength properties as the former"*[3].

In the traditional RG procedure we follow the evolution of a few coupling constants. In contrast, in the functional approach, where the generating functional is defined for all coupling constants, we can follow the evolution of a large number of coupling constants with the change of scale [7]. For instance, in the multiplicative RG schemes, considering the UV cutoff Λ and the substraction (observational) scales μ, if the $n-$point renormalized and bare Green functions are,

$$G_R^{(n)}(p_1,\dots,p_n;g_R,\mu) = Z^{-n/2}G_B^{(n)}(p_1,\dots,p_n;g_B,\Lambda) + \mathcal{O}\left(\frac{\mu^2}{\Lambda^2}\right) + \mathcal{O}\left(\frac{p_j^2}{\Lambda^2}\right),$$

where the ignored $\mathcal{O}(\dots)$ terms are cutoff dependent interactions; then, the **bare RG equation**, which emphasizes that the theory is independent of the cutoff, is

$$\Lambda\frac{d}{d\Lambda}\left(G_R^{(n)}(p_1,\dots,p_n;g_B,\Lambda)\right) = \Lambda\frac{d}{d\Lambda}\left(Z^{-n/2}(g_B,\Lambda)G_B^{(n)}(p_1,\dots,p_n;g_B,\Lambda)\right) = 0\,.$$

It expresses the possibility of compensating the change of μ by the change of coupling constants $g_B(\Lambda)$ [7].

On the other hand, the **renormalized RG equation** emphasizes that the theory is independent of the observational (substraction) scale,

$$\mu\frac{d}{d\mu}\left(G_B^{(n)}(p_1,\dots,p_n;g_R,\mu)\right) = \mu\frac{d}{d\mu}\left(Z^{n/2}(g_R,\mu)G_R^{(n)}(p_1,\dots,p_n;g_R,\mu)\right) = 0\,,$$

where the bare couplings and Λ are kept fixed [7].

The limitations of these equations is that they are only applicable in the regime $\mu^2, p^2 \ll \Lambda^2$. In the sections below, we will develop the functional approach to the RG equations.

Functional methods instead make use of concepts like generating functionals and the effective action. This is useful as functional methods still work, unlike perturbation theory, for strong couplings. So, they work non-perturbatively and include all operators [2].

3 Functional RG

3.1 Basics: the effective action

The previously discussed n-point functions are summarized in the generating functional as follows. The general formula for a n-point function in Euclidean QFT is

$$\langle \varphi(x_1), \ldots, \varphi(x_n) \rangle = N \int D\varphi \, \varphi(x_1) \ldots \varphi(x_n) \, e^{-S[\varphi]}. \tag{1}$$

Here, $N^{-1} = \int D\varphi \, e^{-S[\varphi]}$ is a normalization such that $\langle 1 \rangle = 1$. We can also rewrite this as

$$\langle \varphi(x_1), \ldots, \varphi(x_n) \rangle = \frac{1}{Z[0]} \left(\frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right)_{J=0},$$

where, as usual, the generating functional in the presence of an external source is

$$Z[J] = \int \mathcal{D}\varphi e^{-S[\varphi] + \int d^D x J(x) \varphi(x)}.$$

We can define an energy functional, here denoted as $W[J]$,

$$e^{W[J]} = Z[J].$$

Note that $W[J]$ is *the generating functional of connected correlators.* Indeed,

$$\frac{\delta W[J]}{\delta J(x)} = \frac{\delta}{\delta J(x)} \log(Z[J]) = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J = \phi,$$

where we have denoted with $\langle \ldots \rangle_J$ the vacuum expectation value in the presence of a source J, and we have defined the classical field ϕ, which depends on J.

Now, we take the Legendre transform of $W[J]$ and call this quantity **effective action**, (note the dependance on ϕ)

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J] \right).$$

This effective action is the generator of one-particle-irreducible correlation functions [5]. Let us verify: at $J = J_{sup}$

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -\int_y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \phi(x)} + \int_y \phi(y) \frac{\delta J(y)}{\delta \phi(x)} + J(x) = J(x), \tag{2}$$

which is the *quantum equation of motion*, such that the effective action gives the dynamics of the field expectation value, with **all** quantum fluctuations already taken into account [2].

There is a common anoalogy with thermodynamics, where we consider thermal fluctuations. The generating functional of connected correlators and effective action correspond to the Helmholtz free energy and Gibbs free energy, which are also connected by a Legendre transform. It is well explained in [5].

Critically, let us emphasize on the difficulty of computing $\Gamma[\phi]$. It contains **all quantum fluctuations integrated out**. Instead, we can obtain useful quantities for low energy observers by following Wilson's idea, integrating out modes shell-by-shell. This will lead us to the Exact RG equation.

3.2 Wilsonian Renormalization group

In the Wilsonian picture we assume that there is an overall cutoff Λ (*e.g.* atomic spacing in a metal). "Then, all loops are finite, and it is possible to integrate over a shell of momentum in the path integral and change the couplings of the theory so that the low-energy physics is the same" [9]. The Wilsonian RGE describes the flow of the coupling constants under infinitesimal changes in the cutoff.

Consider the generating functional for a Euclidean theory expressed over a scalar field Φ representing all relevant degrees of freedom,

$$Z(\mathbf{g}) = \int_\Lambda \mathcal{D}[\Phi] e^{-S[\Phi;\mathbf{g}]} \tag{3}$$

where $S[\Phi;\mathbf{g}]$ depends on a set of coupling constants which we collect in a vector $\mathbf{g} = (g_1, g_2, \dots)$. Here, we have imposed the cutoff Λ by setting $\Phi(p) = 0$ for $|p| > \Lambda$.We will calculate the resulting change of $S[\Phi;\mathbf{g}]$ as we integrate over shells of momentum.

3.3 Mode elimination

Let us eliminate the modes associated to short-distance fluctuations. Working in momentum space we integrate over all fields $\Phi(p)$ with wave vectors p up to the cutoff Λ. Now we want to find out how the action changes if we integrate out high momentum degrees of freedom, which are the momenta above some momentum scale k. This can be expressed as integrating over the momentum shell $|k| < |p| < \Lambda$. For the field, this amounts to writing

$$\Phi = \Phi^< + \Phi^> ,$$

where $\Phi^<$ are the slow modes, field configurations containing fluctuations with small wavevectors compared k, and $\Phi^>$ are the fast modes. For now, this distinction between low and high momenta is sharp. Namely,

$$\Phi^<(p) = \Theta(|k| - |p|)\Phi(p) \tag{4}$$

$$\Phi^>(p) = \Theta(|p| - |k|)\Phi(p)\,. \tag{5}$$

Let us define the modified action $S_k[\Phi^<;\mathbf{g}^<]$, now depending on k, after integrating out the fast modes as,

$$e^{-S_k[\Phi^<;\mathbf{g}^<]} = \int_\Lambda \mathcal{D}[\Phi^>] e^{-S[\Phi;\mathbf{g}]}\,. \tag{6}$$

The (vector) of coupling constants $\mathbf{g}^<$ is in general different from the original couplings $\mathbf{g}$. In this way, the generating functional takes the form

$$Z(\mathbf{g}) = \int_\Lambda \mathcal{D}[\Phi] e^{-S[\Phi;\mathbf{g}]} = \int \mathcal{D}[\Phi^<] \int_\Lambda \mathcal{D}[\Phi^>] e^{-S[\Phi;\mathbf{g}]} = \int \mathcal{D}[\Phi^<] e^{-S_k[\Phi^<;\mathbf{g}^<]}. \tag{7}$$

Unfortunately, it is quite difficult to perform the integration over the high momentum degrees of freedom, so one tries to find suitable approximations. Also, the number of coupling constants in this effective action S_k can be much higher that in the bare theory as higher dimension operators can appear.

4 From Wilsonian RG to the Exact Functional RG (FRG)

As discussed above, Wilson's idea was to integrate out the quantum fluctuations not all at once but successively from scale to scale, in momentum space shell by shell [2]. Wegner and Houghton were the first to derive an exact RGE. They took the limit to integrate over infinitesimal momentum shells [10]. "The resulting change of the effective action can be expressed in terms of a formally exact functional differential equation describing the evolution of the effective action as a function of the cutoff" [3]. There are two problems arising from the sharp cutoff at the scale k: First of all, applying this procedure to gauge theories is impossible as a gauge transformation would also act on high momentum modes, which can therefore not be eliminated by a sharp cutoff [1]. Another problem arises when looking at derivative terms of the field. Derivative terms require a spacetime-dependent field which requires a spacetime-dependent source. Momentum has to be conserved in each momentum shell which is not necessarily the case for a spacetime-dependent source with a sharp cutoff.

Generalizing the idea of making the transformation of the action continuous leads us to the *functional renormalization group* (FRG): We make use of functional representations of the various types of generating functionals to obtain *formally exact* generating functional differential equations describing the *change* of the generating functionals due to an infinitesimal change of the *momentum shell parameter* k [3].

Now, we want to derive Polchinski's equation. Instead of using a sharp cutoff at the scale k as Wilsonian RG in section 3.3, we smooth out our cutoff, as presented by Morris [4]. More specifically, we define a smooth cutoff for the propagator of the theory, s.t. its inverse is well defined. We will use an ϵ *-smoothed* step function that reduces to the Heaviside step function as,

$$\Theta_\epsilon(p,k) \to \Theta(p-k) \text{ as } \epsilon \to 0\,, \tag{8}$$

it is bounded from above (below) by 1 (0) as in Figure 1, satisfying,

$$\Theta_\epsilon(p,k) \approx 1 \text{ for } p > k + \frac{\epsilon}{2}\,,$$

$$\Theta_\epsilon(p,k) \approx 0 \text{ for } p < k - \frac{\epsilon}{2}\,,$$

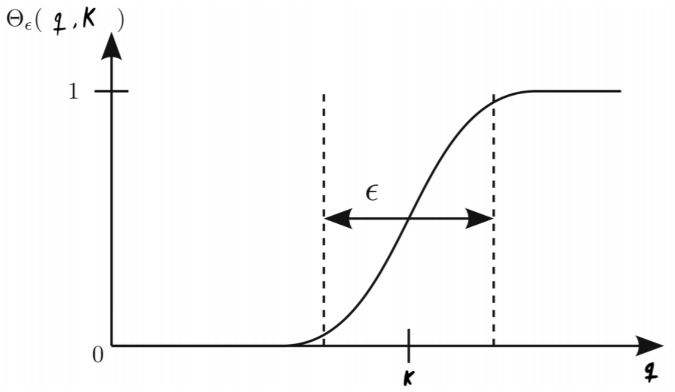

Figure 1: $\epsilon-$ smoothed step function in momentum space with step width of order ϵ (Modified from source [3]).

Furthermore, in the Wilsonian RG we started by integrating out the high energy modes directly in the generating functional $Z[J]$. It is intuitively clear that this will lead to other generating functionals such as $W[J]$ and the effective action without these modes. The formal connection was first spelled out by Morris [4]. Below we will assume an UV cutoff Λ, while k is a momentum-shell parameter, also called IR cutoff, with $k < \Lambda$. We will use the notation,

$$\varphi.J = \varphi_x J_x = \int d^D x \varphi(x) J(x)\,,$$

and $\Delta(x,y)$ for the propagator and other Green functions of two arguments,

$$\varphi.\Delta^{-1}.\varphi = \varphi_x \Delta^{-1}_{x\,y} \varphi_y = \int \frac{d^D p}{(2\pi)^D} \varphi(p)\Delta^{-1}(p)\varphi(-p)\,.$$

Let us write the generating functional for the scalar field φ with the quadratic part manifestly apart from the bare **interaction** $S^{int}_\Lambda[\varphi]$ regulated by the UV cutoff Λ,

$$Z[J] = \int \mathcal{D}\varphi e^{-\frac{1}{2}\varphi.\Delta^{-1}.\varphi - S^{int}_\Lambda[\varphi] + \varphi.J}\,,$$

where, as before, the corresponding propagator is Δ. $Z[J]$ can be rewritten, up to a multiplicative factor, in terms of two propagators and two fields as

$$Z[J] = \int \mathcal{D}\varphi_> \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi_>.\Delta_>^{-1}.\varphi_> - \frac{1}{2}\varphi_<.\Delta_<^{-1}.\varphi_< - S_\Lambda^{int}[\varphi_> + \varphi_<] + (\varphi_> + \varphi_<).J\right). \quad (9)$$

where we have defined a partition of the propagator

$$\Delta = \Delta_> + \Delta_< ,$$

where the individual parts are given by

$$\Delta_<(p) = [1 - \Theta_\epsilon(p,k)]\Delta(p) \text{ and } \Delta_>(p) = [\Theta_\epsilon(p,k) - \Theta_\epsilon(p,\Lambda)]\Delta(p). \quad (10)$$

$\Delta_<(p)$ approximately vanishes for momenta p larger than the momentum-shell parameter k plus an $\epsilon/2$ shift, while $\Delta_>(p)$ does **not** vanish for momenta in between the IR (k) and UV (Λ) cutoff up to an ϵ shift, $\{k + \epsilon/2, \Lambda - \epsilon/2\}$, such that,

$$(1 - \Theta_\epsilon(p,\Lambda))\Delta(p) = \Delta_>(p) + \Delta_<(p) ,$$

where we have made explicit the UV cutoff in the last expression for the full propagator $\Delta(p)$. Additionally to this partition of the propagator, we defined some partition of the field, $\varphi = \varphi_> + \varphi_<$. Notice that the mixed terms are not included in Eq.(9). This is because we can get rid of them using a specific transformation, $\varphi_< = \varphi'_< + (\Delta_</\Delta).\varphi$ and then integrating over $\varphi'_<$.

Notice that in the limit $\epsilon \to 0$ (and only in this limit!) we recover the sharp cutoff for the Heavyside step function, and $\varphi_<(\varphi_>)$ correspond to the low (high) momentum degrees of freedom, just like in section 3.3.

Below we will take the limit $\epsilon \to 0$, but it is key to note that the inverses defining the differential operators $\Delta^{-1}_{<>}$ are only well defined if they are nonzero for all p at intermediate stages of the deduction.

Now, let us perform *the mode-elimination step of the Wilsonian RG exactly*. First, isolate the integral over the high momentum modes $\varphi_>$,

$$Z[J] = \int \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi_<.\Delta_<^{-1}.\varphi_<\right) Z_k[\varphi_<, J] \quad (11)$$

where we have defined $Z_k[\varphi_<, J]$ depending on the low momentum modes and the momentum-shell parameter k,

$$\begin{aligned} Z_k[\varphi_<, J] &= \int \mathcal{D}\varphi_> \exp\left(-\frac{1}{2}\varphi_>.\Delta_>^{-1}.\varphi_> - S_\Lambda^{int}[\varphi_> + \varphi_<] + (\varphi_> + \varphi_<).J\right) \quad (12) \\ Z_k[\varphi_<, J] &= \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}(\varphi - \varphi_<).\Delta_>^{-1}.(\varphi - \varphi_<) - S_\Lambda^{int}[\varphi] + \varphi.J\right), \end{aligned}$$

where we have used $\varphi_> = \varphi - \varphi_<$, noting that with respect to $Z_k[\varphi_<, J]$, the shift of $-\varphi_<$ over the integration variable ($\varphi_>$) is fixed, and hence $\mathcal{D}\varphi_> = \mathcal{D}\varphi$. Now, expanding we find,

$$Z_k[\varphi_<, J] = \exp\left(-\frac{1}{2}\varphi_<.\Delta_>^{-1}.\varphi_<\right) \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi.\Delta_>^{-1}.\varphi + \varphi.(J + \Delta_>^{-1}.\varphi_<) - S_\Lambda^{int}[\varphi]\right)$$

which can be integrated as follows: Let us use the standard result for the normalized generating functional for an interacting theory (See pages 197 − 199 in [8]),

$$Z_k[\varphi_<, J] = \exp\left(-S_\Lambda^{int}\left[\frac{\delta}{\delta J}\right]\right) Z_k^{(0)}[\varphi_<, J],$$

where $Z_k^{(0)}$ is the generarting functional for the free theory. Namely, disregarding for the moment the bare interaction with UV cutoff Λ, $S_\Lambda^{int}[\varphi]$,

$$\begin{aligned} Z_k^{(0)}[\varphi_<, J] &= \exp\left(-\frac{1}{2}\varphi_<.\Delta_>^{-1}.\varphi_<\right) \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi.\Delta_>^{-1}.\varphi + \varphi.(J + \Delta_>^{-1}.\varphi_<)\right) \\ Z_k^{(0)}[\varphi_<, J] &= \mathcal{N} \exp\left(-\frac{1}{2}\varphi_<.\Delta_>^{-1}.\varphi_<\right) \exp\left(\frac{1}{2}(J + \Delta_>^{-1}.\varphi_<).\Delta_>.(J + \Delta_>^{-1}.\varphi_<)\right) \end{aligned}$$

where we have used the standard result for Gaussian integrals,

$$\int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi.\Delta^{-1}.\varphi - B.\varphi - C\right) = \mathcal{N}\exp\left(\frac{1}{2}B.\Delta.B - C\right),$$

using the normalization $\mathcal{N} = \left(\det\frac{\Delta_>^{-1}}{2\pi}\right)^{-1/2}$, with $B = J + \Delta_>^{-1}.\varphi_<$. After integration over φ we find

$$Z_k[\varphi_<, J] = \mathcal{N}\exp\left(-\frac{1}{2}\varphi_<.\Delta_>^{-1}.\varphi_<\right)\exp\left(-S_\Lambda^{int}\left[\frac{\delta}{\delta J}\right]\right)\exp\left(\frac{1}{2}(J+\Delta_>^{-1}.\varphi_<).\Delta_>.(J+\Delta_>^{-1}.\varphi_<)\right).$$

Finally, note the following identity which can be directly verified expanding all products,

$$\exp\left(-\frac{1}{2}\varphi_<.\Delta_>^{-1}.\varphi_<\right) = \exp\left(\frac{1}{2}J.\Delta_>.J + J.\varphi_<\right)F[-1, J],$$

where,

$$F[s, J] = \exp\left(s\frac{1}{2}(J+\Delta_>^{-1}.\varphi_<).\Delta_>.(J+\Delta_>^{-1}.\varphi_<)\right),$$

we can rewrite the above expression as,

$$Z_k[\varphi_<, J] \quad = \quad \mathcal{N}\exp\left(\frac{1}{2}J.\Delta_>.J + J.\varphi_<\right)F[-1, J]\exp\left(-S_\Lambda^{int}\left[\frac{\delta}{\delta J}\right]\right)F[+1, J].$$

Performing all functional derivatives of the bare interaction part, one obtains either $\Delta_>.J + \varphi_<$ or $\Delta_>$. Thus, we obtain a functional depending on the momentum-shell parameter k, $S_k^{int}[\Delta_>.J + \varphi_<]$, such that

$$\exp(-S_k^{int}[\Delta_>.J + \varphi_<]) = F[-1, J]\exp\left(-S_\Lambda^{int}\left[\frac{\delta}{\delta J}\right]\right)F[+1, J],$$

which reduces to the average generating functional depending only on the low momentum modes,

$$Z_k[\varphi_<, J] = \mathcal{N}\exp\left(\frac{1}{2}J.\Delta_>.J + J.\varphi_< - S_k^{int}[\Delta_>.J + \varphi_<]\right). \tag{13}$$

Let us explore our results further by taking the limit $\epsilon \to 0$. There are the following two cases: If J only couples to the low momentum modes, $J(p) = 0$ for all $p > k$, then the only J contribution is $J.\varphi_<$. This means that the generating funtional from Eq.(11) in this limit simplifies to

$$Z[J] \to \int \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi_<.\Delta_<^{-1}.\varphi_< - S_k^{int}[\varphi_<] + J.\varphi_<\right), \tag{14}$$

which is exactly the Wilsonian effective action from section 3.3.

Also note, if we set $\varphi_< \equiv 0$ in Eq.(12) we get a standard partition function for a field $\varphi_>$ with an imposed IR cutoff k. Thus, let us define,

$$W_k[\varphi_<, J] = \ln Z_k[\varphi_<, J] = \frac{1}{2}J.\Delta_>.J + J.\varphi_< - S_k^{int}[\Delta_>.J + \varphi_<],$$

which *only* holds relation with a generating functional of connected Green functions if $\varphi_< \equiv 0$. Namely,

$$W_k[0, J] = \frac{1}{2}J.\Delta_>.J - S_k^{int}[\Delta_>.J].$$

Thus, in the $\epsilon \to 0$ limit, S_k^{int} *separates into*:

- low momenta, corresponding to $\varphi_<$, where S_k equals the Wilsonian effective action and
- high momenta, corresponding to $\Delta_>.J$, related to the generator of connected Green functions.

Note that the derivative respect to k of $\Delta_>(p)$ (the propagator for high energy modes with the ϵ-smoothed step function) takes a *regulator* character,

$$-\frac{d}{dk}\Delta_>(p) = \delta_\epsilon(p,k)\Delta(p) =: K_k(p)\,,$$

where, by definition of $\theta_\epsilon(p,k)$, we recognize the Dirac delta,

$$\delta_\epsilon(p,k) = -\frac{d}{dk}\theta_\epsilon(p,k) \to \delta(p-k) \text{ as } \epsilon \to 0.$$

The former behaviour is similar to the behaviour of the *ad-hoc* regulator introduced by hand in the modern approach to FRG that we detail below (see figure 3).

From the definition for $Z_k[\varphi_<, J]$, (12), and using again that $\varphi - \varphi_< = \varphi_>$,

$$\begin{aligned}
\frac{d}{dk}Z_k[\varphi_<, J] &= \int \mathcal{D}\varphi_> \left(-\frac{1}{2}\varphi_>.\frac{d}{dk}\Delta_>^{-1}.\varphi_>\right)\exp\left(-\frac{1}{2}\varphi_>.\Delta_>^{-1}.\varphi_> - S_\Lambda^{int}[\varphi_> + \varphi_<] + (\varphi_> + \varphi_<).J\right)\\
&= -\frac{1}{2}\int \mathcal{D}\varphi_>(\varphi - \varphi_<).\frac{d}{dk}\Delta_>^{-1}.(\varphi - \varphi_<)\exp\left(-\frac{1}{2}\varphi_>.\Delta_>^{-1}.\varphi_> - S_\Lambda^{int}[\varphi_> + \varphi_<] + (\varphi_> + \varphi_<).J\right)\\
\frac{d}{dk}Z_k[\varphi_<, J] &= -\frac{1}{2}\left(\frac{\delta}{\delta J} - \varphi_<\right).\frac{d}{dk}\Delta_>^{-1}.\left(\frac{\delta}{\delta J} - \varphi_<\right)Z_k[\varphi_<, J] \qquad (15)
\end{aligned}$$

substituting in (13), and defining $\Phi = \Delta_>.J + \varphi_<$, we find for the LHS

$$\begin{aligned}
\frac{d}{dk}Z_k[\varphi_<, J] &= \left(\frac{1}{2}J.\left(\frac{d}{dk}\Delta_>\right).J - \frac{d}{dk}S_k^{int}[\Phi] + \frac{d\mathcal{N}}{dk}\right)Z_k[\varphi_<, J]\\
&= \left(\frac{1}{2}J.\left(\frac{d}{dk}\Delta_>\right).J - \frac{\partial}{\partial k}S_k^{int}[\Phi] - \frac{\delta S_k^{int}}{\delta\Phi}.\left(\frac{d}{dk}\Delta_>\right).J - \frac{1}{2}\frac{d}{dk}\text{Tr}\log(\Delta_>^{-1})\right)Z_k[\varphi_<, J]\\
&= \left(-\frac{1}{2}J.K_k.J - \frac{\partial}{\partial k}S_k^{int}[\Phi] + \frac{\delta S_k^{int}}{\delta\Phi}.K_k.J - \frac{1}{2}\text{Tr}\Big(\frac{d}{dk}\Delta_>^{-1}.\Delta_>\Big)\right)Z_k[\varphi_<, J].
\end{aligned}$$

Notice that the functional derivative wrt. to J acting on Z_k in Eq.(13) gives

$$\frac{\delta}{\delta J}Z_k[\varphi_<, J] = \left(\Delta_>.J + \varphi_< - \Delta_>\frac{\delta S_k^{int}}{\delta\Phi}\right)Z_k[\varphi_<, J],$$

s.t. for the RHS we find

$$\begin{aligned}
&-\frac{1}{2}\left(\frac{\delta}{\delta J} - \varphi_<\right).\left(\frac{d}{dk}\Delta_>^{-1}\right).\left(\frac{\delta}{\delta J} - \varphi_<\right)Z_k[\varphi_<, J]\\
&= -\frac{1}{2}\left(\frac{\delta}{\delta J} - \varphi_<\right).\left(\frac{d}{dk}\Delta_>^{-1}\right).\Delta_>.\left(J - \frac{\delta S_k^{int}}{\delta\Phi}\right)Z_k[\varphi_<, J]\\
&= -\frac{1}{2}\left[\left(J - \frac{\delta S_k^{int}}{\delta\Phi}\right).K_k.\left(J - \frac{\delta S_k^{int}}{\delta\Phi}\right) + \text{Tr}\left(\Big(\frac{d}{dk}\Delta_>^{-1}\Big).\Delta_>\right) - \text{Tr}\left(K_k.\frac{\delta^2 S_k^{int}}{\delta\Phi\delta\Phi}\right)\right]Z_k[\varphi_<, J].
\end{aligned}$$

Equating both sides and dividing by Z_k gives

$$-\frac{1}{2}J.K_k.J - \frac{\partial}{\partial k}S_k^{int}[\Phi] + \frac{\delta S_k^{int}}{\delta\Phi}.K_k.J = -\frac{1}{2}J.K_k.J + \frac{\delta S_k^{int}}{\delta\Phi}.K_k.J - \frac{1}{2}\frac{\delta S_k^{int}}{\delta\Phi}.K_k.\frac{\delta S_k^{int}}{\delta\Phi} + \frac{1}{2}\text{Tr}\left(K_k.\frac{\delta^2 S_k^{int}}{\delta\Phi\delta\Phi}\right).$$

Cancelling equal terms on both sides, we finally obtain Polchinski's equation[6],

$$\frac{\partial}{\partial k}S_k^{int}[\Phi] = \frac{1}{2}\frac{\delta S_k^{int}}{\delta\Phi}.K_k.\frac{\delta S_k^{int}}{\delta\Phi} - \frac{1}{2}\text{Tr}\left(K_k.\frac{\delta^2 S_k^{int}}{\delta\Phi\delta\Phi}\right). \qquad (16)$$

5 The Wetterich equation

The idea of this approach is combine Wilson's strategy to integrate out the fluctuations not all at once but to do it momentum shell by shell with an IR regulator[2]. Our new effective average action which includes this IR regulator should integrate out all fluctuations with momenta larger that the regulator scale[11].

5.1 Average Effective Action and IR Regularization

We define an average effective action Γ_k in the following way: It should interpolate between the effective action Γ including all fluctuations and the bare action,

$$\Gamma_{k\to\Lambda} \simeq S_{bare},\ \Gamma_{k\to 0} = \Gamma. \tag{17}$$

Similar to the effective action, we can write the average effective action as a Legendre transformation of $W_k[J] = ln(Z_k[J])$. The generating functional Z_k is the usual generating functional with another IR regulator term

$$Z_k[J] = e^{W_k[J]} = \int_\Lambda D\varphi\ e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi}. \tag{18}$$

The IR regulator term is

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D}\varphi(-q)R_k(q)\varphi(q). \tag{19}$$

Notice that we defined it to be quadratic in φ s.t. it looks like a mass term. If we require

$$\lim_{q^2/k^2\to 0} R_k(q) > 0 \tag{20}$$

then for $k^2 > q^2$ the regulator is positive and finite, which is actually often written more specifically as $R_k(q) \simeq k^2$ for $k^2 > q^2$. That means the IR modes get heavy and decouple, s.t. we have defined an IR regulator term. When $k^2 \to \Lambda \to \infty$ all fluctuations get suppressed and therefore we recover the bare action in this limit. Now we can find conditions for the regulator function R_k s.t. the average effective action satisfies the starting conditions from Eq.(17). If we set

$$\lim_{k^2/q^2\to 0} R_k(q) = 0, \tag{21}$$

which implies that $R_{k\to 0}(q) = 0$, then $Z_{k\to 0}[J] = Z[J]$. Hence, when all fluctuations are integrated out we recover the effective action, $\Gamma_{k\to 0} = \Gamma$. Eq. (21) also implies that $R_k(q) \to 0$ for $q^2 > k^2$ s.t. momenta higher than k are unaffected.

A sketch how this regulator could look like is in Fig. 3.

5.2 Derivation of the Flow Equation

We will now derive the flow equation of the average effective action. That means we are interested in the derivative $\partial_t \Gamma_k$, where $t = ln(\frac{k}{\Lambda})$ and $\partial_t = k\frac{d}{dk}$. With this definition, we find for the flow of the generating functional of connected correlators

$$\begin{aligned}
\partial_t W_k &= \frac{1}{Z_k}\partial_t Z_k = \frac{1}{Z_k}\partial_t\Big(\int_\Lambda D\varphi\ e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi}\Big) \\
&= \frac{1}{Z_k}\int_\Lambda D\varphi\ (-\partial_t\Delta S_k[\varphi])\ e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi} \\
&= \frac{1}{Z_k}\int_\Lambda D\varphi\ \Big(-\partial_t\ \frac{1}{2}\int\frac{d^D q}{(2\pi)^D}\varphi(-q)R_k(q)\varphi(q)\Big)\ e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi} \\
&= -\frac{1}{2}\frac{1}{Z_k}\int_q(\partial_t R_k(q))\int_\Lambda D\varphi\ \varphi(-q)\varphi(q)\ e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi} \\
&= -\frac{1}{2}\int_q(\partial_t R_k(q))\ \langle\varphi(-q)\varphi(q)\rangle \\
&= -\frac{1}{2}\int_q(\partial_t R_k(q))\ \big(G_k(q) + \langle\varphi(-q)\rangle\langle\varphi(q)\rangle\big) \\
&= -\frac{1}{2}\int_q(\partial_t R_k(q))\ G_k(q) - \partial_t\Delta S_k[\langle\varphi\rangle],
\end{aligned}$$

were we abbreviated $\int \frac{d^D q}{(2\pi)^D} = \int_q$ and defined the connected propagator

$$\begin{aligned} G_k(q) &= \frac{\delta^2 W_k}{\delta J(-q)\delta J(q)} = \frac{\delta}{\delta J(-q)} \frac{1}{Z_k} \frac{\delta Z_k}{\delta J(q)} \\ &= \frac{1}{Z_k} \frac{\delta^2 Z_k}{\delta J(-q)\delta J(q)} - \frac{1}{Z_k^2} \frac{\delta Z_k}{J(-q)} \frac{\delta Z_k}{J(q)} \\ &= \langle \varphi(-q)\varphi(q)\rangle - \langle \varphi(-q)\rangle\langle \varphi(q)\rangle. \end{aligned}$$

Using $\phi = \langle\varphi\rangle$ this calculation can be summarized as

$$\partial_t W_k = -\frac{1}{2} \int_q (\partial_t R_k(q))\, G_k(q) - \partial_t \Delta S_k[\phi]. \tag{22}$$

But our aim was actually to find a flow equation for the effective average action. For this we follow the ideas of 3.1 where we wrote the average effective action as a Legendre transformation with a modification

$$\Gamma_k[\phi] = \sup_J \Big(\int J\phi - W_k[J] \Big) - \Delta S_k[\phi]. \tag{23}$$

Then the supremum is approached at $J = J_{sup}$ and again

$$0 = \frac{\delta}{\delta J(x)} \Big(\int J\phi - W_k[J] \Big) = \phi(x) - \frac{\delta W_k[J]}{\delta J(x)} = \phi(x) - \langle\varphi\rangle_J. \tag{24}$$

From this it also follows that

$$\frac{\delta\phi(x)}{\delta J(y)} = \frac{\delta^2 W_k[J]}{\delta J(x)\delta J(y)} = G_k(x-y). \tag{25}$$

The quantum equation of motion becomes

$$\begin{aligned} \frac{\delta\Gamma_k[\phi]}{\delta\phi(x)} &= J(x) + \cancel{\int_y \frac{\delta J(y)}{\delta\phi(x)} \phi(y)} - \cancel{\int_y \frac{\delta W_k[J]}{\delta J(y)} \frac{\delta J(y)}{\delta\phi(x)}} - \frac{\delta\Delta S_k[\phi]}{\delta\phi(x)} \\ &= J(x) - (R_k\phi)(x) \end{aligned} \tag{26}$$

and therefore

$$\frac{\delta J(x)}{\delta\phi(y)} = \frac{\delta^2\Gamma_k[\phi]}{\delta\phi(x)\delta\phi(y)} + R_k(x,y). \tag{27}$$

Putting together eq.(25) and eq.(27) we find

$$\begin{aligned} \delta(x-x') &= \frac{\delta J(x)}{\delta J(x')} = \int_y \frac{\delta J(x)}{\delta\phi(y)} \frac{\delta\phi(y)}{\delta J(x')} \\ &= \int_y \Big(\frac{\delta^2\Gamma_k[\phi]}{\delta\phi(x)\delta\phi(y)} + R_k(x,y) \Big)\, G_k(y-x'), \end{aligned}$$

which we can also write as

$$\mathbf{1} = (\Gamma_k^{(2)} + R_k) G_k, \tag{28}$$

where

$$\Gamma_k^{(n)} = \frac{\delta^n\Gamma_k[\phi]}{\delta\phi\ldots\delta\phi}. \tag{29}$$

Therefore, the derivative of the average effective action for ϕ fixed at $J = J_{sup}$ becomes

$$\begin{aligned} \partial_t\Gamma_k[\phi] &= \partial_t \Big(\int J\phi - W_k[J] - \Delta S_k[\phi] \Big) \\ &= -\partial_t W_k[J] - \partial_t\Delta S_k[\phi] \\ &\overset{(22)}{=} \frac{1}{2} \int_q (\partial_t R_k(q))\, G_k(q) \\ &\overset{(28)}{=} \frac{1}{2} \int_q \partial_t R_k(q) (\Gamma_k^{(2)} + R_k)^{-1}. \end{aligned}$$

To get the final form of the flow equation, we rewrite the integral over q as a trace

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} Tr\Big[\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1}\Big]. \tag{30}$$

So, we have found a functional differential equation for the effective average action. This equation already defines the whole theory[2].

Equation (30) is exact. With the full propatagor $G_k = (\Gamma_k^{(2)} + R_k)^{-1}$ in the trace it has a one-loop structure, as diagrammatically shown in figure 2.

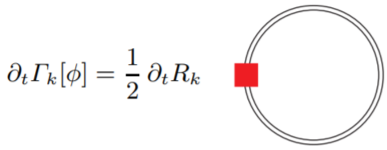

Figure 2: Diagram of the flow equation (30). The double line corresponds to the propagator G_k and the red box to the insertion of $\partial_t R_k$[2].

Notice that with R_k in the denominator of Eq. (30) we have constructed it to be IR regularized. But additionally to that, the derivative $\partial_t R_k$ is supported only in a small momentum shell near $p^2 \sim k^2$ and therefore acts as a UV regulator[2]. This is visualized in figure 3.

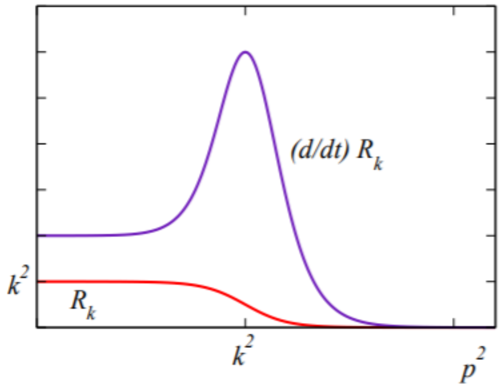

Figure 3: The red curve is the regulator and the blue one is its derivative as a function of p^2.[2].

When solving the flow equation we find a trajectory in the space of possible theories where the two ends of the trajectory correspond to the full effective action and the bare action. A sketch of this can be seen in figure 4.

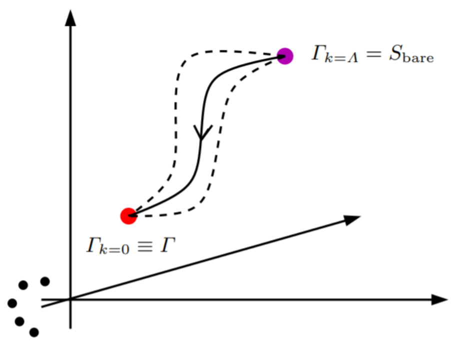

Figure 4: RG flow in theory space. The axis label different operators of the theory. Different IR regulators give different trajectories with same end points[2].

As a consistency check consider the the loop expansion of $\Gamma_k = S + \hbar\Gamma_k^{1-loop} + \mathcal{O}(\hbar^2)$. If we now consider the flow equation at one-loop order, we can replace $\Gamma_k^{(2)}$ with $S^{(2)}$ on the RHS of Eq. (30).

So, the flow equation for the one-loop effective action simplifies to

$$\partial_t \Gamma_k^{1-loop} = \frac{1}{2} Tr \left[\partial_t R_k (S_k^{(2)} + R_k)^{-1} \right] = \frac{1}{2} \partial_t Tr\, ln(S_k^{(2)} + R_k). \tag{31}$$

Hence, we find for the one-loop effective action

$$\Gamma_k^{1-loop} = S + \frac{1}{2} Tr\, ln(S_k^{(2)} + R_k) + \text{const}, \tag{32}$$

which looks familiar as it is just the standard formula for the one-loop effective action.

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