Exact Renormalization Group Equation, and Asymptotic Safety

Mauricio Valencia-Villegas

February 2021

Contents

1 Application of the Exact Renormalization Group Equation (ERGE) to gravity

The objective in this talk is to compute the beta functions of Newton's constant, and of the cosmological constant using the ERGE. In particular, we will use the Wetterich equation, and we will restrict to the Einstein-Hilbert truncation. This talk will be amply based on the references [1, 2, 3, 4].

In the second part of this talk, we will use the beta functions to discuss the proposal of Asymptotic Safety in gravity, as well as the criticisms to such scenario. This discussion will be based in [1, 2, 3, 4, 5, 6, 7, 8]

1.1 Review of ERGE, and notation.

1.1.1 The Wetterich equation, and the regulator

Let us recall the motivation behind the ERGE: we implement the Wilsonian idea of integrating out "one momentum shell at a time", momenta shell-by-shell, instead of all at once [9, 10]. Practical implementations of this idea lead to the Wegner-Houghton equation [11], the Polchinski equation, which solved some issues introducing a smooth screening of modes [12, 13], and finally, lead us to the Wetterich equation [14, 15, 16].

For the Wetterich equation, we use the effective action and smooth regulators. Furthermore, we introduce a momentum-shell parameter, k, and we write a k-dependent effective action, also called *averaged effective action*,

$$\Gamma_k(\phi) = \sup_J \left(\int J\phi - W_k(J) \right) - \Delta S_k(\phi)\,,$$

where $W_k(J) = \log Z_k(J)$ is the generator of connected correlators, which also depends on the scale k through the generating functional $Z_k(J)$, computed with the action $S + \Delta S_k$. Let us recall that $\Delta S_k(\phi)$ is the regulator whose general properties we specify below. In the effective action, ϕ is the vacuum expectation value of the field.

However, *resorting to clarity from the context, below we will write both the fields, and their vacuum expectation values with the same symbol.*

Finally, let us recall that Γ_k interpolates between the bare action $S(\phi)$ and the full effective action $\Gamma(\phi)$,

$$\lim_{k\to\Lambda_{UV}} \Gamma_k \approx S_{\text{bare}} \qquad\qquad \lim_{k\to 0} \Gamma_k = \Gamma\,,$$

where we emphasize on the notation Λ_{UV} for an UV cutoff, as oposed to the symbol Λ, which we will use below to denote the Cosmological constant in the Einstein-Hilbert action.

The regulator is quadratic in the fields. It has the general form,

$$\Delta S_k(\phi) = \frac{1}{2}\int d^dx\,\sqrt{g}\,\phi R_k(\Delta)\phi\,,$$

where Δ is the cutoff operator [1]. Hence, R_k has been written as a kernel, but we will also write the functional form $R_k(z)$, where z can be thought as eigenvalues of Δ. **In flat spacetime** z is just the momentum squared.

Let us recall the general properties of the regulator [1, 15]:

- As the scale $k \to 0$, $R_k \to 0$, in order to recover the usual 1PI action from Γ_k at the deep IR.
- For $z > k^2$, then $R_k \to 0$ sufficiently fast, in order to leave undisturbed the integration over UV modes.
- For $z \leq k^2$, then $R_k(z) = k^2$, in order to "freeze" the IR degrees of freedom by giving them an effective mass k.

Finally, let us recall the general form of the Wetterich ERGE, or flow equation,

$$\frac{d\Gamma_k(\phi)}{dt} = \frac{1}{2}\text{Tr}\left[\partial_t R_k\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\right] \tag{1}$$

where $t = \ln\left(\frac{k}{\Lambda_{UV}}\right)$.

1.1.2 Extracting the beta functions from the ERGE

Let us now find the Beta functions using the ERGE equation (1) for the averaged effective action, Γ_k.

To approximate the solution to the ERGE we will take a derivative expansion,

$$\Gamma_k(\phi) = \sum_i g_i(k)\mathcal{O}_i(\phi)\,, \tag{2}$$

where $\mathcal{O}_i(k)$ are integrals of monomials in the field and its derivatives, and $g_i(k)$ are running couplings. Now, it is clear how to obtain the Beta functions from the ERGE. First, plugging-in the derivative expansion (2) on the left hand side of (1), we obtain,

$$\frac{d\Gamma_k[\phi]}{dt} = \sum_i \beta_i\mathcal{O}_i\,, \tag{3}$$

where we have identified the Beta functions as,

$$\beta_i(g_j, k) = \frac{dg_i}{dt}\,, \tag{4}$$

where the g_j dependance for β_i denotes that the latter can be functions of all, or a subset of the couplings, and they depend on the scale[1] k [1]. Now, we expand the trace in the right hand side

[1]Let us stress that although k is usually denoted IR cutoff, curing IR divergences is not the primary purpose of the scale k. It denotes the momentum scale above which the momentum modes are integrated out.

of the ERGE (1) on the basis of operators $\{\mathcal{O}_i\}$, such that we can extract the β_i by identifying the coefficient of $\mathcal{O}_i$.

An example of such a procedure was shown in the Anharmonic oscillator in the second session of the current topic. However, the operator basis chosen for this example contained no derivatives (Local potential approximation). Below, we will perform a derivative expansion in powers of the curvature invariants.

1.2 The flow equation for gravity

In order to apply ERGE to gravity, it is convenient not only technically but also conceptually to use the background field method. We consider the (euclidean) Einstein-Hilbert action,

$$S(g) = \frac{1}{4\pi G}\int d^d x\,\sqrt{g}\,(2\Lambda - R)\,, \tag{5}$$

and we decompose the metric ($g_{\mu\nu}$) into a fixed background ($\bar{g}_{\mu\nu}$) and a quantum fluctuation ($h_{\mu\nu}$),

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,. \tag{6}$$

Below, we take the convention that indices are raised and lowered with $\bar{g}$. Furthermore, the inverse takes the form,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_\alpha^\nu + \dots\,.$$

The *conceptual* usefulness of the background field method to compute the ERGE is the following: in curved, dynamical spacetime there is no preferred definition of distance between points, nor of norm of momentum vectors, hence, it is difficult to define coarse graining without a background reference $\bar{g}_{\mu\nu}$. Since the ERGE explicitly depends on the momentum scale k, it is hard to follow an alternative method [1].

The technical usefulness is related to the definition of the gravitational path integral. This is briefly discussed as additional material in the section below 1.2.1. Only a few results will be used below in the Wetterich equation; hence, it is possible to refer to these particular equations only when they are needed.

Additional material.

1.2.1 The basics: The gravitational path integral, and the background field method

The gravitational path integral

We would like to write the generating functional integrating over geometries $\{g\}$, with a diffeomorphism-invariant action, modulo diffeomorphisms [1]. However, there is a simpler approach using the background field method, which, in any case, we will need to use to compute the ERGE. In this case, we can use the standard Faddeev-Popov procedure. Hence, the euclidean functional integral with the regulator is,

$$Z_k(j,\bar{J},J;\bar{g}) = \int dh\,dC\,d\bar{C}\exp\Big(-S(h;\bar{g}) - S_{GF}(h;\bar{g}) - S_{gh}(\bar{C},C;\bar{g}) - \Delta S_k(h,\bar{C},C;\bar{g}) + \int d^dx\,\sqrt{\bar{g}}j^{\mu\nu}h_{\mu\nu} + J^\mu\bar{C}_\mu + \bar{J}^\mu C_\mu\Big)\,,$$

where $\bar{C}^\mu$ and C_μ are the Faddeev-Popov ghosts, anti-commuting Grassmann-valued fields. The gauge fixing term is quadratic in h,

$$S_{GF}(h;\bar{g}) = \frac{1}{2\alpha}\frac{1}{16\pi G}\int d^dx\sqrt{\bar{g}}\left(\bar{\nabla}^\alpha h_{\alpha\mu} - \frac{1}{2}\bar{\nabla}_\mu h\right)\bar{g}^{\mu\nu}\left(\bar{\nabla}^\alpha h_{\alpha\nu} - \frac{1}{2}\bar{\nabla}_\nu h\right)\,, \tag{7}$$

where α is the gauge parameter, and we have denoted,

$$h = \bar{g}^{\mu\nu}h_{\mu\nu}\,,$$

up to complementary discussions where we avoid explicitly writing Lorentz indices. The ghost action is,

$$S_{gh}(\bar{C},C;\bar{g}) = -\int d^dx\,\sqrt{\bar{g}}\,\bar{C}^\mu\,(\Delta_{(gh)})_\mu{}^\nu C_\nu\,, \tag{8}$$

where,

$$(\Delta_{(gh)})_\mu{}^\nu = \delta^\nu_\mu\bar{\nabla}^\sigma\nabla_\sigma + \bar{\nabla}^\nu\nabla_\mu - \bar{\nabla}_\mu\nabla^\nu\,, \tag{9}$$

where $\bar{\nabla}$ and ∇ are the Levi-Civita connections compatible with the background metric $\bar{g}$, and the complete metric g, respectively. Hence, this part of the action introduces ghost-h mixing terms (For a detailed deduction, see for instance [1] section 3.3).
In the effective action, these terms are valued on the vacuum expectation values of the fields. Below, we will take an approximation where $\langle h\rangle = 0$, hence, both covariant derivatives coincide, compatible with the background metric, such that the kinetic operator for the ghost term is written as,

$$(\Delta_{(gh)})_\mu{}^\nu = -\delta^\nu_\mu\bar{\nabla}^2 - \bar{R}_\mu{}^\nu\,. \tag{10}$$

Perturbations around a general background

To implement the background field method, we need the expansion of the action around a general background $\bar{g}_{\mu\nu}$,

$$S(g) = S(\bar{g}) + S^{(1)}(h;\bar{g}) + S^{(2)}(h;\bar{g}) + \,\ldots\,,$$

where the $S^{(n)}(h;\bar{g})$ term is,

$$S^{(n)}(h;\bar{g}) = \frac{1}{n!}\int dx_1\ldots dx_n\,\left.\frac{\delta^n S}{\delta g_{\mu_1\nu_1}(x_1)\,\ldots\,\delta g_{\mu_n\nu_n}(x_n)}\right|_{\bar{g}} h_{\mu_1\nu_1}(x_1)\,\ldots\,h_{\mu_n\nu_n}(x_n)\,.$$

For (1), we must compute the modified propagator of the form,

$$\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\,,$$

hence, we will be mostly interested in $S^{(2)}(h;\bar{g})$, which is quadratic in h. Furthermore, let us recall that to obtain the ERG Wetterich equation (1), we introduce a regulator that is also quadratic in the field h, much as the gauge fixing term (7).

Now, expanding (5) at quadratic order in h, and collecting terms together with the gauge fixing term (7), in the Feynman gauge $\alpha = 1$ (See further deatils in [1] section 3.4),

$$S^{(2)}(h;\bar{g}) + S_{GF} = \frac{1}{32\pi G}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}H^{\mu\nu\rho\sigma}h_{\rho\sigma} \tag{11}$$

$$\begin{aligned} H^{\mu\nu\rho\sigma} &= K^{\mu\nu\rho\sigma}(-\bar{\nabla}^2 - 2\Lambda) + U^{\mu\nu\rho\sigma} \\ K^{\mu\nu\rho\sigma} &= \frac{1}{4}(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} + \bar{g}^{\mu\sigma}\bar{g}^{\nu\rho} - \bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}) \\ U^{\mu\nu}_{\rho\sigma} &= \bar{R}K^{\mu\nu}_{\rho\sigma} + \frac{1}{2}(\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{R}^{\mu\nu}\bar{g}_{\rho\sigma}) - \delta^{(\mu}_{(\rho}\bar{R}^{\nu)}_{\sigma)} - \bar{R}^{(\mu\ \ \nu)}_{\ (\rho\ \ \sigma)}\,, \end{aligned} \tag{12}$$

where H is the Hessian (There are also notations where Hessian refers to the determinant). For our discussion of ERGE, we will not discuss the necessary, important changes that need to be done to H, which would take us far from our main objective (Plase note a detailed discussion in page 58 [1]). The Hessian to be used in this talk is,

$$H^{\mu\nu\rho\sigma} = K^{\mu\nu\alpha\beta}\Delta^\Lambda_{(h)\alpha\beta}{}^{\rho\sigma}\,, \tag{13}$$

where,

$$\Delta^\Lambda_{(h)\rho\sigma}{}^{\mu\nu} = -(\bar{\nabla}^2 + 2\Lambda)\mathbf{1}^{\mu\nu}_{\rho\sigma} + W_{\rho\sigma}{}^{\mu\nu} \tag{14}$$

$$W_{\rho\sigma}{}^{\mu\nu} = 2U^{\mu\nu}_{\rho\sigma} - \frac{d-4}{d-2}\bar{g}_{\rho\sigma}\left(\bar{R}^{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}^{\mu\nu}\right) \tag{15}$$

To define the gravitational averaged effective action Γ_k we add the following regulator, quadratic in h and the ghosts $\bar{C}C$,

$$\Delta S_k(h, C, \bar{C}; \bar{g}) = \frac{1}{2}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}\mathcal{R}_k^{\mu\nu\rho\sigma}(\bar{g})h_{\rho\sigma} + \int d^dx\sqrt{\bar{g}}\bar{C}_\mu\bar{g}^{\mu\nu}\mathcal{R}_k^{(\mathrm{gh})}(\bar{g})C_\nu\,. \tag{16}$$

Now, the effective averaged action is written as,

$$\Gamma_k(h, \bar{C}, C; \bar{g}) = -W_k(j, \bar{J}, J; \bar{g}) + \int d^dx\sqrt{\bar{g}}(j^{\mu\nu}h_{\mu\nu} + J^\mu\bar{C}_\mu + \bar{J}^\mu C_\mu) - \Delta S_k(h, \bar{C}, C; \bar{g})\,, \tag{17}$$

and the Wetterich equation can be written as follows,

$$\frac{d}{dt}\Gamma_k(\Phi; \bar{g}) = \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\Phi\delta\Phi}\right)^{-1}\frac{d}{dt}\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2\Delta S_k}{\delta\Phi\delta\Phi}\,, \tag{18}$$

where, as we emphasized before, we have denoted the quantum part of the fields as $\Phi = (h_{\mu\nu}, \bar{C}_\mu, C_\mu)$, and for simplicity, resorting to clarification from the context, we also denote with Φ their vacuum expectation values $\langle\Phi\rangle$.

In order to expand the ERGE, let us note that computing the $2-$ point vertex function $\Gamma_k^{(2)}$ we will find a Hessian with the following structure,

$$\begin{bmatrix} \frac{\delta^2\Gamma_k}{\delta h\delta h} & \frac{\delta^2\Gamma_k}{\delta h\delta C} & \frac{\delta^2\Gamma_k}{\delta h\delta\bar{C}} \\ \frac{\delta^2\Gamma_k}{\delta\bar{C}\delta h} & 0 & \frac{\delta^2\Gamma_k}{\delta\bar{C}\delta C} \\ \frac{\delta^2\Gamma_k}{\delta C\delta h} & \frac{\delta^2\Gamma_k}{\delta C\delta\bar{C}} & 0 \end{bmatrix},$$

where the mixed terms $h - C$ and $h - \bar{C}$ terms come from expanding the covariant derivatives in the ghost action (8). Let us also note that the regulator term only modifies the $h - h$ and $\bar{C} - C$ terms. This structure is similar for the inverse. Finally, let us also note that from the right-most term in (18) and the quadratic form of the regulator (16) there are no mixed terms $h - C$ and $h - \bar{C}$, hence, the trace can be written as the sum of two terms, although the mixed terms are still considered in the traces,

$$\frac{d\Gamma_k(\Phi; \bar{g})}{dt} = \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\Phi\delta\Phi}\right)^{-1}_{hh}\frac{d\mathcal{R}_k}{dt} - \mathrm{Tr}\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\Phi\delta\Phi}\right)^{-1}_{\bar{C}C}\frac{d\mathcal{R}_k^{gh}}{dt}\,. \tag{19}$$

The minus sign in front of the ghost contribution arises because of the anti-commuting nature of the ghosts, Grassmann-valued fields.

1.2.2 Single metric approximation for the ERGE

In order to compute the ERGE there are a number of possible approximations, for the most of this discussion we will use the *single metric approximation*.

Let us recall that the effective average action for gravity necessarily depends on two fields. Namely, not only the gauge fixing, but also, for instance, the regulator term (16) introduce separate dependances on the background $\bar{g}_{\mu\nu}$ and quantum fluctuation $h_{\mu\nu}$. For that reason, we have written explicitly $\Gamma_k[\Phi; \bar{g}]$. The approximation in this section consists in setting the *vev* of the fluctuation $h_{\mu\nu}$ to zero,

$$\langle h_{\mu\nu}\rangle = 0 \leftrightarrow \langle g_{\mu\nu}\rangle = \langle\bar{g}_{\mu\nu}\rangle\,, \tag{20}$$

such that we are left with a gauge invariant functional of the background alone $\bar{g}_{\mu\nu}$, which we denote as,

$$\bar{\Gamma}_k(\bar{g}) = \Gamma_k(0, 0, 0; \bar{g})\,.$$

We can also split the average effective action as,

$$\Gamma_k(h, \bar{C}, C; \bar{g}) = \bar{\Gamma}_k(\bar{g} + h) + \hat{\Gamma}_k(h, \bar{C}, C; \bar{g})\,, \tag{21}$$

where the first term on the right hand side depends only on the full field $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. In other words, the first term cannot carry h-dependance only, by the regulator quadratic on h, neither the gauge fixing. It is clear that,

$$\hat{\Gamma}_k(0, 0, 0; \bar{g}) = 0\,.$$

Now, the problem within the *single metric approximation* reduces to write the flow equation for $\bar{\Gamma}_k$ by putting all the vacuum expectation values of the fluctuations to zero. In such a case,

$$\nabla_\mu \to \bar{\nabla}_\mu \,,$$

in the ghost operator (9), thus obtaining (10), and the mixed terms in the Hessian for Γ_k do not appear anymore,

$$\begin{bmatrix} \frac{\delta^2\Gamma_k}{\delta h \delta h} & 0 & 0 \\ 0 & 0 & \frac{\delta^2\Gamma_k}{\delta \bar{C} \delta C} \\ 0 & \frac{\delta^2\Gamma_k}{\delta C \delta \bar{C}} & 0 \end{bmatrix} ,$$

such that (19) can be written as,

$$\frac{d\bar{\Gamma}_k(\bar{g})}{dt} = \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2\Gamma_k}{\delta h \delta h} + \mathcal{R}_k\right)^{-1}\frac{d\mathcal{R}_k}{dt} - \mathrm{Tr}\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2\Gamma_k}{\delta \bar{C} \delta C} + \mathcal{R}^{gh}\right)^{-1}\frac{d\mathcal{R}_k^{(gh)}}{dt} \,. \tag{22}$$

where the first term on the right hand side is entirely due to the metric fluctuations and the second due to the ghosts. Note, however, that the right hand side depends on Γ_k, while we wish a closed equation for $\bar{\Gamma}_k$. To do this, let us approach the same problem from a different perspective: the 1−loop approximation to the effective action. Below, we will identify the necessary assumptions to get a closed equation for $\bar{\Gamma}_k$ starting from (22).

Additional material. A review, and a "*renormalization group improvement*":

As a first plausible approximation for the flow equation of only $\bar{\Gamma}_k$, let us recall the structure of the full effective action (See for instance [17], section 11.4), without a regulator, and in flat space for simplicity,

$$\Gamma(\Phi) = S(\Phi) + \frac{1}{2}\mathrm{Tr}\,\log\left(\frac{\delta^2 S}{\delta\Phi\delta\Phi}\right) + \mathcal{O}(\hbar^2)\text{ Connected diagrams} + \text{Counter terms}\,,$$

where the second term on the right hand side includes the 1−loop contribution,

$$\Gamma^{(1-\mathrm{loop})}(\Phi) = S(\Phi) + \frac{1}{2}\mathrm{Tr}\,\log\left(\frac{\delta^2 S}{\delta\Phi\delta\Phi}\right) .$$

Furthermore, let us recall that the averaged effective action with the momentum scale (IR cutoff) k is obtained by the replacement, $S \to S + \Delta S_k$ and the substraction of the regulator ΔS_k from the Legendre transform. Hence, the 1−loop averaged effective action takes the form,

$$\begin{aligned} \Gamma_k^{(1-\mathrm{loop})}(\Phi) &= (S(\Phi) + \Delta S_k) + \frac{1}{2}\mathrm{Tr}\,\log\left(\frac{\delta^2(S+\Delta S_k)}{\delta\Phi\delta\Phi}\right) - \Delta S_k \\ \Gamma_k^{(1-\mathrm{loop})}(\Phi) &= S(\Phi) + \frac{1}{2}\mathrm{Tr}\,\log\left(\frac{\delta^2 S}{\delta\Phi\delta\Phi} + \mathcal{R}_k\right) , \end{aligned} \tag{23}$$

where we have used for the argument of the logarithm, the property that the regulator is quadratic in the field.
Let us also note that taking ∂_t at both sides of (23) we obtain an analogous structure as Wetterich ERGE, with Γ_k replaced by the scale independent S. This is of course, a 1−loop approximation of ERG flow equation (1),

$$\partial_t\Gamma_k^{(1-\mathrm{loop})} = \frac{1}{2}\mathrm{Tr}\,\left(\left(\frac{\delta^2 S}{\delta\Phi\delta\Phi} + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right) ,$$

hence, the replacement $S \to \Gamma_k$ on the right hand side of the latter equation is sometimes called "*renormalization group improvement*" in the literature [1], which would bring us to (1). We will use this replacement below.

Now, let us come back to gravity. Let us use (23) to write the 1−loop approximation of $\bar{\Gamma}_k$ at vanishing fluctuation h, $\bar{\Gamma}^{(1l)}$,

$$\begin{aligned}
\bar{\Gamma}_k^{(1l)}(\bar{g}) &= S(\bar{g}) + \frac{1}{2}\mathrm{Tr}\,\log\,\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2(S+S_{GF}+\Delta S_k)}{\delta h\delta h}\right)\Bigg|_{h=0} - \mathrm{Tr}\,\log\,\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2(S+\Delta S_k^{(gh)})}{\delta\bar{C}\delta C}\right)\\
\bar{\Gamma}_k^{(1l)}(\bar{g}) &= S(\bar{g}) + \frac{1}{2}\mathrm{Tr}\,\log\,\left(\frac{1}{\sqrt{\bar{g}}}\left(\frac{\delta^2 S(\bar{g})}{\delta\bar{g}\delta\bar{g}} + \frac{\delta^2 S_{GF}}{\delta h\delta h}\Bigg|_{h=0}\right) + \mathcal{R}_k\right) - \mathrm{Tr}\,\log\,\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2 S_{gh}}{\delta\bar{C}\delta C} + \mathcal{R}_k^{(gh)}\right),
\end{aligned}$$

where, from the first to second line we have used the fact that the regulators are quadratic in h or $\bar{C}C$, and besides, that $S = S(g) = S(\bar{g}+h)$, hence, the functional derivative with respect to h is simply with respect to the argument g at $h = 0$. Now, we derive the flow equation for the 1−loop approximation, also within the single metric approximation, taking ∂_t at both sides,

$$\partial_t\bar{\Gamma}_k^{(1l)}(\bar{g}) = \frac{1}{2}\mathrm{Tr}\,\left(\left(\frac{1}{\sqrt{\bar{g}}}\left(\frac{\delta^2 S(\bar{g})}{\delta\bar{g}\delta\bar{g}} + \frac{\delta^2 S_{GF}}{\delta h\delta h}\Bigg|_{h=0}\right) + \mathcal{R}_k\right)^{-1}\right)\partial_t\mathcal{R}_k - \mathrm{Tr}\,\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2 S_{gh}}{\delta\bar{C}\delta C} + \mathcal{R}_k^{(gh)}\right)^{-1}\partial_t\mathcal{R}_k^{(gh)},$$

and finally, as defined in the Box below equation (23), it is plausible to take a "renormalization group improvement" from the 1−loop approximation of the ERGE, replacing $S \to \bar{\Gamma}_k$, which leads us to the

single metric flow equation:

$$\begin{aligned}
\partial_t\bar{\Gamma}_k(\bar{g}) &= \frac{1}{2}\mathrm{Tr}\,\left(\left(\frac{1}{\sqrt{\bar{g}}}\left(\frac{\delta^2\bar{\Gamma}_k}{\delta\bar{g}\delta\bar{g}} + \frac{\delta^2 S_{GF}}{\delta h\delta h}\Bigg|_{h=0} + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right)\right.\\
&- \mathrm{Tr}\,\left(\left(\frac{1}{\sqrt{\bar{g}}}\frac{\delta^2 S_{gh}}{\delta\bar{C}\delta C} + \mathcal{R}_k^{(gh)}\right)^{-1}\partial_t\mathcal{R}_k^{(gh)}\right). \qquad (24)
\end{aligned}$$

Let us note that (24) was derived from the 1−loop contribution to the effective action, plus the "*RG improvement*"; hence, it is in principle not the ERGE. To justify calling (24) the ERGE in the single metric approximation $\partial_t\bar{\Gamma}_k(\bar{g})$, we must plug-in (21) in the derivation of the ERGE with the single metric approximation (22), and identify

$$\hat{\Gamma}_k(h, C, \bar{C}; \bar{g}) = S_{GF}(h; \bar{g}) + S_{gh}(\bar{C}, C; \bar{g})\,. \qquad (25)$$

To summarize, we reach the **single metric flow equation** (24) if starting from the ERGE (1) (or more precisely, for the gravitational path integral, the starting ERGE is (18)), we take two approximations,

- the single metric approximation (20) where we assume $\langle h_{\mu\nu}\rangle = 0$, such that we are left with the gauge invariant functional of the background alone $\bar{\Gamma}_k(\bar{g})$,
- we ignore the running of $\hat{\Gamma}_k$ in (21), which is evident on that the right hand side of (25) does not depend on k.

Furthermore, within these approximations, we have written the averaged effective action (21) as,

$$\Gamma_k(h, \bar{C}, C; \bar{g}) = \bar{\Gamma}_k(\bar{g}+h) + S_{GF}(h; \bar{g}) + S_{gh}(\bar{C}, C; \bar{g}) \qquad (26)$$

Furthermore, let us note that we reach the 1−loop flow equation $\partial_t\bar{\Gamma}_k^{(1-\mathrm{loop})}$, if we neglect the running of the couplings in the Hessian $\delta^2\bar{\Gamma}/\delta\bar{g}^2$ on (24). On the other hand, because of the single metric approximation (24) is not the full ERGE. Although there are better approximations to address asymptotic safety in gravity (See for instance, chapter 7 in [1]), we will restrict this discussion to (24).

1.3 Computation of the flow equation

1.3.1 The Einstein-Hilbert truncation

The simplest truncation is to consider that only the terms in the Einstein-Hilbert action contribute to the operator expansion (2) for $\bar{\Gamma}_k$, with the couplings depending on the scale k. Namely, we assume that $\bar{\Gamma}_k$ has the form of the Einstein-Hilbert action with the cosmological constant Λ, and the Newton constant G depending on k,

$$\bar{\Gamma}_k(g) =: \frac{1}{16\pi G(k)} \int d^d x \sqrt{g}(2\Lambda(k) - R)\,. \tag{27}$$

1.3.2 The choice for the regulator

Now, for the Wetterich equation there is a free choice for the regulator, which however, must satisfy the conditions reviewed in section 1.1. In order to mantain *background gauge invariance*, the cutoff cannot be simply a function of momentum squared but must be defined by means of the background covariant derivative $\bar{\nabla}_\mu$. Let us choose for (16),

$$\Delta S_k(h, C, \bar{C}; \bar{g}) = \frac{1}{2}\int d^d x \sqrt{\bar{g}} h_{\mu\nu} \mathcal{R}_k^{\mu\nu\rho\sigma}(\bar{g}) h_{\rho\sigma} + \int d^d x \sqrt{\bar{g}} \bar{C}_\mu \bar{g}^{\mu\nu} \mathcal{R}_k^{(\mathrm{gh})}(\bar{g}) C_\nu\,,$$

$$\begin{aligned} \mathcal{R}_k^{\mu\nu\rho\sigma} &= Z_N K^{\mu\nu\rho\sigma} R_k(\Delta_2) && (28)\\ \mathcal{R}_k^{(gh)\mu}{}_\nu &= \delta^\mu{}_\nu R_k(\Delta_{gh})\,, && (29) \end{aligned}$$

where we have denoted,

$$\frac{1}{16\pi G} = Z_N$$

as the field renormalization constant, and we have defined the following operators,

$$\begin{aligned} \Delta_{2\rho\sigma}{}^{\mu\nu} &= \Delta^{\Lambda}_{(h)\rho\sigma}{}^{\mu\nu} + 2\Lambda \mathbf{1}^{\mu\nu}_{\rho\sigma} = -\bar{\nabla}^2 \mathbf{1}^{\mu\nu}_{\rho\sigma} + W_{\rho\sigma}{}^{\mu\nu} && (30)\\ \Delta_{(gh)\mu}{}^{\nu} &= -\bar{\nabla}^2 \delta^\nu_\mu - \bar{R}^\nu_\mu\,, && (31) \end{aligned}$$

where $K^{\mu\nu\rho\sigma}$ is a tensor that depends only on the background metric $\bar{g}$. It was defined in (12). Similarly, $W_{\rho\sigma}{}^{\mu\nu}$ defined in (15) depends only on $\bar{g}$ through different contractions of the Ricci tensor, which depend only on $\bar{g}$ (hence, we write them as $\bar{R}$). $\Delta^{\Lambda}_{(h)\rho\sigma}{}^{\mu\nu}$ is the kinetic operator of the graviton after gauge fixing and , which was introduced in (14).

1.3.3 The computation

The left hand side of the flow equation (24) in the Einstein-Hilbert truncation, contains only a constant field operator and the first power of the Ricci scalar. Namely, taking the ∂_t derivative of $\bar{\Gamma}_k$ in the Einstein-Hilbert truncation (27), and taking the single metric approximation,

$$\partial_t \bar{\bar{\Gamma}}_k(g) =: \int d^d x \sqrt{\bar{g}} \left(\partial_t \left(\frac{2\Lambda(k)}{16\pi G(k)} \right) - \partial_t \left(\frac{1}{16\pi G(k)} \right) \bar{R} \right)\,. \tag{32}$$

The right hand side of the flow equation (24)

Now, let us proceed to compute the terms on the right hand side of the single metric flow equation (24). We shall only be interested in approximating the right hand side of (24) up to the same order in the derivative expansion as in the in the righ hand side (32). Namely, in the Einstein-Hilbert truncation, up to $\mathcal{O}(R)$.

For (24) we need the 2-point vertex functions including gauge fixing, ghosts and regulator. Using the Hessian (13) and the definition (30),

$$\begin{aligned}\frac{1}{\sqrt{\bar g}}\frac{\delta^2\Gamma_k}{\delta h_{\mu\nu}\delta h_{\rho\sigma}}+\mathcal{R}_k^{\mu\nu\rho\sigma} &= \frac{1}{\sqrt{\bar g}}\left(\frac{\delta^2\bar\Gamma_k}{\delta\bar g_{\mu\nu}\delta\bar g_{\rho\sigma}}+\frac{\delta^2 S_{GF}}{\delta h_{\mu\nu}\delta h_{\rho\sigma}}\right)+\mathcal{R}_k^{\mu\nu\rho\sigma}\\ \frac{1}{\sqrt{\bar g}}\frac{\delta^2\Gamma_k}{\delta h_{\mu\nu}\delta h_{\rho\sigma}}+\mathcal{R}_k^{\mu\nu\rho\sigma} &= Z_N K^{\mu\nu\rho\sigma}(\Delta_2+R_k(\Delta_2)-2\Lambda)\,,\end{aligned}\tag{33}$$

where we have used (26) on the left hand side, also within the single metric approximation (20) taking $\langle h_{\mu\nu}\rangle = 0$ (Let us recall that the effective action depends only on vacuum expectation values $\langle\Phi\rangle$ which we have denoted as Φ). In particular, $S_{(GF)}$ (7) is quadratic in h, hence, we have dropped the evaluation on $|_{h=0}$.

Also using (26),

$$\frac{1}{\sqrt{\bar g}}\frac{\delta^2\Gamma_k}{\delta\bar C_\mu\delta C^\nu}+\mathcal{R}_k^{(gh)\mu}{}_\nu=\frac{1}{\sqrt{\bar g}}\frac{\delta^2 S_{gh}}{\delta\bar C_\mu\delta C^\nu}+\mathcal{R}_k^{(gh)\mu}{}_\nu \;=\; (\Delta_{(gh)}+R_k(\Delta_{(gh)}))\delta^\mu_\nu\,,\tag{34}$$

where $\Delta_{(gh)}$ was defined in (10). Furthermore, using (29) and (28),

$$\partial_t\mathcal{R}_k^{(gh)\mu}{}_\nu \;=\; \delta^\mu{}_\nu\partial_t R_k(\Delta_{gh})\,,\tag{35}$$

and,

$$\begin{aligned}\partial_t\mathcal{R}_k^{\mu\nu\rho\sigma} &= \frac{d}{dt}Z_N K^{\mu\nu\rho\sigma}R_k(\Delta_2)\\ &= K^{\mu\nu\rho\sigma}(\partial_t Z_N R_k(\Delta_2)+Z_N\partial_t R_k(\Delta_2))\\ \partial_t\mathcal{R}_k^{\mu\nu\rho\sigma} &= Z_N K^{\mu\nu\rho\sigma}(-\eta_N R_k(\Delta_2)+\partial_t R_k(\Delta_2))\,,\end{aligned}\tag{36}$$

where, in the last line, we have used the definition for the *anomalous dimension*,

$$\eta_N = -\partial_t \log Z_N\,.$$

Now, let us plug-in (33), (34), (35), (36) in the single metric approximation to the flow equation (plus the "improved[2]" 1−loop approximation) (24),

$$\begin{aligned}\partial_t\bar\Gamma_k(\bar g) &= \frac{1}{2}\mathrm{Tr}\left(\left(\frac{1}{\sqrt{\bar g}}\left(\frac{\delta^2\bar\Gamma_k}{\delta\bar g\delta\bar g}+\frac{\delta^2 S_{GF}}{\delta h\delta h}\bigg|_{h=0}\right)+\mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right)-\mathrm{Tr}\left(\left(\frac{1}{\sqrt{\bar g}}\frac{\delta^2 S_{gh}}{\delta\bar C\delta C}+\mathcal{R}_k^{(gh)}\right)^{-1}\partial_t\mathcal{R}_k^{(gh)}\right)\\ &= \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\Delta_2+R_k(\Delta_2)-2\Lambda}(K^{\mu\nu\rho\sigma})^{-1}Z_N^{-1}Z_N K^{\mu\nu\rho\sigma}(-\eta_N R_k(\Delta_2)+\partial_t R_k(\Delta_2))\right)\\ &- \mathrm{Tr}\left(\frac{1}{\Delta_{(gh)}+R_k(\Delta_{(gh)})}\partial_t R_k(\Delta_{gh})\right)\end{aligned}$$

$$\boxed{\partial_t\bar\Gamma_k(\bar g)=\frac{1}{2}\mathrm{Tr}\left(\frac{-\eta_N R_k(\Delta_2)+\partial_t R_k(\Delta_2)}{\Delta_2+R_k(\Delta_2)-2\Lambda}\right)-\mathrm{Tr}\left(\frac{\partial_t R_k(\Delta_{gh})}{\Delta_{(gh)}+R_k(\Delta_{(gh)})}\right)}\tag{37}$$

where we have used,

$$(K^{\mu\nu\alpha\beta})^{-1}K^{\alpha\beta\rho\sigma}=\frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu+\delta^\alpha_\nu\delta^\beta_\mu)=\mathbf{1}^{\alpha\beta}_{\mu\nu}\,.$$

Evaluating (37) with the heat kernel expansion,

Now, let us evaluate (37) using the general heat kernel expansion for the trace of a functional W of an operator Δ,

$$\mathrm{Tr}\;W(\Delta) \;=\; \frac{1}{(4\pi)^{\frac{d}{2}}}\left[Q_{\frac{d}{2}}(W)B_0(\Delta)+Q_{\frac{d}{2}-1}(W)B_2(\Delta)+\ldots\right]\,.\tag{38}$$

[2]Let us recall the *renormalization group improvement* that was discussed below equation (23).

Defining the functionals,

$$W_2(\Delta_2) \quad = \quad \frac{-\eta_N R_k(\Delta_2) + \partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda} \tag{39}$$

$$W_{(gh)}(\Delta_{(gh)}) \quad = \quad \frac{\partial_t R_k(\Delta_{gh})}{\Delta_{(gh)} + R_k(\Delta_{(gh)})}\,, \tag{40}$$

we can rewrite (37) using (38), (39), (40), as,

$$\begin{aligned}
\partial_t \bar{\Gamma}_k(\bar{g}) &= \frac{1}{2}\mathrm{Tr}\ (W_2(\Delta_2)) - \mathrm{Tr}\ \left(W_{(gh)}(\Delta_{(gh)})\right) \\
\partial_t \bar{\Gamma}_k(\bar{g}) &= \frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\left[Q_{\frac{d}{2}}\left(W_2\right) B_0(\Delta_2) + Q_{\frac{d}{2}-1}\left(W_2\right) B_2(\Delta_2) + \dots\right] \\
&- \frac{1}{(4\pi)^{\frac{d}{2}}}\left[Q_{\frac{d}{2}}\left(W_{(gh)}\right) B_0(\Delta_{(gh)}) + Q_{\frac{d}{2}-1}\left(W_{(gh)}\right) B_2(\Delta_{(gh)}) + \dots\right].
\end{aligned} \tag{41}$$

where $B_n(\Delta)$ are heat kernel coefficients that can be found in the literature [1, 3],

$$B_n(\Delta) = \int d^d x\, \sqrt{g}\, b_n(\Delta)\,, \tag{42}$$

$$b_0(\Delta_2) = \frac{d(d+1)}{2} \qquad\qquad b_0(\Delta_{(gh)}) = d \tag{43}$$

$$b_2(\Delta_2) = \frac{d(7-5d)}{12}\bar{R} \qquad\qquad b_2(\Delta_{(gh)}) = \frac{d+6}{6}\bar{R}\,. \tag{44}$$

Plugging-in and collecting in powers of $\bar{R}$, we can rewrite the right hand side of (41) as,

$$\begin{aligned}
\partial_t \bar{\Gamma}_k(\bar{g}) = \frac{1}{(4\pi)^{\frac{d}{2}}}\int d^d x \sqrt{\bar{g}}\Bigg(&\left[\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(W_2\right) - d\, Q_{\frac{d}{2}}\left(W_{(gh)}\right)\right] \\
&+ \left[\frac{d(7-5d)}{24}Q_{\frac{d}{2}-1}\left(W_2\right) - \frac{d+6}{6}\,Q_{\frac{d}{2}-1}\left(W_{(gh)}\right)\right]\bar{R}\Bigg)\,,
\end{aligned} \tag{45}$$

where in the Einstein-Hilbert truncation, we have explicitly diregarded the ellipsis, which denoted higher order terms in the derivative expansion (Higher powers of scalars built with the Ricci tensor).

Evaluation of $Q-$functionals in (45)

Now, let us evaluate the Q functionals. They are are generally given by,

$$Q_n(W) \quad = \quad \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1} W(z)\,, \tag{46}$$

if $n > 0$.

In general, the $Q-$functionals depend on the regulator that modifies the operators and the eigenvalues z (z has k^2 units). There are a few that only depend on the general properties of the regulators that were reviewed in section 1.1. These will be computed at the end. However, let us first show a particularly illuminating computation that we will use to evaluate (45), where we will use the properties of smooth regulators.

Consider the $Q-$functional corresponding to $W_2(\Delta_2)$ (39)

$$Q_n(W_2(\Delta_2)) = Q_n\left(\frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda}\right) - Q_n\left(\frac{\eta_N R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda}\right)\,, \tag{47}$$

for the first term on the right hand side,

$$Q_n\left(\frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda}\right) = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{\partial_t R_k(z)}{z + R_k(z) - 2\Lambda} \tag{48}$$

For the so-called "optimal choice", Litim regulator,

$$R_k(z) = (k^2 - z)\theta(k^2 - z) \qquad\qquad \partial_t R_k(z) = 2k^2\theta(k^2 - z) - 2z(k^2 - z)\delta(k^2 - z)\,.$$

Let us stress that the the Dirac δ on the right hand side must only be evaluated inside an integral. This will make evident the sort of ambiguous expressions we would get if we were to use sharp cutoffs [16]. Therefore, we must resort to the C^1 character of the Litim regulator, and interpret the step function, and derived Dirac deltas in a proper way. Namely, plugging in (48),

$$\begin{aligned} Q_n\left(\frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda}\right) &= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{2k^2\theta(k^2-z) - 2z(k^2-z)\delta(k^2-z)}{z + (k^2-z)\theta(k^2-z) - 2\Lambda} \\ &= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{2k^2}{z + (k^2-z)\theta(k^2-z) - 2\Lambda}\theta(k^2-z) \\ &- \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{2z(k^2-z)}{z + (k^2-z)\theta(k^2-z) - 2\Lambda}\delta(k^2-z)\,, \end{aligned} \tag{49}$$

we find that the last line of (49) includes an expression of the type,

$$\int dx f(\theta(x))\delta(x)\,,$$

which is not well defined. Hence, we must use Morris' lemma to make sense out of (49),

$$f(\theta(x))\delta(x) = \delta(x)\int_0^1 dt\, f(t)\,,$$

which is proven considering general, smoothed step and delta functions [13] (See also pages 184-185 in [16]). This is an example of why we have chosen smooth regulators. In particular, using Morris's lemma, expanding the last line of (49) it clearly vanishes, and defining

$$\tilde{\Lambda} = \frac{\Lambda}{k^2}\,,$$

we are left with the first line,

$$Q_n\left(\frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda}\right) = \frac{1}{\Gamma(n)}\int_0^{k^2} dz\, z^{n-1}\frac{2k^2}{z + k^2 - z - 2\Lambda} = \frac{2}{\Gamma(n)(1-2\tilde{\Lambda})}\int_0^{k^2} dz\, z^{n-1}$$

$$Q_n\left(\frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) - 2\Lambda}\right) = \frac{2k^{2n}}{n\Gamma(n)\left(1 - 2\tilde{\Lambda}\right)}\,, \tag{50}$$

which holds for $n > 0$ (It also holds for $n = 0$, but the deduction is slightly different).

Now, the computation of the right most term in (47) follows in the same way, without the need to use Morris' lemma. Collecting terms, we have,

$$Q_n(W_2(\Delta_2)) = \frac{k^{2n}}{n\Gamma(n)(1-2\tilde{\Lambda})}\left(2 - \frac{\eta}{n+1}\right)\,, \tag{51}$$

furthermore, the deduction for the $Q-$functionals for $W_{(gh)}(\Delta_{(gh)})$ follow in the same way. In fact, they can be obtained by directly setting $\tilde{\Lambda}$ and η to zero in the last expression,

$$Q_n(W_{(gh)}(\Delta_{(gh)})) = \frac{2k^{2n}}{n\Gamma(n)}\,. \tag{52}$$

Final result of the flow equation for gravity in the Einstein-Hilbert truncation

Plugging-in (51) and (52) in the truncated flow equation (45), with some algebra, we obtain the final result:

$$\partial_t \bar{\Gamma}_k = \frac{k^d}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int d^dx \sqrt{\bar{g}} \left[\left(-4 + \frac{d+1}{1-2\tilde{\Lambda}}\left(\frac{\eta}{d+2}\right)\right) - \frac{1}{12k^2}\left(4(d+6) - \frac{d(7-5d)}{1-2\tilde{\Lambda}}\left(1-\frac{\eta}{d}\right)\right)\bar{R}\right] \tag{53}$$

1.3.4 The one-loop beta functions

Now, let us extract the beta functions. Plugging (32) on the left hand side of (53), we identify the factors on both sides, according to the powers of $\bar{R}$,

$$\partial_t\left(\frac{2\Lambda(k)}{16\pi G(k)}\right) = \frac{k^d}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(-4 + \frac{d+1}{1-2\tilde{\Lambda}}\left(\frac{\eta}{d+2}\right)\right) \tag{54}$$

$$\partial_t\left(\frac{1}{16\pi G(k)}\right) = \frac{k^{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\frac{1}{12}\left(4(d+6) - \frac{d(7-5d)}{1-2\tilde{\Lambda}}\left(1-\frac{\eta}{d}\right)\right) \tag{55}$$

2 Next session: Asymptotic Safety (AS), and the critiques to this scenario in quantum gravity

References

[1] R. Percacci, *An introduction to covariant quantum gravity and asymptotic safety*, vol. 3. World Scientific, 2017.

[2] A. Codello, R. Percacci, and C. Rahmede, "Investigating the ultraviolet properties of gravity with a wilsonian renormalization group equation," *Annals of Physics*, vol. 324, no. 2, pp. 414–469, 2009.

[3] M. Reuter, "Nonperturbative evolution equation for quantum gravity," *Physical Review D*, vol. 57, no. 2, p. 971, 1998.

[4] M. Niedermaier and M. Reuter, "The asymptotic safety scenario in quantum gravity," *Living Reviews in Relativity*, vol. 9, no. 1, pp. 1–173, 2006.

[5] J. F. Donoghue, "A critique of the asymptotic safety program," *arXiv preprint arXiv:1911.02967*, 2019.

[6] R. Percacci *et al.*, "Asymptotic safety," *Approaches to quantum gravity*, pp. 111–128, 2009.

[7] A. Eichhorn, "An asymptotically safe guide to quantum gravity and matter," *Frontiers in Astronomy and Space Sciences*, vol. 5, p. 47, 2019.

[8] A. Bonanno, A. Eichhorn, H. Gies, J. M. Pawlowski, R. Percacci, M. Reuter, F. Saueressig, and G. P. Vacca, "Critical reflections on asymptotically safe gravity," *arXiv preprint arXiv:2004.06810*, 2020.

[9] K. G. Wilson and J. Kogut, "The renormalization group and the epsilon expansion," *Physics reports*, vol. 12, no. 2, pp. 75–199, 1974.

[10] K. G. Wilson, "The renormalization group and critical phenomena," *Reviews of Modern Physics*, vol. 55, no. 3, p. 583, 1983.

[11] F. J. Wegner and A. Houghton, "Renormalization group equation for critical phenomena," *Physical Review A*, vol. 8, no. 1, p. 401, 1973.

[12] J. Polchinski, "Renormalization and effective lagrangians," *Nuclear Physics B*, vol. 231, no. 2, pp. 269–295, 1984.

[13] T. R. Morris, "The exact renormalization group and approximate solutions," *International Journal of Modern Physics A*, vol. 9, no. 14, pp. 2411–2449, 1994.

[14] C. Wetterich, "Average action and the renormalization group equations," *Nuclear Physics B*, vol. 352, no. 3, pp. 529–584, 1991.

[15] H. Gies, "Introduction to the functional rg and applications to gauge theories," in *Renormalization Group and Effective Field Theory Approaches to Many-Body Systems*, pp. 287–348, Springer, 2012.

[16] P. Kopietz, L. Bartosch, and F. Schütz, *Introduction to the functional renormalization group*, vol. 798. Springer, 2010.

[17] M. Peskin, *An introduction to quantum field theory.* CRC press, 2018.