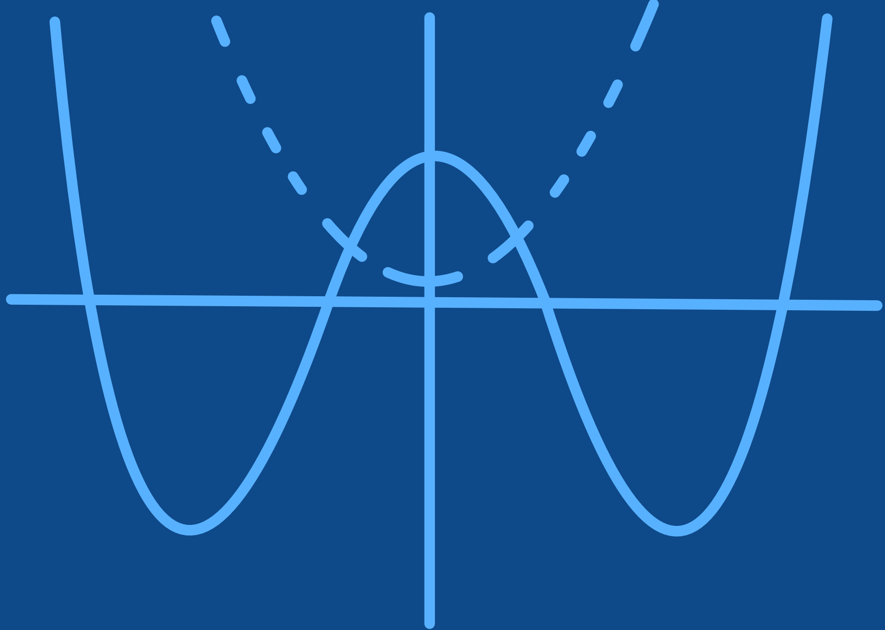
Euclidean Anharmonic Oscillator



Juan Sebastian Valbuena

2020

Euclidean Anharmonic Oscillator.

Lets consider a simple example: 0+1 dim. real scalar f.t.-

$$\phi(x^\mu) \longrightarrow x(t).$$

The bare action that we want to quantize is:

$$S = \int d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{4} x^4 \right),$$

where $\omega^2, \lambda > 0$. (for the case $\omega^2 < 0$ see references [1], [2])

We are mainly interested in determining the ground state energy, E_0.

Our expectation is that E_0 is (predominantly) influenced by the effective potential. Hence we consider the truncation:

$$\Gamma_k[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V_k(x) \right). \Rightarrow \text{Local potential approx.}$$

For a concrete computation of $V_k(x)$, we need:

i) An explicit regulator $R_k(p^2)$. An optimal choice for the present problem (since it improves the stability of the flow equation) is the Litim regulator [3], show in Figure 1.

$$R_k(p^2) = (k^2 - p^2)\,\Theta(k^2 - p^2)$$

It fulfills the required asymptotic behaviour:

1) $\frac{p^2}{k^2} \to 0$: $R_k(p^2) \simeq k^2 > 0$; → IR regulator screens the IR modes in mass like fashion ($m^2 \sim k^2$)

2) $\frac{k^2}{p^2} \to 0$: $R_k(p^2) \to 0$; → UV regulator vanishes. In the limit $k \to 0$, we recover $\Gamma_{k\to 0} = \Gamma$

3) $k^2 \to \Lambda \to \infty$: $R_k(p^2) \to \infty$; → Functional integral dominated by stationary point of the action. In the limit $\Lambda \to \infty$; $\Gamma_{k\to\Lambda} \longrightarrow S + \text{constant}$.

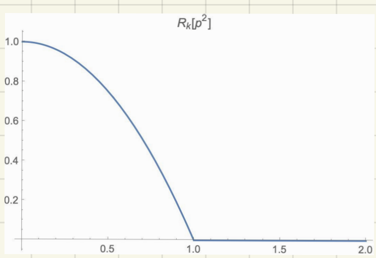


Figure 1: Litim propagator $R_k(p^2)$

The choice of the Litim regulator implies that

$$\partial_t R_k(p^2) = 2K^2\Theta(K^2-p^2) + 2K^2(K^2-p^2)\delta(K^2-p^2). \quad (i)$$

Lets recall the Wetterich equiation:

Flow Equation $\Rightarrow$ $$\boxed{\partial_t \Gamma_K[\phi] = \frac{1}{2}\, Tr\left[\partial_t R_K\left(\Gamma_K^{(2)}[\phi] + R_K\right)^{-1}\right],}$$

In momentum space $Tr = \int \frac{d^d q}{(2\pi)^d}$

Recall: $t = \ln\frac{K}{\Lambda}, \partial_t = K\frac{\partial}{\partial K}$

Using the local potential aprox.:

$$\Gamma_K[x] = \int d\tau\left(\frac{1}{2}\dot{x}^2 + V_K(x)\right)$$
$$= \int d\tau\left(-\frac{1}{2}x\,\partial_\tau^2 x + V_K(x) + \frac{1}{2}\partial_\tau(x\,\partial_\tau x)\right),$$

we get $\Gamma_K^{(2)} = (-\partial_\tau^2 + V_K''(x))\delta(\tau-\tau')$. After a Fourier transform

$$\Gamma_K^{(2)^{-1}} = \frac{1}{p^2 + V_K''(x)}. \quad (ii)$$

We can now project the flow of Γ_K onto the flow of V_K.

It suffices to consider the constant field configuration, i.e. $x = const$, $\dot{x} = 0$

$$\partial_t \Gamma_K = \int d\tau\; \partial_t V_K(x)$$

and replacing (i) and (ii) on the RHS of the flow equation, we get

$$\frac{1}{2}\,Tr\left[\frac{\partial_t R_K}{\Gamma_K^{(2)}[x] + R_K}\right] = \frac{1}{2}\int d\tau \int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{2K^2\Theta(K^2-q^2) + 2K^2(K^2-q^2)\delta(K^2-q^2)}{q^2 + V_K''(x) + (K^2-q^2)\Theta(K^2-q^2)}$$

$$= \frac{1}{2}\int d\tau \int_{-K}^{K}\frac{dq}{2\pi}\,\frac{2K^2}{K^2+V_K''}$$

$$= \int \frac{d\tau}{2\pi}\,\frac{K^2}{K^2+V_K''}\,(2K) = \int \frac{d\tau\, K^3}{\pi(K^2+V_K'')}$$

Thus, the Wetterich equation can be rewritten as:

$$\int d\tau\; \partial_t V_K(x) = \int d\tau\; \frac{1}{\pi}\,\frac{K^3}{(K^2+V_K'')}$$

We obtained the flow equation for the effective potential:

$$\partial_t V_K(x) = \frac{K^3}{\pi(K^2 + V''_K)}$$

$$\boxed{\frac{d}{dK} V_K(x) = \frac{1}{\pi} \frac{K^2}{K^2 + V''_K\big|_{x_c}}} \quad \text{(iii)}$$

→ P.D.E for the effective potential $V_K(x)$

Now a second to solve the flow equation for $V_K(x)$, we consider a further approximation:

ii) Guided by our goal of computing the ground state energy, and considering the case $\lambda, w > 0$, then it suffices to consider a polynomial expansion of $V_K(x)$, around its ground state $(x=0)$.

$$\boxed{V_K(x) = \frac{1}{2} w_K^2 x^2 + \frac{1}{24} \lambda_K x^4 + \ldots + \tilde{E}_K}$$

This expansion is also known as vertex expansion.

Notice that the coupling constants $(w_K, \lambda_K, \ldots)$ are promoted to be scale dependent, and they encode the effects of quantum fluctuations. We consider here a further truncation of $V_K(x)$ up to $O(x^4)$ operators for pedagogical reasions, but it can be generalized.

At $K = \Lambda$, we shall recover the bare action, thus:

$$\left.\begin{aligned} w_\Lambda &= w \\ \lambda_\Lambda &= \lambda \end{aligned}\right\} \text{bare parameters}$$

On the other hand, $\tilde{E}_K$ is not identical to the desired ground state energy. To understand why it is so, consider the following decomposition

$$\tilde{E}_K = \underbrace{E_{0,K}}_{\substack{\text{Ground} \\ \text{state} \\ \text{Energy} \\ \text{of } V_K}} + \underbrace{\Delta_K}_{\substack{x\text{-indep.} \\ \text{contribution}}}$$

In the Limit $K \to \Lambda$:

$$R_K(p) \to (\Lambda^2 - p^2)\Theta(\Lambda^2 - p^2)$$

$$\Delta S_K = \frac{1}{2}\int \frac{dq}{2\pi}\, x(-q) R_K(q) x(q) \longrightarrow \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} \underbrace{x(-q)\Lambda^2 x(q)}_{\text{Mass term} \Rightarrow \text{Contributes to } \Delta_\Lambda \text{ as} \sim \frac{1}{2}\Lambda} - x(-q)\, q^2\, x(-q).$$

True ground state energy flow:

In order to extract the true ground $E_{0,K}$ from the flow of $\tilde{E}_K$ we can proceed as explained below.

From the vertex expansion of V_K, we have

$$\frac{d}{dK} V_K(x) = \frac{1}{2}\frac{d}{dK} \omega_K^2 \; x^2 + \frac{1}{24}\frac{d\lambda}{dK} x^4 + \frac{d\tilde{E}_K}{dK},$$

$$V_K''(x) = \omega_K^2 + \frac{1}{2}\lambda_K x^2,$$

thus the flow equation (iii) can be rewritten as

$$\frac{1}{2}\frac{d}{dK}\omega_K^2 x^2 + \frac{1}{24}\frac{d\lambda}{dK} x^4 + \frac{d\tilde{E}_K}{dK} = \frac{1}{\pi}\frac{K^2}{K^2 + \omega_x^2 + \frac{1}{2}\lambda x^2}. \qquad (iv)$$

At $x=0$, the ground state, the equation (iv) reduces to the flow eq. of $\tilde{E}_K$:

$$\frac{d\tilde{E}_K}{dK} = \frac{1}{\pi}\frac{K^2}{K^2 + \omega_K^2},$$

from which, the flow equation $E_{0,K}$ can be derived:

$$\frac{dE_{0,K}}{dK} = \frac{1}{\pi}\frac{K^2}{K^2 + \omega_K^2} - \frac{d\Delta_K}{dK}.$$

To avoid the build-up of unphysical contributions, we can perform a controlled substraction of $\frac{d\Delta_K}{dK}$. In the limit $\lambda = \omega = 0$, the ground state energy has to stay zero, $E_{0,K} = 0$, for all momenta K. Then

$$0 = \frac{1}{\pi} - \frac{d\Delta_K}{dK} \to \frac{d\Delta_K}{dK} = \frac{1}{\pi},$$

and the flow equation for the true ground energy becomes.

$$\hookrightarrow \boxed{\frac{dE_{0K}}{dK} = \frac{1}{\pi}\left(\frac{K^2}{K^2 + \omega_K^2} - 1\right)} \quad (I)$$

Coupling constants flow equations.

In this section we will derive the flow equations of the coupling constants ω_K and λ_K. Starting from the flow equation of $V_K(x)$, and doing a truncation up to $\mathcal{O}(x^4)$ operators, we got equation (iv):

$$\frac{1}{2}\frac{d}{dK}\omega_K^2 x^2 + \frac{1}{24}\frac{d\lambda}{dK}x^4 + \frac{d\tilde{E}_K}{dK} = \frac{1}{\pi}\frac{K^2}{K^2+\omega_K^2+\frac{1}{2}\lambda x^2}.$$

After doing a Taylor expansion of the RHS, around $x=0$, we obtain

$$\frac{1}{2}\frac{d}{dK}\omega_K^2 x^2 + \frac{1}{24}\frac{d\lambda}{dK}x^4 + \frac{d\tilde{E}_K}{dK} = \frac{1}{\pi}\frac{K^2}{K^2+\omega_K^2}\Bigg(1 - \frac{1}{2}\frac{\lambda_K}{K^2+\omega_K^2}x^2 + \left(\frac{1}{2}\frac{\lambda_K}{K^2+\omega_K^2}\right)^2 x^4 + \underbrace{\mathcal{O}(x^6)}_{\text{Neglected}}\Bigg).$$

We remark here again that $\mathcal{O}(x^6)$ operators are neglected for simplicity.

Identifying the coefficients of x^2 and x^4 on the LHS and RHS, leads to the flow equations of ω_K and λ_K

$$\mathcal{O}(x^2) \rightarrow \boxed{\frac{d\omega_K^2}{dK} = \frac{-2}{\pi}\frac{K^2}{(K^2+\omega_K^2)^2}\left(\frac{\lambda_K}{2}\right)} \quad (\mathrm{II})$$

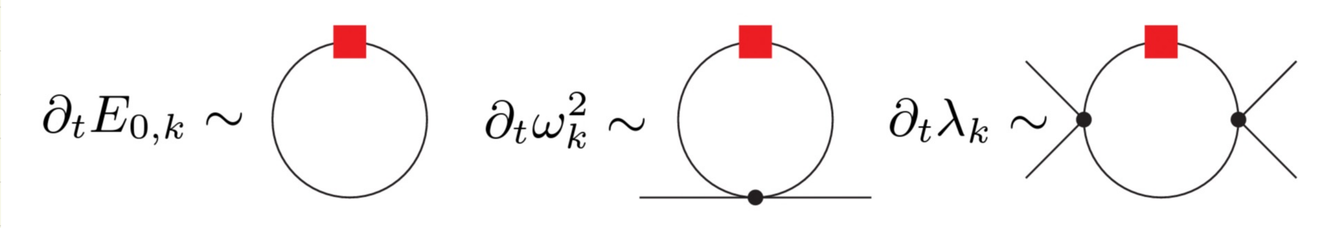
$$\mathcal{O}(x^4) \rightarrow \boxed{\frac{d\lambda_K}{dK} = \frac{24}{\pi}\frac{K^2}{(K^2+\omega_K^2)^3}\left(\frac{\lambda_K}{2}\right)^2} \quad (\mathrm{III})$$

As a sumary we have transformed the flow equation of Γ_K into a couple set of first-order ordinary differential equtions. which can be viewed as the RG β functions of $E_{0,K}$, ω_K, λ_K

$$\Gamma_K\text{-Flow equation} \xrightarrow{} \text{RG-}\beta \text{ equations} \left\{\begin{matrix}(\mathrm{I})\\(\mathrm{II})\\(\mathrm{III})\end{matrix}\right\} \text{ of the generalized coupling } \left\{\begin{matrix}E_{0,K}\\ \omega_K\\ \lambda_K\end{matrix}\right\}$$

- Litim Regulator
- Local potential approximation
- Constant field configurations
- Vertex expansion
- $\mathcal{O}(x^4)$-truncation
- Controlled subtraction

Diagramatically, the flow equations can be represented as.



Caution: The diagrams look very similar to one-Loop perturbative diagrams, but there are important differences:

- → All internal lines and vertices correspond to full propagators and full vertices. → Fully dressed quantities
- → One propagator in each loop carries a regulator insertion, implying the replacement

$$G \longrightarrow G_k \partial_t R_k G_k$$

Solutions to the flow equations:

It is instructive to solve the flow equations.

$$\frac{dE_{0,k}}{dk} = \frac{1}{\pi}\left(\frac{k^2}{k^2+\omega_k^2} - 1\right) \qquad (I)$$

$$\frac{d\omega_k^2}{dk} = \frac{-2}{\pi}\frac{k^2}{(k^2+\omega_k^2)^2}\left(\frac{\lambda_k}{2}\right) \qquad (II)$$

$$\frac{d\lambda_k}{dk} = \frac{24}{\pi}\frac{k^2}{(k^2+\omega_k^2)^3}\left(\frac{\lambda_k}{2}\right)^2 \qquad (III)$$

with various approximations.

a) First case: $\lambda_k = 0$. Thus the flow equations reduce to:

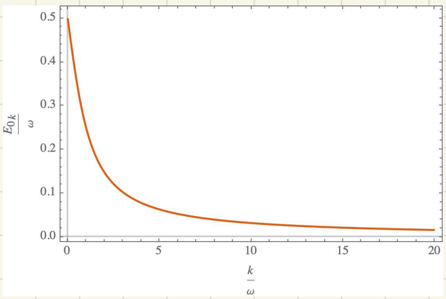
$$(II) \quad \frac{d\omega_k^2}{dk} = 0; \qquad \rightarrow \omega_k^2 = \omega^2$$

$$(I) \quad \frac{dE_{0,k}}{dk} = \frac{-1}{\pi}\frac{\omega^2}{k^2+\omega^2} \rightarrow E_{0,k} = \int \frac{dk}{\pi}\frac{-\omega^2}{k^2+\omega^2} = -\frac{\omega}{\pi}\arctan\left(\frac{k}{\omega}\right) + c.$$

where $E_{0,0} = c$, and $E_{0,\infty} = 0$

$$E_{0,\infty} = -\frac{\omega}{2} + c = 0 \Rightarrow c = \frac{\omega}{2}$$

As expected, we recover the QM result: $\boxed{E_{0,k=0} = \frac{1}{2}\omega.}$.



→ Runing of $E_{0,k}$, for $\lambda = 0$.

b) The second case we consider here is the week coupling regime. Since $[\lambda]_m = 3$ and $[w]_m = 1$, the regime we are interested in is when $\lambda \ll \omega^3$.

The minimal non-trivial approximation is dropping the running of the anharmonic coupling $\lambda_k \to \lambda$. The flow equations in this case reduce to:

$$(\mathrm{II}) \quad \frac{d\omega_k^2}{dk} = -\frac{1}{\pi}\frac{k^2}{(k^2+\omega^2)^2}\lambda$$

; (*) here we take $\omega_k = \omega$ on the rhs as a further approx.

$$(\mathrm{I}) \quad \frac{dE_0}{dk} = \frac{1}{\pi}\left(\frac{k^2}{k^2+\omega_k^2} - 1\right)$$

after integrating (II), we replace the solution ω_k in (I) and integrate the energy $E_{0,k}$. Expanding the result perturbatively in λ, we find:

$$E_0 = \frac{1}{2}\omega + \frac{3}{4}\omega\left(\frac{\lambda}{24\omega^3}\right) - \frac{3(35+4\pi^2)}{16\pi^2}\omega\left(\frac{\lambda}{24\omega^3}\right)^2 + \dots$$

c) A further approximation we implemented, was fixing ω on the rhs of equations (II) and (III). The flow equations in this approximation become:

$$(\mathrm{III}) \quad \frac{d\lambda_k}{dk} = \frac{24}{\pi}\frac{k^2}{(k^2+\omega^2)^3}\left(\frac{\lambda_k}{2}\right)^2$$

$$(\mathrm{II}) \quad \frac{d\omega_k^2}{dk} = -\frac{1}{\pi}\frac{k^2}{(k^2+\omega^2)^2}\lambda_k$$

$$(\mathrm{I}) \quad \frac{dE_0}{dk} = \frac{1}{\pi}\left(\frac{k^2}{k^2+\omega_k^2} - 1\right)$$

After integrating the system we find:

$$E_0 = \frac{1}{2}\omega + \frac{3}{4}\omega\left(\frac{\lambda}{24\omega^3}\right) - \frac{3}{16\pi^2}(128\pi^2 - 139)\,\omega\left(\frac{\lambda}{24\omega^3}\right)^2$$

We can compare the previous approximatios (a), (b), (c) with direct 2nd-order perturbation theory [4]

$$E_0^{PT} = \frac{1}{2}\omega + \frac{3}{4}\omega\left(\frac{\lambda}{24\omega^3}\right) - \frac{105}{40}\omega\left(\frac{\lambda}{24\omega^3}\right)^2 + \dots$$

The "one-Loop" result agrees with perturbation theory. while the second order "two-Loop" coefficient comes out with a $\sim 0,6\,\%$ error.

However, Keep in mind that we have obtained this estimate from a cheap calculation, involving only a one-Loop integral with an RG improved propagator.

(d) Strong coupling regime:

The asymptotic behaviour is Know to be.

$$E_0 = \left(\frac{\lambda}{24}\right)^{1/3}\left[\alpha_0 + \mathcal{O}(\lambda^{-2/3})\right]$$

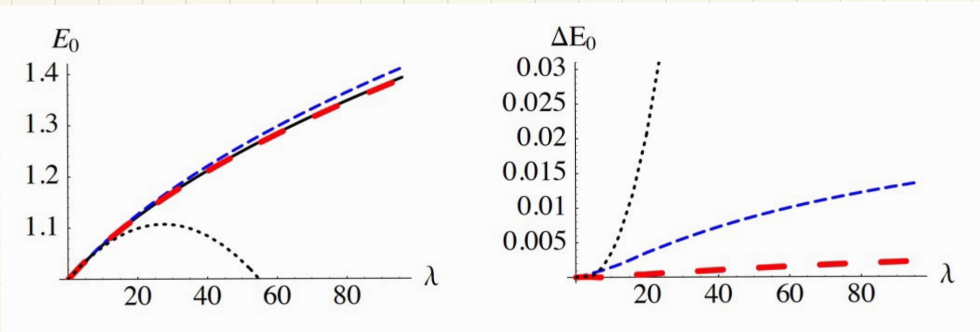
with $\alpha_0 \cong 0,66798$. [4]

Solving equations I, II, III, gives $\alpha_0 \cong 0,6620$ corresponding to a 1% error.

To conclude the discussion, we present a plot of the ground state energy E_0 as a function of λ, for $\omega = 2$.

On the left panel, the solid black line corresponds to the exact solution, the dotted-black line to the 2-loop perturbation theory. The blue dashed line corresponds to the flow based on equations (I)&(II) (see method b) and the red dashed line corresponds to the method c.

On the right panel is plotted the relative error of the different estimates



The quality of the results is remarkable in view of the simple approximation of the full flow. The take home message is that it does not matter whether the terms dropped during the truncation are small compared to the terms kept. It only matters whether their influence on the terms that belong to the truncation is small or large.

References:

[1] A.S. Kapyannis and N. Tetradis, Phys. Lett. A. 276, 225 (2000)

[2] D. Zapata, Phys. Lett. B 486, 92 (2000); Phys. Rev 184, 1231 (1969)

[3] D.F. Litim, Phys. Lett. B 486, 92 (2000); Phys. Rev. D 64, 105007 (2001)

[4] Holger Gies, Introduction to the functional rg and applications to gauge theories. In Renormalization Group and Effective Field Theory Approaches to Many-Body Systems, pages 287-348. Springer, 2012