Amplitudes from Spinor-Helicity and Unitarity Methods

↳ Part III:

Overview of what we will show:

- Self Interacting Theories of massless spin 1, 2 particles are unique
- There is only 1 Graviton which couples universally to matter
- Yang-Mills is the unique theory of interacting massless spin 1 particles
- Charge is conserved
- Recap: Weinberg Soft Theorems

Maybe:
- Excluding Consistent Theories by Pole Counting
- Loop-Calculations
- Massive Particles

In the previous lectures we saw that the 3^{pt} Amplitudes are fully determined by Poincaré Symmetry (up to coupling constants), and that higher point Amplitudes can be built up from lower point amplitudes using recursion relations (BCFW, CSW, ...).

Recall Seminar I: 3^{pt} Amp.'s are completely determined by Poincaré Inv.

$$A(h_1,h_2,h_3) = \begin{cases} \langle 12\rangle^{h_3-h_1-h_2}\ \langle 23\rangle^{h_1-h_2-h_3}\ \langle 31\rangle^{h_2-h_3-h_1} \\ [12]^{h_1+h_2-h_3}\ [23]^{h_2+h_3-h_1}\ [31]^{h_3+h_1-h_2} \end{cases}$$

↳ this allows us to write down any arbitrary 3^{pt} Amp. we could think of

e.g. $h_1 = -h_2 = 72$; $h_3 = -1$

$$\hookrightarrow A_3 = g\,\frac{\langle 23\rangle^{145}}{\langle 12\rangle\langle 31\rangle^{143}}$$ ... has a marginal coupling $[g]=0$

Can we use any arbitrary 3^{pt} Amplitude to build higher point Amplitudes?
Are they physical?

So far we only imposed Little Group Covariance & Locality on 3^{pt} Amplitudes and found them to be completely fixed.

Now we also have to impose Locality & Unitarity on higher point Amplitudes.

Which Self-Interacting Theories are Consistent?

First of all, what does it mean to be consistent?

↳ We have to impose Locality & Unitarity on Amplitudes

reflected in its pole structure: only simple, propagator-like poles are allowed

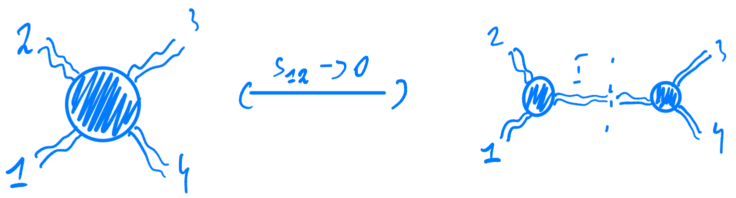
↳ $\frac{1}{(\sum_{i\in I} p_i)^2}$

i) the residue on each and every pole must have an interpretation as a physical factorisation channel

ii) any individual factorisation channel, if it is a legitimate bridge between lower point amp.'s in the theory, must be a residue of a fully legitimate amplitude w/ the same external states

Example: given a fact. channel $A_3(1^{-2}, 2^{-2}, I^{+2})\,\frac{1}{s_{12}}\,A_3(-I^{-2}, 3^{+2}, 4^{+2})$

by unitarity this must be a factorisation channel of $A_4(1^{-2}, 2^{-2}, 3^{+2}, 4^{+2})$



We know that 3^{pt} Amplitudes are completely fixed by Poincaré-Invariance, except for the coupling constant

↳ c.f. previous lectures

Employing Constructive Methods such as BCFW (which use the fixed 3^{pt} Amp.'s as an input) we can construct any higher point Amplitude.

But we can write down any 3^{pt}-Amplitude we want, how do we know what leads to physical theories?

↳ this is where we impose Locality & Unitarity; we will see that they impose strong restrictions

Let's see what consistent self-interacting theories of Helicity s particles we can build.

↳ To investigate these theories we consider $A(1_a^{-s}, 2_b^{-s}, 3_c^{+s})$ and its Parity Conjugate as the primitive building blocks of the theory

$$A(1_a^{-s}, 2_b^{-s}, 3_c^{+s}) = \underline{f_{abc}}\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^s \; ; \; A(1_a^{+s}, 2_b^{+s}, 3_c^{-s}) = f_{abc}\left(\frac{[12]^3}{[23][31]}\right)^s$$

↳ part. labels in case we want to consider multiple spin s particles

Why? Note that $[A_3]_m \overset{!}{=} 1 \rightsquigarrow$ $s=1 : [f]=0$, $s=2 : [f]=-1$ } familiar from YM/GR

$$A(1^-, 2^-, 3^-) \sim (\langle 12\rangle\langle 23\rangle\langle 31\rangle)^s \rightsquigarrow [f] = 1-3s$$ (corr. to higher derivative oper.'s)

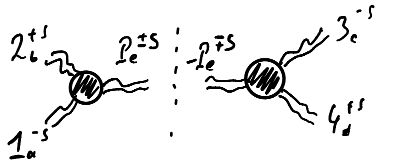
For which values of s is $A_4(1_a^- 2_b^+ 3_c^- 4_d^+)$ consistent w/ Locality & Unitarity?

simple poles in s,t,u — fact. on the poles

Def.: $s=(1+2)^2$; $t=(1+3)^2$; $u=(1+4)^2$

Consider the Residue in the s-channel (Note: this means $s=\langle 12\rangle[12]=\langle 34\rangle[34]\to 0$)

$$\lim_{s\to 0} s A_4 =$$



$$= \sum_{h=\pm s} A_3(1_a^{-s}\, 2_b^{+s}\, P_e^{h})\, A_3(-P_e^{-h}\, 3_c^{-s}\, 4_d^{+s})$$

$$= A_3(1_a^{-s}\, 2_b^{+s}\, P_e^{-s})\, A_3(-P_e^{+s}\, 3_c^{-s}\, 4_d^{+s}) + A_3(1_a^{-s}\, 2_b^{+s}\, P_e^{+s})\, A_3(-P_e^{-s}\, 3_c^{-s}\, 4_d^{+s})$$

$$= f_{abe} f_{edc}\left(\left(\frac{\langle P1\rangle^3}{\langle 12\rangle\langle 2P\rangle}\cdot\frac{[4\,{-P}]^3}{[{-P}3][34]}\right)^s + \left(\frac{[2P]^3}{[12][P1]}\cdot\frac{\langle {-P}3\rangle^3}{\langle 34\rangle\langle 4\,{-P}\rangle}\right)^s\right)$$

Note: $|{-i}\rangle = -|i\rangle$; $|{-i}] = +|i]$

↳ we use momentum conservation & spinor Algebra to get rid of the P-brackets: $P=-1-2=3+4$

$$\langle 1P\rangle^3[4P]^3 = -\langle 1|\not{P}|4]^3 = \langle 1|\not{2}|4]^3 = \langle 12\rangle^3[42]^3$$

$$\langle 2P\rangle[P3] = -\langle 2|\not{P}|3] = \langle 2|\not{1}|3] = \langle 21\rangle[31]$$

$$\langle 3P\rangle^3[P2]^3 = -\langle 3|\not{P}|2]^3 = \langle 3|\not{1}|2]^3 = \langle 13\rangle^3[12]^3$$

$$\langle 4P\rangle[P1] = -\langle 4|\not{P}|1] = \langle 4|\not{2}|1] = \langle 42\rangle[12]$$

$$= f_{abe} f_{edc}\left(\left(\frac{\langle P1\rangle^3}{\langle 12\rangle\langle 2P\rangle}\cdot\frac{[4{+}P]^3}{[{+}P3][34]}\right)^s + \left(\frac{[2P]^3}{[12][P1]}\cdot\frac{\langle {+}P3\rangle^3}{\langle 34\rangle\langle 4{+}P\rangle}\right)^s\right)$$

$$= f_{abe} f_{edc}\left(\left(\frac{\langle 12\rangle^{\cancel{3}}[42]^3}{\langle \cancel{12}\rangle[31]\langle\cancel{12}\rangle[34]}\right)^s + \left(\frac{\langle 13\rangle^3[12]^{\cancel{3}\,1}}{\langle 34\rangle[\cancel{12}]\langle 42\rangle[\cancel{12}]}\right)^s\right)$$

The Limit $s\to 0$: $s = \langle 12\rangle[12] \overset{!}{=} 0$ we get to choose whether $\langle 12\rangle = 0$ or $[12]=0$

↳ corr. to choosing A_c to be holomorphic or antiholomorphic

This is where our choice ($\langle 12\rangle = 0$ or $[12]=0$) becomes "meaningful" and eliminates one of the contributions.

Note the quotation marks, it only eliminates one of the 2 terms, but it doesn't change the result; both of the 2 expressions are the same after some Algebra.

$$\frac{\langle 12\rangle[42]^3}{[31][34]} \overset{\langle 12\rangle[42] = \langle 13\rangle[34]}{=} \frac{\langle 13\rangle[24]^2}{[31]}\cdot\frac{\langle 13\rangle}{\langle 13\rangle} = -\frac{1}{t}\langle 13\rangle^2[24]^2 = \frac{1}{u}\langle 13\rangle^2[24]^2$$

$$\frac{\langle 13\rangle^3[12]}{\langle 34\rangle\langle 42\rangle} \overset{\langle 13\rangle[12] = \langle 34\rangle[42]}{=} \frac{\langle 13\rangle^2[42]}{\langle 42\rangle}\cdot\frac{[42]}{[42]} = \frac{1}{u}\langle 13\rangle^2[24]^2$$

↳ after enforcing $s\to 0$:

$$\boxed{\mathrm{Res}_s\, A_4(1_a^{-s}2_b^{+s}3_c^{-s}4_d^{+s}) = f_{abe}f_{ecd}\left(\frac{1}{u}\langle 13\rangle^2[24]^2\right)^s}$$

The result does not depend on the choice we make; as shown above

completely analogous calculation for the t-, u-channel yields:

↳ t-channel is the easiest, since we don't have to deal w/ pol. sums

$$\boxed{\mathrm{Res}_u\, A_4 = f_{dae}f_{bce}\left(\frac{1}{t}\langle 13\rangle^2[24]^2\right)^s \;;\quad \mathrm{Res}_t\, A_4 = f_{cae}f_{bde}\left(\frac{1}{s}\langle 13\rangle^2[24]^2\right)^s}$$

We'll be assuming the 4^{pt} Amplitude to be of the Form

$$A_4 = \underbrace{\left(\langle 13\rangle^2[24]^2\right)^s}_{\text{ensures correct hel. scaling}} \underbrace{\tilde{F}}_{\text{takes care of the correct pole structure}}(s,t,u)$$

Consider now:

·) Spin 1 particle:

↳ single spin-1 particle: $f_{abc} \to f$

Ansatz: $\tilde{F}(s,t,u) = \frac{c_{su}}{su} + \frac{c_{st}}{st} + \frac{c_{ut}}{ut}$ ⇝ why is this simple structure general enough? why not add e.g. $f_{su}(s/t)$? Answer: would ruin our requirement of simple poles more explicitly c.f. Schwartz 27.5.2

s/t ... only 1 indep. dim.less ratio

we require: $\lim_{s\to 0} s\tilde{F} = \frac{1}{u}f^2$; $\lim_{u\to 0} u\tilde{F} = \frac{1}{t}f^2$; $\lim_{t\to 0} t\tilde{F} = \frac{1}{s}f^2$

by consistent Factorisation

$$\Rightarrow \quad c_{su} - c_{st} = f^2 \qquad c_{ut} - c_{su} = f^2 \qquad c_{st} - c_{ut} = f^2$$

↳ summing the equ.'s: $0 = 3f^2$ ⇝ no 3^{pt} Amp. for one massless spin 1 particle

Note: this recovers a result from last week

$$\boxed{\rightsquigarrow \text{It is impossible to have a single self interacting massless spin-1 particle}}$$

↳ multiple spin-1 particles: $\tilde{F}^{abcd}(s,t,u) = \frac{c^{abcd}_{su}}{su} + \frac{c^{abcd}_{st}}{st} + \frac{c^{abcd}_{ut}}{ut}$

Imposing the same conditions:

$$\begin{aligned}
s:&\quad c^{abcd}_{su} - c^{abcd}_{st} = f^{abe}f^{ecd}\\
u:&\quad c^{abcd}_{ut} - c^{abcd}_{su} = f^{dae}f^{ebc}\\
t:&\quad c^{abcd}_{st} - c^{abcd}_{ut} = f^{cae}f^{ebd}
\end{aligned}$$

⇝ summing the equations yields

$$\boxed{f^{abe}f^{ecd} + f^{bce}f^{eda} + f^{cae}f^{ebd} = 0}$$

We found that a theory of several massless spin 1 particles can be non-trivial only if the dimensionless coupling constants are the structure constants of a Lie Algebra

·) Several Spin-2 particles

$$\mathrm{Res}_s A_4 = \kappa_{abe}\kappa_{ecd}\left(\frac{1}{u}\langle 13\rangle[24]\right)^2 \qquad \mathrm{Res}_t A_4 = \kappa_{cae}\kappa_{ebd}\left(\frac{1}{s}\langle 13\rangle^2[24]^2\right)^2$$

$$\mathrm{Res}_u A_4 = \kappa_{dae}\kappa_{ebc}\left(\frac{1}{t}\langle 13\rangle^2[24]^2\right)^2$$

⇝ looks problematic at first glance since we have "double poles" in $\{s,t,u\}$ in each residue; however noting that $u = -t$ $(s \to 0)$ we can resolve this "problem" and thus arrive at the following

Ansatz: $A_4 = \langle 13\rangle^4[24]^4\, \tilde{f}^{abcd}(s,t,u)$ w/ $\tilde{f}(s,t,u) = \dfrac{C^{abcd}}{stu}$

we require:

$$\lim_{s\to 0} s\,\tilde{f}^{abcd}(s,t,u) = \frac{1}{tu}C^{abcd} \overset{!}{=} \kappa^{abe}\kappa^{ecd}\frac{1}{u^2}$$

$$\lim_{u\to 0} u\,\tilde{f}^{abcd}(s,t,u) = \frac{1}{st}C^{abcd} \overset{!}{=} \kappa^{dae}\kappa^{bce}\frac{1}{t^2}$$

$$\lim_{t\to 0} t\,\tilde{f}^{abcd}(s,t,u) = \frac{1}{us}C^{abcd} \overset{!}{=} \kappa^{cae}\kappa^{bde}\frac{1}{s^2}$$

$$\Rightarrow \boxed{\kappa^{abe}\kappa^{ecd} = \kappa^{dae}\kappa^{bce} = \kappa^{cae}\kappa^{bde}} \qquad (*)$$

due to symmetry properties of κ^{abc} this implies that all other products are equivalent

(*) implies that the Algebra generated by must be commutative e.g.:

↳ $(T^c T^a)_{db} = \kappa^{dce}\kappa^{eab} = \kappa^{dae}\kappa^{ecb} = (T^a T^c)_{db}$ ⇝ $[T^a, T^c] = 0$

↳ It turns out that those Algebras are reducible and the theory reduces to that of several non-interacting massless spin-2 particles more explicit Discussion of this point c.f. 0705.4305

Here we see that consitent factorisation forbids a non-abelian generalisation of a spin-2 Theory. Due to the commutators vanishing it is always possible to say that the spin 2 particles are effectively in different universes w/ no mutual interactions; in each of these decoupled sectors the gravitons can be coupled to their own spectrum of particles. ⇝ We'll see the later part of this statement soon

-) Spin > 2 : There is no non-trivial way of having a consistent 4^{pt} Ampl. with only simple poles in $\{s,t,u\}$

Recall: $$\mathrm{Res}_s A_4 = f_{abe} f_{ecd} \left(\frac{1}{u}\langle 13\rangle^2 [24]^2\right)^s$$

$s=0$: no other poles

$s=1$: requires us to introduce a u-channel pole

$s=2$: using $u^2 = -ut$ (for $s=0$) this forces us to introduce poles in u- and t-channels

$s>2$: we do not have enough distinct kinematic invariants to achieve proper factorisation while still having only simple poles

Recap: So far we have shown (using consistent fact. w/ locality and unitarity) that no consistent self-int. theories of massless particles exist for $s>2$. $s=1$ requires YM structure and $s=2$ requires the grav.'s to interact only amongst their own "colour".

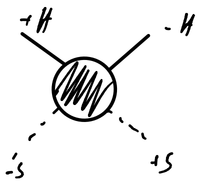
Note: the $s=0$ case doesn't yield more constraints than we already knew from exchange sym. (c.f. last week) $\rightsquigarrow$ ω_{abc} is totally symmetric

Interacting with others?

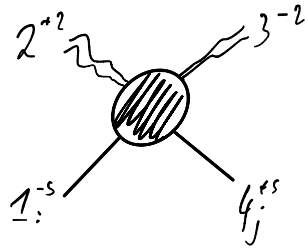
Do our theories play well with others?

What self-consistent interaction can our consistent $s=1,2$ theories have w/ other particles?

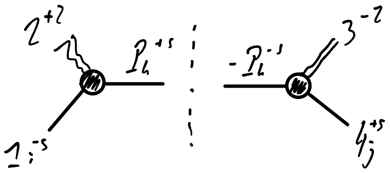
In order to achieve results we consider Amplitudes of the form "Compton-Form"

$$A(1^{-s} 2^{h} 3^{-h} 4^{+s}) =$$ 

·) $\underline{\mathcal{H}=2}$: Which Particles can consistently interact w/ Gravitons?

↳ consider: $A(\underline{1}_i^{-s}\; 2^{+2}\; 3^{-2}\; 4_j^{+s}) =$ 

Residue in s-channel:

Amplitude has to factorise as 

w/ $P = -\underline{1}-2 = 3+4$

↳ $\lim_{s\to0} sA_4 = A(\underline{1}_i^{-s}\; 2^{+2}\; P_k^{+s})\, A(3^{-2}\; 4_j^{+s}\; P_k^{-s}) = T^a_{ik}T^b_{kj}\, \frac{[2P]^{2+2s}}{[12]^{2s-2}[P1]^2}\cdot\frac{\langle P3\rangle^{2+2s}}{\langle 4P\rangle^2\langle 34\rangle^{2s-2}}$

$\langle 3P\rangle[P2] = -\langle 3|\not{1}|2] = \langle 31\rangle[21]$

$\langle 4P\rangle[P1] = -\langle 4|\not{2}|1] = \langle 42\rangle[12]$

$= T_{ik}T_{kj}\frac{\langle 31\rangle^{2+2s}[21]^{\cancel{2+2s}}}{[21]^{\cancel{2s}-2}\langle 43\rangle^{2s-2}\langle 42\rangle^2\cancel{[12]^2}} = T_{ik}T_{kj}\frac{\langle 31\rangle^{2+2s}[12]^2}{\langle 43\rangle^{2s-2}\langle 42\rangle^2} = T_{ik}T_{kj}\langle 13\rangle^{2s}[24]^{2s}\frac{\langle 13\rangle^2[12]^2}{\langle 42\rangle^2\langle 43\rangle^{2s-2}[24]^{2s}}$

$= T_{ik}T_{kj}\frac{1}{\langle 24\rangle^2[24]^2}\langle 13\rangle^{2s}[24]^{2s}\langle 13\rangle^2[12]^2\left(\langle 43\rangle[24]\right)^{2-2s}$

↪ absorb into redef. of T_{ij}

$\langle 43\rangle[24] = -\langle 3|\not{4}|2] = \langle 3|\not{1}|2] = \frac{1}{2}\langle 3|\not{1}-\not{4}|2]$

$\langle 13\rangle^2[12]^2 = \langle 34\rangle^2[42]^2 = \langle 3|\not{4}|2]^2 = \frac{1}{4}\langle 3|\not{1}-\not{4}|2]^2$

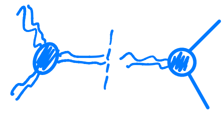
$\lim_{s\to0} sA_4 = -T^a_{ik}T^b_{kj}\,\frac{1}{tu}\,\langle 13\rangle^{2s}[24]^{2s}\left(\langle 3|\not{1}-\not{4}|2]\right)^{4-2s}$

In an analogous fashion we arrive at the Residues in the u-/t-channel

$\lim_{u\to0} uA_4 = -T^b_{ik}T^a_{kj}\,\frac{1}{st}\,\langle 13\rangle^{2s}[24]^{2s}\left(\langle 3|\not{1}-\not{4}|2]\right)^{4-2s}$

↩ recall $3^{pt}\sim\kappa_{abc}$ ⤳ diagonalizable; totally symm.

$\lim_{t\to0} tA_4 = -\kappa T^a_{ij}\,\frac{1}{us}\,\langle 13\rangle^{2s}[24]^{2s}\left(\langle 3|\not{1}-\not{4}|2]\right)^{4-2s}$

Note that the t-channel is distinct from the others: $\lim_{t\to 0} t A_4 =$ 

↳ it requires a self-interaction of the Graviton and thus intr. the coupling const. κ

Ansatz: $$A_4 = \langle 13\rangle^{2s}[24]^{2s}\left(\langle 3|1-4|2]\right)^{4-2s}\tilde F^{ab}_{ij}(s,t,u)$$

w/ $\tilde F_{ij}(s,t,u) = \frac{C^{ab}_{ij}}{stu}$, $[C_{ij}]_m \overset{!}{=} -2$

↳ consistency:

$$\lim_{s\to 0} stu\,\tilde F^{ab}_{ij}(s,t,u) = C^{ab}_{ij} \overset{!}{=} -T^a_{ik}T^b_{kj}$$

$$\lim_{u\to 0} stu\,\tilde F^{ab}_{ij}(s,t,u) = C^{ab}_{ij} \overset{!}{=} -T^b_{ik}T^a_{kj}$$

$$\lim_{t\to 0} stu\,\tilde F^{ab}_{ij}(s,t,u) = C^{ab}_{ij} \overset{!}{=} -\kappa^a T^a_{ij}$$

↳ we see $[T^a,T^b]=0$ ⇝ we can diagonalize

⇝ $\kappa T_{ij} \overset{!}{=} T_{ik}T_{kj}$

as a Matrix equ. $(\kappa\mathbb{1})T = T^2$

↳ $/T^{-1}$ ⇝ $\kappa\mathbb{1} = T$

forces grav. to couple to it's own matter

Note: A key assumption here is here is that the 3pt exists non-trivially $T_{ij}\neq 0$

↳ then we find C_{ij} must be symmetric ⇝ T_{ij} is symmetric Matrix w/ non-zero eval.'s ⇝ invertible

These constraints are satisfied for T_{ij} being proportional to the Identity Matrix. The prop. const. being the strength of the grav. self coupling!

↳ Gravity couples universally (to itself & other part.'s)

Another Important insight of $T = \kappa\mathbb{1}$ is that Gravity doesn't change the part. flavour or its colour!

↳ only the 3pt vert. preserving colour & flavour is consistent

Also: $$A_4 = \kappa^2\frac{1}{stu}\langle 13\rangle^{2s}[24]^{2s}\left(\langle 3|1-4|2]\right)^{4-2s}$$

Looking at this part of the Amplitude we observe that the Amplitude develops spurious poles for $s > 2$

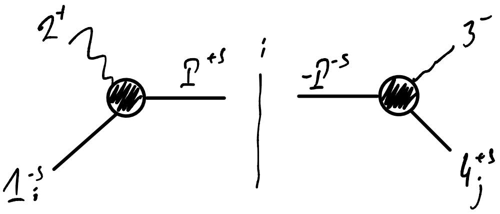
⇒ Gravity can only consistently couple to $s = 0, 1/2, 1, 3/2$ part.'s

·) $H = 1$:

·) only 1 particle

↳ from our analysis before we remember that there is no self-interaction ⇝ no t-chanel

s-channel:



$$\lim_{s\to0} s A_4 = A_3(1_i^{-s}\, 2^+\, P^{+s})\, A_3(-P^{-s}\, 3^-\, 4_j^{+s}) = T^a_{ih}\, \frac{[2P]^{2s+1}}{[12]^{2s-1}[P1]} \cdot T^b_{hj}\, \frac{\langle -P3\rangle^{2s+1}}{\langle 4\,{-P}\rangle\langle 34\rangle^{2s-1}}$$

⇝ ... ⇝

$$↳ \quad \lim_{s\to0} s A_4 = \frac{1}{u}\, T^a_{ih} T^b_{hj}\, (\langle13\rangle[24])^{2s} \, (\langle 3|1-4|2])^{2-2s}$$

analogously:

$$↳ \quad \lim_{u\to0} u A_4 = \frac{1}{s}\, T^b_{ih} T^a_{hj}\, (\langle13\rangle[24])^{2s} \, (\langle 3|1-4|2])^{2-2s}$$

There is no Residue in the t-chanel (no γ-self-int.)

$$↳ \quad A_4 = (\langle13\rangle[24])^{2s} \, \underbrace{(\langle 3|1-4|2])^{2-2s}} \, \frac{C^{ab}_{ij}}{su}$$

⇝ Note: Consistency again forces $[T^a, T^b] = 0$

↳ see below

Note: only allows for values of $s < 3/2$!

→ sym/anti-sym.? $s = 1 \Rightarrow C_{ij} \overset{!}{=} 0$ no γ self int.; seen before

$s = 0$ symmetric in $1 \leftrightarrow 4$

$s = 1/2$ antisym. in $1 \leftrightarrow 4$

⟶ C_{ij} is a symmetric Matrix $\begin{pmatrix} e_{e_e} & \\ & \ddots \end{pmatrix}$

↳ since it is a symmetric matrix we can Diagonalise it and thus see that the interaction is flavour conserving.

No consistent theories for charged spin-3/2 particles

What about multiple massless spin-1 particles?

$\rightsquigarrow$ s- and u-channel Residues remain the same up to multiplication w/ a coupling constant incorporating the colour structure

$$\mathrm{Res}_s A_4 \mapsto (T^a T^b)_{ij}\, \mathrm{Res}_s A_4$$

$$\mathrm{Res}_u A_4 \mapsto (T^b T^a)_{ij}\, \mathrm{Res}_u A_4$$

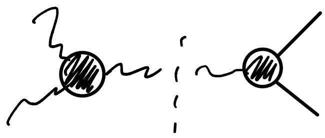
Note: Now the Factorisation is only consistent if $(T^a T^b)_{ij} = (T^b T^a)_{ij}$

$\hookrightarrow [T^a, T^b] = 0$

(Abelian Gauge Group)

if $[T^a, T^b] \neq 0$ we have to introduce a self interaction for the massless spin-1 particles in order to achieve consistent factorisation

<u>t-channel:</u>



Familiar calculation of the residues yields:

$$\lim_{s\to 0} s A_4 = \frac{1}{u} (T^a T^b)_{ij} \left(\langle 13\rangle[24]\right)^{2s} \left(\langle 3\, \not{1} - \not{4}\, 2]\right)^{2-2s}$$

$$\lim_{u\to 0} u A_4 = \frac{1}{s} (T^b T^a)_{ij} \left(\langle 13\rangle[24]\right)^{2s} \left(\langle 3\, \not{1} - \not{4}\, 2]\right)^{2-2s}$$

$$\lim_{t\to 0} t A_4 = \left(\frac{1}{s} - \frac{1}{u}\right) f_{abc} T^c_{ij} \left(\langle 13\rangle[24]\right)^{2s} \left(\langle 3\, \not{1} - \not{4}\, 2]\right)^{2-2s}$$

Requiring the Amplitude A_4 to factorises correctly into 3^{pt} Amplitudes on its poles leads us to the following Ansatz

$$A_4 = \left(\langle 13\rangle[24]\right)^{2s}\left(\langle 3|\not{1}-\not{4}|2]\right)^{2-2s}\left(\left(T^aT^b+T^bT^a\right)_{ij}\frac{1}{us}+f_{abc}T^c_{ij}\left(\frac{1}{st}-\frac{1}{tu}\right)\right)$$

imposing the consistency conditions

$$\lim_{s\to 0} us A_4 \sim \left(T^aT^b+T^bT^a\right)_{ij} - f^{abc}T^c_{ij} \overset{!}{=} \left(T^aT^b\right)_{ij}$$

$$\lim_{u\to 0} su A_4 \sim \left(T^aT^b+T^bT^a\right)_{ij} + f^{abc}T^c_{ij} \overset{!}{=} \left(T^bT^c\right)_{ij}$$

$$\lim_{t\to 0} ts A_4 \sim 2f_{abc}T^c_{ij} \overset{!}{=} 2f_{abc}T^c_{ij}$$

✓ always satisfied

adding up these eqn.s yields

$$2f^{abc}T^c_{ij} \overset{!}{=} \left(T^bT^a - T^aT^b\right)_{ij}$$

↳ redefining the arbitrary constants yields : $\boxed{[T^a,T^b] = f^{abc}T^c}$

Theories like this are only consistent if the Particles coupling constants are given as representations of Lie-Algebra Generators

↳ Thus we find that Gauge Theories Based on Lie Algebras are the unique interacting theories spin 1 particles

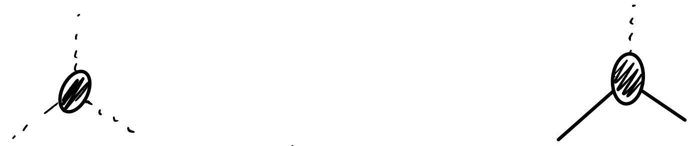
Is charge Conserved?

We've seen above that a photon can only take part in the following amplitudes

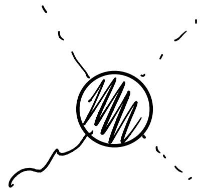
and that it conserves colour/flavour and hence it necessarily conserves charge.

To see if charge is conserved it is sufficient to check if the 3^{pt}-Amp.'s conserve it since we can construct the higher point Amp.'s from them.

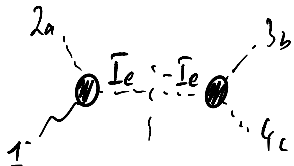
We know that all graviton vertices conserve colour/flavour and therefore charge, so we only have to check



In order to probe these Amplitudes we employ the consistency of 4^{pt} Amplitudes yet again

$A(1^- 2_a 3_b 4_c) =$ 

<u>s-channel:</u>



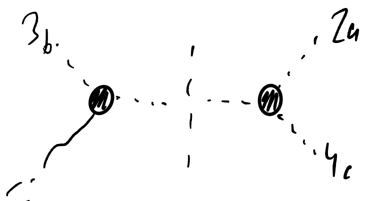
$$\sim e_2 w_{abc} \frac{\langle 12\rangle\langle I1\rangle}{\langle 2I\rangle} \cdot \frac{[3I]}{[3I]} = -e_2 w_{abc} \frac{\langle 12\rangle\langle 14\rangle}{\langle 24\rangle}$$

$$= -e_2 w_{abc} \langle 12\rangle\langle 14\rangle[24] \frac{1}{t}$$

$$\langle 2I\rangle[3I] = \langle 2 \cancel{I} 3] = \langle 24\rangle[34]$$
$$\langle I1\rangle[3I] = -\langle 1 \cancel{I} 3] = \langle 14\rangle[43]$$

$$\langle 12\rangle[24] = -\langle 13\rangle[34]$$

<u>t-channel:</u>



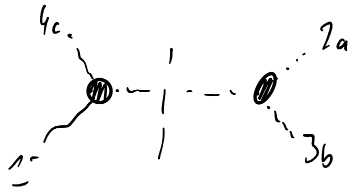
$$\sim e_3 w_{abc} \frac{\langle 13\rangle\langle I'1\rangle}{\langle 3I'\rangle} \cdot \frac{[4I']}{[4I']} = +e_3 w_{abc} \frac{\langle 13\rangle\langle 12\rangle[24]}{\langle 23\rangle[24]}$$

$$= -e_3 w_{abc} \langle 13\rangle\langle 12\rangle[23] \frac{1}{s}$$

$$= +e_3 w_{abc} \langle 14\rangle\langle 12\rangle[24] \frac{1}{s}$$

$$\langle 3I'\rangle[4I'] = \langle 3 \cancel{I} 4] = \langle 32\rangle[42]$$
$$\langle I'1\rangle[4I'] = -\langle 1 \cancel{I} 4] = \langle 12\rangle[24]$$
$$\langle 13\rangle[23] = \langle 14\rangle[42]$$

u-channel:



$$\sim e_4 w_{abc} \frac{\langle 14\rangle\langle I''1\rangle}{\langle 4I''\rangle}\cdot\frac{[3I'']}{[3I'']} = -e_4 w_{abc}\frac{\langle 14\rangle\langle 12\rangle\cancel{[23]}}{\langle 42\rangle\cancel{[32]}}$$

$$\langle I''1\rangle[3I''] = -\langle 1|\cancel{I''}|3] = \langle 12\rangle[23]$$

$$\langle 4I''\rangle[3I''] = \langle 4|\cancel{I''}|3] = \langle 42\rangle[32]$$

$$= +e_4 w_{abc}\langle 12\rangle\langle 14\rangle[24]\frac{1}{t}$$

$$A_4 = \langle 12\rangle\langle 14\rangle[24]\tilde{F}(s,t,u)$$

$\hookrightarrow \lim_{s\to 0} us\,\tilde{F}(s,t,u) \overset{!}{=} e_2 w_{abc}$; $\lim_{t\to 0} st\,\tilde{F}(s,t,u) \overset{!}{=} e_3 w_{abc}$; $\lim_{u\to 0} tu\,\tilde{F}(s,t,u) \overset{!}{=} e_4 w_{abc}$

Ansatz: $$\tilde{F}(s,t,u) = w_{abc}\left(\frac{C_1}{us} + \frac{C_2}{st} + \frac{C_3}{tu}\right)$$

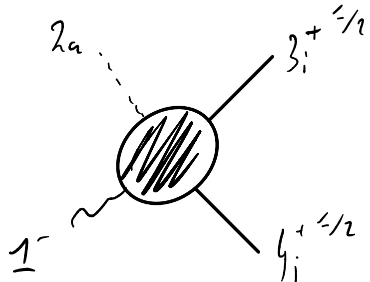
$\hookrightarrow \lim_{s\to 0} us\,\tilde{F}(s,t,u) = w_{abc}(C_1 - C_2) \overset{!}{=} w_{abc}\, e_2$

$\hookrightarrow \lim_{t\to 0} st\,\tilde{F}(s,t,u) = w_{abc}(C_2 - C_3) \overset{!}{=} w_{abc}\, e_3$

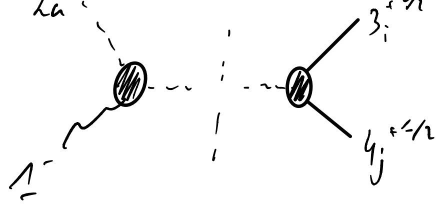
$\hookrightarrow \lim_{u\to 0} tu\,\tilde{F}(s,t,u) = \underbrace{w_{abc}(C_3 - C_1)}_{} \overset{!}{=} w_{abc}\, e_4$

$$\Sigma \quad 0 \overset{!}{=} w_{abc}\underbrace{(e_2 + e_3 + e_4)}_{}$$ <u>charge cons.!</u>

$$A(1^- 2_a 3_i^{+1/2} 4_j^{+1/2}) =$$



s-channel:



$$= e_2\delta_{ae}T^e_{ij}\frac{\langle 12\rangle\langle I1\rangle}{\langle 2I\rangle}\cdot[34]\cdot\frac{[3I]}{[3I]}$$

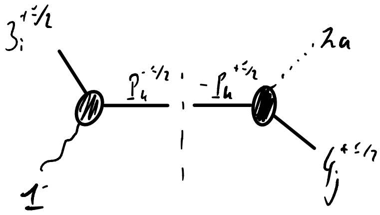
$$\langle I1\rangle[3I] = -\langle 1|\cancel{I}|3] = \langle 14\rangle[43]$$

$$\langle 2I\rangle[3I] = \langle 2|\cancel{I}|3] = \langle 24\rangle[34]$$

$$= e_2 T^a_{ij}\frac{\langle 12\rangle\langle 14\rangle[43]\cancel{[34]}}{\langle 24\rangle\cancel{[34]}} = \frac{1}{t}e_2 T^a_{ij}\langle 12\rangle\langle 14\rangle[43][24]$$

$$= \frac{1}{t}e_2 T^a_{ij}\langle 1|\cancel{2}|4]\langle 1|\cancel{4}|3]$$

t-channel:



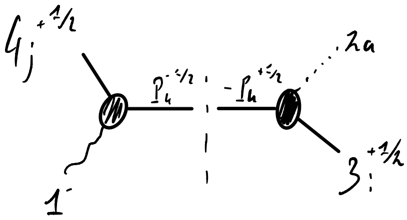
$$= e_3\,\delta_{ik}\,\frac{\langle P1\rangle^2}{\langle 3P\rangle}\cdot T^a_{jk}\,[4+P_4] \qquad (-P)] = P]$$

$$\langle P1\rangle^2[4P]^2 = \langle 1\not{P}4]^2 = \langle 12\rangle^2[24]^2$$

$$\langle 3P\rangle[4P] = \langle 3\not{P}4] = \langle 32\rangle[42]$$

$$= e_3 T^a_{ji}\,\frac{\langle 12\rangle^2[24]^{\cancel{2}}}{\langle 32\rangle\cancel{[42]}} = \frac{1}{u}\,e_3 T^a_{ij}\langle 12\rangle[24]\,\langle 12\rangle[32] = \frac{1}{u}\,e_3 T^a_{ij}\langle 1\not{2}4]\langle 1\not{4}3]$$

u-channel:



$$= e_4\,\delta_{jk}\,\frac{\langle P1\rangle^2}{\langle 4P\rangle}\cdot T^a_{ik}\,[3+P]$$

$$\langle P1\rangle^2[3P]^2 = \langle 1\not{P}3]^2 = \langle 12\rangle^2[23]^2$$

$$\langle 4P\rangle[3P] = \langle 4\not{P}3] = -\langle 42\rangle[23]$$

$$= -e_4 T^a_{ij}\,\frac{\langle 12\rangle^2[23]^{\cancel{2}}}{\langle 42\rangle\cancel{[23]}} = -e_4 T^a_{ij}\,\frac{1}{t}\,\langle 12\rangle[23]\langle 12\rangle[42] = \frac{1}{s}\,e_4 T^a_{ij}\langle 1\not{2}4]\langle 1\not{4}3]$$

$$A_4 = \langle 1\not{2}4]\langle 1\not{4}3]\,\tilde{F}(s,t,u)$$

$$\lim_{s\to0} s\,\tilde{F}(s,t,u) \overset{!}{=} e_2 T^a_{ij} \;;\quad \lim_{t\to0} tu\,\tilde{F}(s,t,u) \overset{!}{=} e_3 T^a_{ij} \;;\quad \lim_{u\to0} us = e_4 T^a_{ij}$$

Ansatz: $$\tilde{F}(s,t,u) = T^a_{ij}\left(\frac{c_1}{us} + \frac{c_2}{st} + \frac{c_3}{tu}\right)$$

$$\lim_{s\to0} st\,\tilde{F}(s,t,u) = T^a_{ij}\,(c_2 - c_1) \overset{!}{=} T^a_{ij}\,e_2$$

$$\lim_{t\to0} tu\,\tilde{F}(s,t,u) = T^a_{ij}\,(c_3 - c_2) \overset{!}{=} T^a_{ij}\,e_3$$

$$\lim_{u\to0} tu\,\tilde{F}(s,t,u) = \underbrace{T^a_{ij}\,(c_1 - c_3)}_{\Sigma\quad 0} \overset{!}{=} T^a_{ij}\,e_4$$

$$\Sigma \qquad 0 \overset{!}{=} T^a_{ij}\,\underbrace{(e_2 + e_3 + e_4)}$$

charge conservation!

Recap:

Now we have seen that YM & GR are the unique theories of interacting spin-1 and -2 particles as well as the statements that charge is conserved, gravity couples universally, & there are no consistent theories for spin > 2 particles

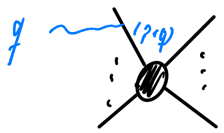
↳ in conventional Field theory these statements arise from the Weinberg soft theorem
here we have reproduced these results and gained even more!

Weinbergs Argument:

c.f. Schwartz 9.5

Asking for consistency of physical processes → we'll learn here charge conservation, univ. of Grav. & no spin ≥ 3

Ampl. $\mathcal{M}(p_1,\dots,p_n)$



emitting an extra massless particle

↳ $\mathcal{M}(p_1,\dots,p_n;q)$ what happens as $q \to 0$?

↳ we get a prop. in the Amp.: $\frac{1}{(p_i+q)^2 - M_i^2} = \frac{1}{2q\cdot p_i}$ ← singular as $q \to 0$

only terms that become large: emissio on 1 one of the ext. legs; if it was inside the Diagram all prop. would be off-shell → no cancelation $p_i^2 - M_i^2 = 0$

What is the Numerator? Weinberg assumed the most leading possible int. is $\sim \varepsilon_\mu(q)\cdot p_{i\mu}$

nothing can be more leading

$$\hookrightarrow \quad \mathcal{M}(p_i,q) = \mathcal{M}(p_i) \times \underbrace{\sum_i \frac{\varepsilon^\mu(q)\cdot p_{i\mu}}{2p_i\cdot q}\, e_i}_{\equiv S}$$

b/c of Lorentz inv. + Unitarity: $\varepsilon \to \varepsilon + \alpha p$ must give the same answer

$\Rightarrow S \to \frac{1}{2}\sum_i e_i \overset{!}{=} 0$ ⇝ charge must be conserved

↳ only way to have consistent interactions w/ massless spin-1

Analogously we find for the Graviton $\mathcal{M}(p_i,q) = \mathcal{M}(p_i) \times \sum_i \frac{\varepsilon^{\mu\nu}(q)\, p_{i\mu} p_{i\nu}\, \kappa_i}{2 p_i \cdot q}$

red.: $h_{\mu\nu} \mapsto h_{\mu\nu} + \alpha_{(\mu} q_{\nu)}$

$$\mathcal{M}(p_i,q) \mapsto \mathcal{M}(p_i) \times \sum_i \frac{\alpha p_i (q \cdot p_i)}{2 p_i \cdot q} \kappa_i \overset{!}{=} 0$$

$$\hookrightarrow \sum_i \alpha p_i \kappa_i = \alpha^\mu \underbrace{\sum_i p_{i\mu} \kappa_i}_{} \overset{!}{=} 0$$

now: $\cdot$) scattering only at discrete angles

$\cdot$) $\underbrace{\kappa_i = \kappa \; \forall i}_{}$ $\rightsquigarrow$ $\sum_i p_{i\mu} = 0$ by momentum cons.

Consequence of properly understanding Soft Limit of GR — principle of equivalence

Weinbergs soft theorem is derived in Field Theory by requiring that gauge / diffeo. transf.'s do not affect the physics

We start w/ Fields, which transform as representations of the Lorentz Group (while physical particles only transform as representations of the Little Group). The Feynman Amplitudes calculated in Field Theory also have the wrong transformation properties. They transform as Lorentz Tensors $\rightsquigarrow$ this is why we have to introduce "polarisation-vectors" which transform as bi-fundamentals of the Lorentz- and Little Group (e.g. $\varepsilon^\mu_\sigma(\Lambda p) = \Lambda^\mu{}_\nu \, \varepsilon^\nu_{\sigma'}(p)\, D_{\sigma\sigma'}(W)$ for a massive spin-1 particle). A problem arises when we try to find such bi-fundamentals for massless particles $\rightsquigarrow$ they do not exist (only gauge-equivalence classes like $\{\varepsilon^\mu_\sigma \,|\, \varepsilon^\mu_\sigma + \alpha p^\mu\}$) exist. This is one key simplification in the Amplitudes Formalism, we avoid the introduction of redundancy due to the fact that only on-shell physical states are considered.
We achieved the same results as Weinberg (even more!) only through the application of Little Group Covariance, Unitarity & Locality

Great! Now we have shown something in this new approach that Weinberg already showed but as a quote attributed to Feynman says: "Every theoretical physicist should know six or seven different theoretical representations for exactly the same physics."
Well, two is a good start and it is amazing how vastly different they are.

Fun w/ Pole Counting

$\rightsquigarrow$ c.f. 1311.2938

Useful to introduce the following Definitions for 3^{pt} Amp.'s

$$\boxed{A = |h_1 + h_2 + h_3| \quad ; \quad H = \max\{|h_1|, |h_2|, |h_3|\}}$$

↳ Ruling out Constructible Theories by Pole Counting

Pedestrian Pole Counting (mandated by constructibility) strongly constrains on-shell Amp.'s

↳ The number of Poles in an Amplitude must be less or equal to the number of accessible physical factorization channels.

↳ 4^{pt} Amp.'s naturally split into three parts: (Note: we consider Amp.' contr. from $A_3, \bar{A}_3$)

- ·) a Numerator $\mathcal{N}$ accounting for all the Little Group weights of the Helicity states
- ·) a Factor $F(s,t,u)$ which encodes the pole structure
- ·) the coupling constants which encode the species dependent part of the int.'s

$\rightsquigarrow A_4 = \kappa_3^2 \, \underline{\mathcal{N}} \, \frac{1}{F(s,t,u)}$ w/ $\underline{[\mathcal{N}/F(s,t,u)]_m = 2A-1}$ shown below

↳ it can be shown that the minimal Numerator accomplishing these requirements has $2H$ anti-/holomorphic brackets none of which cancels against poles from $F(s,t,u)$

$$\mathcal{N} \sim \langle\ \rangle_1 \ldots \langle\ \rangle_{2H} [\ ]_1 \ldots [\ ]_{2H} \rightsquigarrow [\mathcal{N}] = [(k^2)^{2H}] = 4H$$

↳ has exactly $2H$ holomorphic ; $2H$ anti-holomorphic brackets

this statement is proven in 1311.2938

Can we count the number of poles a 4^{pt} Amp. has for given 3^{pt} Amp.'s ?

↳ Yes!

Note: here we consider 4^{pt} Amplitudes constructed from A_3 and its parity Conjugate

We observe that the net-total number of brackets in A_3 is given by A; useful observation since this number gives its mass dimension.

$\rightsquigarrow$ As argued for last week: $[A_n]_m = 4 - n$

$$\rightsquigarrow [A_3]_m = [\kappa_3]_m + A \overset{!}{=} 1 \rightsquigarrow [\kappa_3]_m = 1 - A$$

knowing that $[A_4]_m \overset{!}{=} 0 \rightsquigarrow [\kappa_3^2]_m + [N/F_{(s,t,u)}]_m \overset{!}{=} 0$

$$\rightsquigarrow [N/F]_m = 2(A-1)$$

also proven in the paper: $[N]_m = 4H$ and stated above

$$\rightsquigarrow \text{we find: } [F] \overset{!}{=} 2(2H + 1 - A)$$

F is a function of the Mandelstam invariants and as argued for in the paper it cannot contain any terms which could cancel against N.

↳ since $[\{s,t,u\}] = 2 \rightsquigarrow$ we find $N_p \overset{!}{=} 2H + 1 - A$

by locality we can only have simple poles in $\{s,t,u\}$

knowing that we can have a maximum of 3 poles

↳ we find the additional constr. $\boxed{2H + 1 - A \leq 3}$

Therefore we arrive at the powerful constraint:

A Theory is necessarily inconsistent if

$$2H + 1 - A > 3$$

Number of Poles > Cardinality of $\{s,t,u\}$

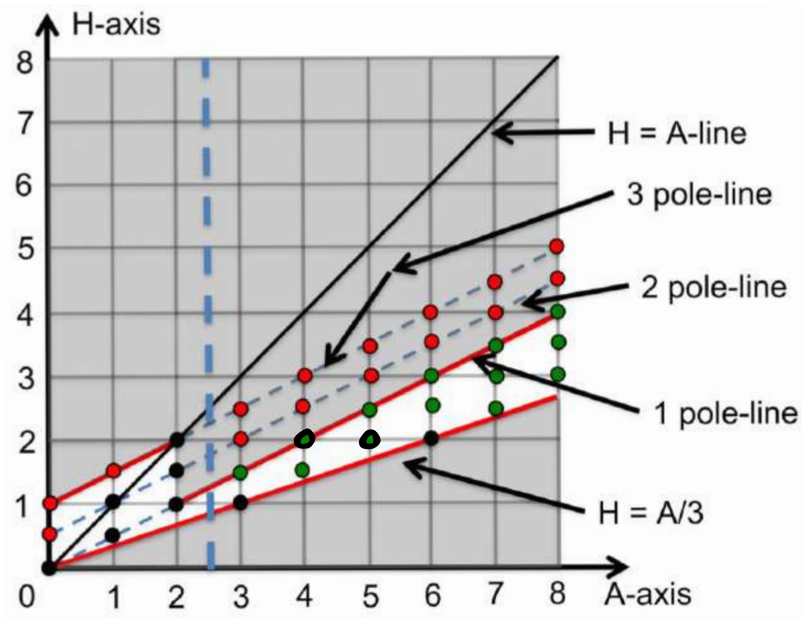
just using pedestrian pole counting we can already exclude an insane amount of theories

$\hookrightarrow$ all Amplitudes in the grey areas are excluded

The Number of Poles in a four point Amp. increases w/ the highest spin particle in the theory

Note: this constraint already rules out theories w/ relevant int. for massless spin $3/2$, 2

$$A = 0 \rightsquigarrow 2H + 1 \leq 3 \rightsquigarrow H \leq 1$$



taken from 1311.2938

Recall: $A = |\sum h|$; $H = \max\{h_i\}$

Black Dots ... 3pt Amp.'s that define self consistent S-Matrices that can couple to grav.

Green Dots ... 3pt Amp.'s that define S-Matrices that cannot couple to S-Matrices defined by Black Dots

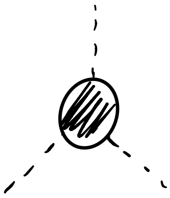
Red Dots ... cannot ever form a consistent S-Mat.

pole counting already labels a lot of theories as inconsistent, we can eliminate even more by means of consistent factorisation; and even more are eliminated when considering coupling to gravitons

$\rightsquigarrow$ only the Black Dots are consistent

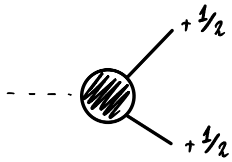
Building Blocks for Consistent Amplitudes

$A = H = 0$: $A(0,0,0)$

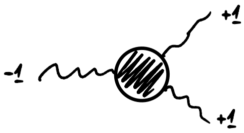


Note: $A(0, +\tfrac{1}{2}, -\tfrac{1}{2})$ is inconsistent

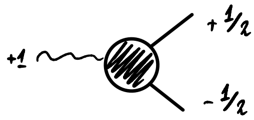
$A = 1;\ H = \tfrac{1}{2}$: $A(\tfrac{1}{2}, \tfrac{1}{2}, 0)$



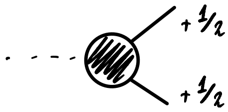
$A = 1;\ H = 1$: $A(+1, +1, -1)$



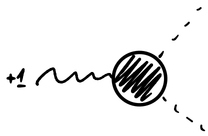
$A(+1, +\tfrac{1}{2}, -\tfrac{1}{2})$



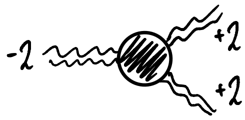
$A(0, +\tfrac{1}{2}, +\tfrac{1}{2})$



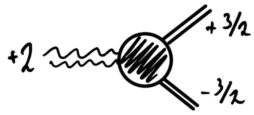
$A(+1, 0, 0)$



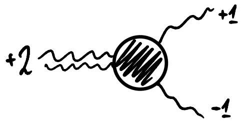
$A = H = 2$: $A(+2, +2, -2)$



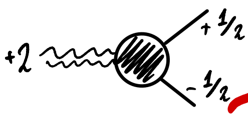
$A(2, \tfrac{3}{2}, -\tfrac{3}{2})$



$A(2, 1, -1)$

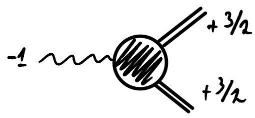


$A(2, \tfrac{1}{2}, -\tfrac{1}{2})$

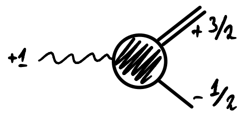


gravitational int.'s

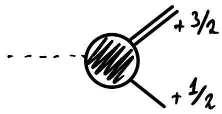
$A = 2;\ H = \tfrac{3}{2}$: $A(\tfrac{3}{2}, \tfrac{3}{2}, -1)$



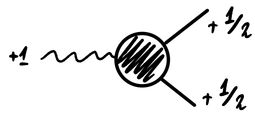
$A(\tfrac{3}{2}, 1, -\tfrac{1}{2})$



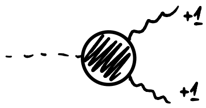
$A(\tfrac{3}{2}, \tfrac{1}{2}, 0)$



$A = 2;\ H = 1$: $A(1, \tfrac{1}{2}, \tfrac{1}{2})$

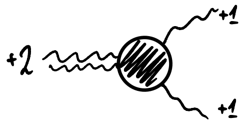


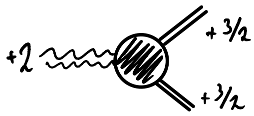
$A(1, 1, 0)$

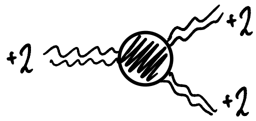


N = 8 SUGRA Amp.'s

$A = 2$:

- $\mathcal{N} = 4$: $A(2,1,1)$ 

- $\mathcal{N} = 5$: $A(2,\,^3/_2,\,^3/_2)$ 

- $\mathcal{N} = 6$: $A(2,2,2)$ 

↳ higher derivative 3pt-Amp.s

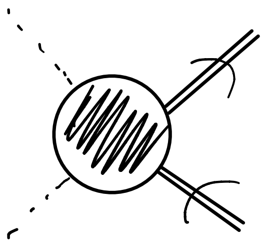
"It is remarkable that the architecture of fundamental physics emerges from these concrete algebraic consistency conditions" ~ Nima Arkani-Hamed

Fun Fact:

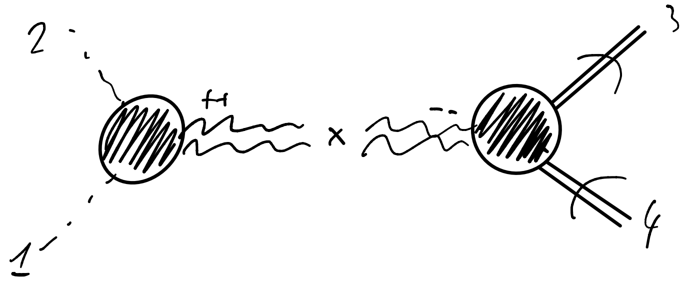
We discover the need for Supersymmetry when massless spin $3/2$ particles are present.

Consider:

$$A(\underline{1}^0, 2^0, 3^{-3/2}, 4^{3/2}) =$$

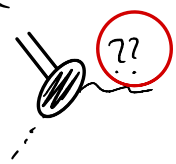


Again using our ol' reliable consistency check by Factorisation we find the residue in the s-channel:



$$\rightsquigarrow \dots \rightsquigarrow \frac{1}{8}\delta_{ab}\delta_{ij}\frac{\langle 3|\underline{1}-\underline{2}|4]^3}{t}$$

it has a pole in the t-channel! in order to have a consistent theory we have to introduce an amplitude like



what gives consistent constr. theories?

↳ comparing w/ results from before:

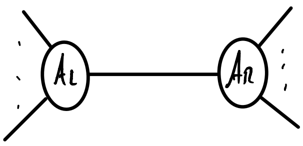
we see that we have to introduce a massless spin $1/2$ Fermion w/ the same coupling strength as the graviton! (same procedure as always)

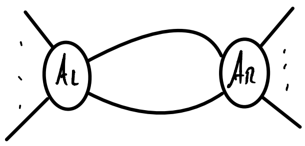
↳ we find: $A(\underline{1}, \underline{2}, 3^{-3/2}, 4^{3/2}) = \kappa^2 \dfrac{\langle 3|\underline{1}-\underline{2}|4]^3}{st}$

Outlook:

·) Loop Calculations:

So far we restricted to working with the pole structure (which corresponds to working at tree level in conventional Field Theory). The Branch Cut structure of Amplitudes corresponds to Loop Calculations in full analogy to the tree level procedure





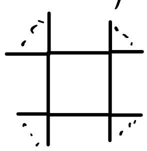
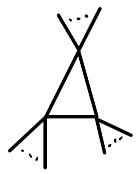
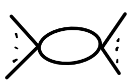
$\rightsquigarrow$ exists a pole w/ $\operatorname{Res} A = A_L \times A_R$

$\rightsquigarrow$ existence of a branch cut w/ $\operatorname{Disc} A = \int d\,\underline{LIPS}\; A_L \times A_R$

From Field Theory we know that we can express 1-Loop Amplitudes through a set of scalar basis integrals multiplied by coefficients encoding all the kinematics and colour structure

↳ Passarino - Veltmann Decomposition

rational pieces

$A_n^{1\text{-loop}} = \sum_i d_i$  $+ \sum_i c_i$  $+ \sum_i b_i$  $+ R_n$

↳ no tadpoles for massless particles

LHS is defined by the sum over all 1-loop Feynman Diagrams, RHS is an alternative representation of the same analytic function $\rightsquigarrow$ since they are the same analytic function they must have the same analytic continuation meaning they must have the same Branch cuts and Discontinuities across them

↳ use this information to get coeff.'s

for Details cf. 0808.1446

The idea is to perform generalised unitarity cuts.

generalisation of the Cutkosky procedure to complex momenta & multi-line cuts

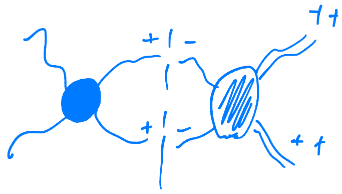
This idea is very powerful, & provides us (after some leg work to arrive here ⇝ c.f. 2005.07129) with a new way of calculating the anomalous dimension

$$\gamma_{ij} A_{O_i}(1,\dots,n) = -\frac{1}{4\pi^3}\frac{C_{O_i}}{C_{O_j}}\int d\,\mathrm{LIPS} \sum_{\substack{\text{ext. legs distr.}\\ l_1,l_2}} A_L(\dots,l_1,l_2)\, A(-l_1,-l_2,\dots)$$

we do not have to do any loop integrations; just phase-space integrations; additionally it is fairly easy to see non-renormalisation (if one of the Amplitudes in the integral vanishes)

e.g. $\underline{\tilde F^4 \to C^2 \tilde F^2}$:

↳ vanishes



$A_R = 0$

·) Massive Particles:

So far we only dealt with with massless particles, what about massive particles? Are they gonna be a massive problem? Turns out they can be dealt with without too much compl.

Recall 1st Seminar: $p^\mu \to \sigma_{\mu a\dot a} p^\mu = p_{a\dot a}$ w/ $\det p_{a\dot a} = m^2$

for massless momenta $p_{a\dot a}$ is a rank 1 matrix and we could write it as the exterior products of 2 spinors $\lambda_a \tilde\lambda_{\dot a}$

For massive momenta $p_{a\dot a}$ has rank 2 and we can simply write it as the sum of two rank 1 matrices $\lambda^1_a \tilde\lambda_{\dot a 1} + \lambda^2_a \tilde\lambda_{\dot a 2}$

$\rightsquigarrow p_{a\dot a} = \lambda^I_a \tilde\lambda_{\dot a I} = |\mathbf{p}\rangle[\mathbf{p}|$

satisfying the equations $\not p\, |\mathbf{1}] = |\mathbf{1}\rangle$; $\not p\, |\mathbf{1}\rangle = |\mathbf{1}]$

the pol. of a massive spin s particle can be expressed as the symmetric comb. of 2s these SU(2) tensors

↳ e.g. $M^{I_1\dots I_{2s}} = |\mathbf{p}\rangle^{I_1}_{\alpha_1}\dots|\mathbf{p}\rangle^{I_{2s}}_{\alpha_{2s}}\, M^{\alpha_1\dots\alpha_{2s}}$

key take away: we can also treat massive particles in the on-shell formalism.

↳ For more info: c.f. 1709.04891