# On-shell EFTs II: pNRQCD

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January 13, 2021

## 1 Second matching: from NRQCD to pNRQCD

As already mentioned in previous subsections, pNRQCD is obtained from NRQCD by integrating out all soft modes and potential massless modes. The resulting EFT has as degrees of freedom only potential quarks and ultrasoft gluons, hence power counting in terms of $v$ can be made non-ambiguous at the level of the Lagrangian. An important first observation is that all new operators involving heavy quark fields in the pNRQCD Lagrangian will be non-local in space, since the momentum of the heavy quark ($\mathbf{p} \sim m_Q v$) is of the order of the soft scale that is integrated out. It will still be local in time. We will understand this more a bit later.

### 1.1 Construction of the Lagrangian

After identifying the degrees of freedom of the theory, we can construct the pNRQCD Lagrangian by writing down the most general Lagrangian involving these fields and compatible with the symmetries of NRQCD. Schematically, it will take the form

$$\mathcal{L}_{pNRQCD} = \mathcal{L}'_{NRQCD} + \mathcal{L}_{non-local} \tag{1}$$

which splits the Lagragian in a local and a non-local part. The local terms are those of NRQCD, but the heavy quarks are now exclusively potential and gluons and light quark fields are always ultrasoft. Explicitly, the leading orders of the pNRQCD Lagrangian are given by

$$\begin{aligned}\mathcal{L}_{pNRQCD} = &\psi^\dagger(x)\left(i\partial_0 + g_s A_0(t,\mathbf{0}) - g_s\, \boldsymbol{x}\cdot\boldsymbol{E}(t,\mathbf{0}) + \frac{\boldsymbol{\partial}^2}{2m_Q}\right)\psi(x) + \\ &+ \chi^\dagger(x)\left(i\partial_0 + g_s A_0(t,\mathbf{0}) - g_s\, \boldsymbol{x}\cdot\boldsymbol{E}(t,\mathbf{0}) - \frac{\boldsymbol{\partial}^2}{2m_Q}\right)\chi(x) \\ &+ \int d^3\mathbf{r}\, \psi_a^\dagger\psi_b(x+\mathbf{r})\, V_{ab;cd}(r,\boldsymbol{\partial})\, \chi_c^\dagger\chi_d(x) + \mathcal{L}_g,\end{aligned} \tag{2}$$

where $\mathcal{L}_g$ is the now purely ultrasoft gluon and light quark Lagrangian and $E^i = G^{i0}$ the chromo-electric field.
The first two rows come from the NRQCD terms $\psi^\dagger\left(iD_0 + \frac{\boldsymbol{D}^2}{2m_Q}\right)\psi + \chi^\dagger\left(iD_0 - \frac{\boldsymbol{D}^2}{2m_Q}\right)\chi$, but look different because the ultrasoft gluon field was multipole-expanded according to[1]

$$A^\mu(t,\boldsymbol{x}) = A^\mu(t,\mathbf{0}) + \boldsymbol{x}\partial_{\boldsymbol{x}} A^\mu(t,\mathbf{0}) + \dots \tag{3}$$

[1]This implies to choose the origin of our coordinate system to coincide with the c.o.m. of the $Q\bar{Q}$-system.

The reason why we (have to) do this is that we wanted to have a non-ambiguous power counting in $v$ at the level of $\mathcal{L}_{pNRQCD}$. However, in the interaction of an ultrasoft gluon with potential heavy (anti)quarks ($\psi^\dagger A\psi$-type vertices), the heavy (anti)quark momentum $\mathbf{p} \sim m_Q v$ changes by an amount of the ultrasoft gluon momentum $\mathbf{l} \sim m_Q v^2$, hence the momentum space expression will allow for an expansion in $\mathbf{l}/\mathbf{p}$. These different orders in $v$ in the $\psi^\dagger A\psi$-type vertex are separated by the multipole expansion of the ultrasoft gluon fields appearing in these types of vertices. A more physical explanation for this is that the relative distance of the $Q\bar{Q}$ system is given by the inverse of the soft scale $\sim 1/(m_Q v)$ and is therefore small compared to the remaining dynamical lengths in the system. This allows us to multipole expand the gluon fields with respect to this relative distance. Explicitly, the expansion gives

$$\psi^\dagger \left( iD_0 + \frac{\boldsymbol{D}^2}{2m_Q} \right) \psi = \psi^\dagger \left( i\partial_0 + g_s A_0(t,\boldsymbol{r}) - \frac{ig_s}{2m_Q}(\vec{\partial} - \overleftarrow{\partial}) \cdot \vec{A}(t,\boldsymbol{r}) - \frac{g_s^2}{2m_Q}\vec{A}^2(t,\boldsymbol{r}) + \frac{\vec{\partial}^2}{2m_Q} \right) \psi =$$
$$= \psi^\dagger \left( i\partial_0 + g_s A_0(t,\mathbf{0}) + g_s x_i \partial_i A_0(t,\mathbf{0}) - \frac{ig_s}{2m_Q}(\vec{\partial} - \overleftarrow{\partial}) \cdot \vec{A}(t,0) + \frac{\vec{\partial}^2}{2m_Q} \right) \psi$$

where in the last line we only kept terms up to order $v^{13/2}$ and analogously for the heavy antiquark. The first, second and last terms are already those that appear in the Lagrangian (2) above and they scale as $\sim v^5, v^{11/2}, v^5$ respectively. The third and fourth terms, together with another operator arising from potential gluon exchange between a heavy quark and an heavy antiquark with emission of an ultrasoft gluon from the potential gluon line in NRQCD[2], can be combined to give the chromo-electric operator $\boldsymbol{x} \cdot \boldsymbol{E}(t,\mathbf{0})$, which scales as $v^{13/2}$. This requires the use of integration by parts and the leading order EOM.

Lastly, note that the as a consequence of the multipole expansion, the Lagrangian in equation (2) is no longer gauge invariant under full gauge transformations, but only under gauge transformation of the form $U(t,\mathbf{0})$ (since the first two terms can be combined to give the ultrasoft covariant derivative $D_0 = \partial_0 - ig_s A_0(t,\mathbf{0})$). This is because gauge transformations with non-zero spatial arguments mix different orders of $v$ and we wanted to have power counting manifest at the level of the Lagrangian. Hence to recover full gauge symmetry, we would need to resum the multipole expansions of the gauge fields to all orders.

Let's now take a look at the third row of the Lagrangiang in equation (2) which contains non-local four fermion operator terms. The expression $V_{ab;cd}(r,\boldsymbol{\partial})$ collects the spatially non-local (although instantaneous) matching coefficients between NRQCD and pNRQCD after integrating out the soft scale. They are called "potentials" and can be written as a sum of a LO term and higher order corrections:

$$V_{ab;cd}(r,\boldsymbol{\partial}) = T^A_{ab} T^A_{cd} V_0(r) + \delta V_{ab;cd}(r,\boldsymbol{\partial}). \tag{4}$$

## 1.2 Matching of pNRQCD to NRQCD and emergence of the potentials

In order to understand the emergence of non-local operators and how to compute the potentials $V_{ab;cd}(r,\boldsymbol{\partial})$, let's look at the NRQCD graph displayed on the left side of Figure 1. A propagating heavy quark and anti-quark exchange a potential gluon via the interaction terms $-g_s\psi^\dagger A^0\psi$, which are leading order in the potential region of NRQCD. The amputated on-shell amplitude of this process, leaving out the external heavy (anti)quark spinors, takes the form

$$\mathcal{M}_{NRQCD} = (-ig_s T^A_{ab}) \frac{-ig^{00}}{k^2} (-ig_s T^A_{cd}) = \; iT^A_{ab} T^A_{cd} \, \frac{-4\pi\alpha_s}{\mathbf{k}^2} + O(v) \tag{5}$$

[2]For the curious: This term has the form $\delta V_{ab;cd}(r,\boldsymbol{\partial}) = T^B_{ab} T^C_{cd} (-\alpha_s/r) g_s f^{ABC} r^i A^{iA}(t,\mathbf{o})$ when embedded in the third line of the pNRQCD Lagrangian above.


Figure 1: Diagrams in NRQCD (left) vs. pNRQCD (right).

As we can see, the leading order term contains two inverse powers of the difference of the external incoming and outgoing heavy quark three-momentum. Clearly, there is no way to write down a local effective four fermion operator with such a Feynman rule. All we can do with local operators to get momenta of external particles in the vertex are adding derivatives and these give only positive powers. Negative momentum powers are reserved for propagators in local QFTs. More physically speaking: we try to shrink a line to a point (local vertex), which carries momenta of orders that appear in our theory. Such lines therefore have to be resolved and our attempt is bound to fail.
The way out are spatially non-local effective operators, as we will demonstarte now. As the above graph is the leading order interaction of NRQCD, we match it to what we introduced before as the leading order potential of pNRQCD, more precisely the four fermion interaction

$$\int d^3\mathbf{r}\, \psi_a^\dagger \psi_b(x+\mathbf{r})\, T^A_{ab} T^A_{cd} V_0(r)\, \chi_c^\dagger \chi_d(x). \tag{6}$$

It is straight forward to compute its Feynman rule, which reads

$$i \int d^3\mathbf{r} T^A_{ab} T^A_{cd} V_0(r) e^{i\mathbf{r}\cdot\mathbf{k}} \equiv i T^A_{ab} T^A_{cd} \tilde{V}_0(\mathbf{k}), \tag{7}$$

where again $\mathbf{k}$ is the difference of the outgoing and incoming heavy quark 3-momenta. Notice that the non-locality of the vertex causes an additional exponential factor, which lets us identify the vertex as a Fourier transformation of $V_0(r)$. Of course in no way coincidental, but by construction of the non-local vertex, the Fourier transformed $\tilde{V}_0$ depends exactly on $\mathbf{k}$.
This Feynman rule gives then directly the amputated on-shell amplitude of the pNRQCD graph displayed to the right in Figure 1 (leaving out the external heavy (anti)quark spinors). It is matched to the leading order term of the NRQCD graph and we find by comparison

$$\tilde{V}_0(\mathbf{k}) = \frac{-4\pi\alpha_s}{\mathbf{k}^2}. \tag{8}$$

Fourier transforming back to position space, we get

$$V_0(r) = -4\pi\alpha_s \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2} = -\frac{\alpha_s}{r}, \tag{9}$$

the well known Coulomb potential.
Corrections to the leading Coulomb potential, collected in $\delta V_{ab;cd}(r, \boldsymbol{\partial})$, originate in several ways. For once, above we only tackled the leading order in $v$ of the NRQCD graph. There will be higher order potentials which summed up to all orders in $v$ give the full NRQCD graph. Second, we

can have diagrams in NRQCD of the same form as the above but with other operators, e.g. the chromo-magnetic operator, inserted at the vertices. Third, we can go beyond tree level and compute NRQCD loops as for example the one depicted in Figure 2 with soft momentum running in the loop (you may notice that it is the NRQCD equivalent to the loop graph that we considered before in full QCD).

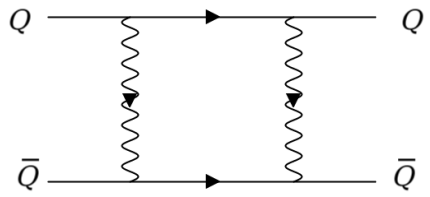


Figure 2: One loop correction to the Coulomb potential

### 1.3 Leading order Lagrangian and Coulomb Green function

Let's now restrict to only the leading order of the pNRQCD Lagrangian. From previous subsections, we know already that they should scale as $\sim v^5$. To identify it, we can use the velocity scaling rules also derived in that subsection: the heavy quarks can only be potential and scale as $\sim v^{3/2}$, while gluon fields are only ultrasoft and scale as $\sim v^2$. Additionally, we need the scaling rules $g_s \sim v^{1/2}$ and $r \sim 1/v$ [3] implying $d^3\mathbf{r} \sim 1/v^3$. The unperturbed pNRQCD Lagrangian is then found to be

$$\mathcal{L}_{pNRQCD} = \psi^\dagger \left( i\partial_0 + \frac{\boldsymbol{\partial}^2}{2m_Q} \right) \psi + \chi^\dagger \left( i\partial_0 - \frac{\boldsymbol{\partial}^2}{2m_Q} \right) \chi + \\ + \int d^3\mathbf{r}\, \psi_a^\dagger \psi_b(x+\mathbf{r})\, T^A_{ab} T^A_{cd} V_0(r)\, \chi_c^\dagger \chi_d(x). \tag{10}$$

The scaling of the first two terms is obvious. For the potential term we have $V_0 = -\frac{\alpha_s}{r} \sim v^2$, such that it scales in total also as $v^5$. This implies that the leading order Coulomb potential, $V_0$, is not a perturbation, hence can't be treated perturbatively in pNRQCD[4] and contributes to the LO equation of motion, even at weak coupling.
The gluon field strength tensor $G^A_{\mu\nu}$ now scales as $\sim v^4$ for ultrasoft gluon fields. Therefore, none of the terms in the gluon Lagrangian contribute at LO.
How do we deal with the Coulomb potential not being a perturbation? Whenever we calculate in pNRQCD, Coulomb potential interactions have to be summed to all orders - sounds tedious. Instead of doing that, there is an alternative conceptually similar to what we usually do with mass terms: we somehow include it in the propagators of the heavy quarks.
However, when doing this, there is a complication from the start: the Coulomb potential term is bilinear in both the heavy quark and heavy antiquark field and mixes them. In order to proceed, we make a change in the field degrees of freedom that we consider. Since heavy quark and antiquark particle numbers are separately conserved in our theory, we will not consider separate propagation of a heavy quark or a heavy antiquark, but instead propagation of a heavy quark-antiquark pair. This allow us to find an expression for the pair propagator, called Coulomb Green function. Implicitly, this means that we restrict the Fock space to the one $Q\bar{Q}$ pair sector.

[3] Remember: during the matching in the previous subsection, we have seen that $\mathbf{r}$ is the Fourier conjugated variable to differences in external heavy quark momenta $\sim m_Q v$ and hence scales with the exact inverse powers.

[4] We already noticed that the corresponding interaction in NRQCD contributes at leading order as well and hence also there needs to be summed to all orders.

We will only sketch the derivation of the Coulomb Green function. It is intuitively clear, that the propagator should in principle be given by summing up the momentum space amplitudes of all ladder diagram of a heavy quark-antiquark pair propagation with any number of Coulomb interaction insertions (see Figure 3).

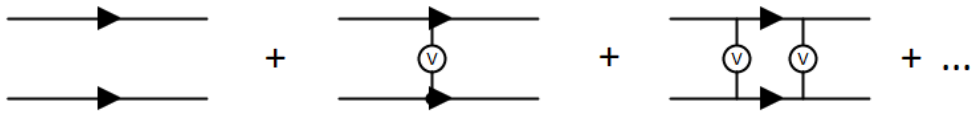


Figure 3: Ladder diagrams to be summed to obtain the Coulomb Green function.

We consider the scattering $Q(p_1)\bar{Q}(p_2) \to Q(p_1')\bar{Q}(p_2')$ with an arbitrary number of Coulomb interactions and then we will sum over all these exchanges. In the centre of mass frame, the external momenta are given by $p_1 = (E/2, \boldsymbol{p})$, $p_2 = (E/2, -\boldsymbol{p})$, $p_1' = (E/2, \boldsymbol{p'})$, $p_2' = (E/2, -\boldsymbol{p'})$. Here, $E$ stands for the non-relativistic centre of mass energy of the pair and $\mathbf{p}$ ($\mathbf{p}'$) for the 3-momentum of the incoming (outgoing) heavy quark. The amplitude given by the sum of these diagrams in $d$-dimensions is labelled by $G_0(\mathbf{p}, \mathbf{p}', E)$. But first we are interested in computing the amputated amplitude of the diagrams in Figure 3, denoted by $H(\mathbf{p}, \mathbf{p}', E)$

$$H(\mathbf{p}, \mathbf{p}', E) = \sum_{n=0}^{\infty} C_F^{n+1} \int \left[\prod_{i=1}^{n} \frac{d^d k_i}{(2\pi)^d}\right] \frac{i(ig_s)^2}{(\boldsymbol{k}_1 - \boldsymbol{k}_0)^2} \cdots \frac{i(ig_s)^2}{(\boldsymbol{k}_{n+1} - \boldsymbol{k}_n)^2} \cdot \\ \cdot \prod_{i=1}^{n} \frac{i}{\frac{E}{2} + k_i^0 - \frac{(\boldsymbol{p}+\boldsymbol{k}_i)^2}{2m}} \frac{-i}{\frac{E}{2} - k_i^0 - \frac{(\boldsymbol{p}+\boldsymbol{k}_i)^2}{2m}} \tag{11}$$

where $k_0 = 0$ and $k_{n+1} = \boldsymbol{p'} - \boldsymbol{p}$.

Note that the first term in the sum ($n = 0$) corresponds to the Coulomb potential (generated by the exchange of a potential gluon). To get the Coulomb Gren function we need to include the external quark/antiquark pairs, add the graph with no gluon exchange and also multiply by a factor of $(-i)$

$$G_0(\mathbf{p}, \mathbf{p}', E) = -\frac{(2\pi)^{d-1}\delta^{d-1}(\boldsymbol{p} - \boldsymbol{p'})}{E - \frac{\boldsymbol{p}^2}{m}} + \frac{1}{E - \frac{\boldsymbol{p}^2}{m}} iH(\mathbf{p}, \mathbf{p}', E) \frac{1}{E - \frac{\boldsymbol{p'}^2}{m}} \tag{12}$$

In order to prove that this is indeed the Coulomb Green function we need to show that this amplitude, $G_0(\mathbf{r}, \mathbf{r}', E)$, satisfies the d-dimensional leading order equation of motion, namely the Lippmann-Schwinger equation of a quark pair in the SU(3) colour singlet representation

$$\left(\frac{\boldsymbol{p}^2}{m} - E\right) G_0(\mathbf{p}, \mathbf{p}', E) - \tilde{\mu}^{2\epsilon} \int \frac{d^{d-1}\boldsymbol{k}}{(2\pi)^{d-1}} \frac{4\pi C_F \alpha_s}{\boldsymbol{k}^2} G_0(\mathbf{p} - \mathbf{k}, \mathbf{p}', E) = \\ = (2\pi)^{d-1}\delta^{d-1}(\boldsymbol{p} - \boldsymbol{p'}) \tag{13}$$

In $d = 4$ dimensions this equation reduces to the Schrödinger equation

$$\left(-\frac{\nabla_{\mathbf{r}}^2}{m} - C_F V_0(r) - E\right) \tilde{G}_0(\mathbf{r}, \mathbf{r}', E) = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \tag{14}$$

where $\tilde{G}_0(\mathbf{r}, \mathbf{r}', E)$ is the Fourier transform of equation (12). To prove that equation (14) is indeed satisfied, we need to Fourier transform it and then insert back the result from equation (12).

The details of this computation are left out. However, this justifies why the matching coefficients $V_{ab;cd}(r, \boldsymbol{\partial})$ are called potentials

Perturbation theory in pNRQCD then proceeds schematically as follows:

- Assuming that we consider a process without any kind of ultrasoft gluon interaction, a general diagram consists of a propagating heavy quark-antiquark pair with perturbation potential insertions.
- For every propagating heavy quark-antiquark pair, insert the Coulomb Green function propagator $iG_0(\mathbf{p}, \mathbf{p}', E)$ times a colour factor and for every perturbation potential insertion the respective momentum space Feynman rule factor $\delta V(\mathbf{p}, \mathbf{p}')$ times a colour factor.
- Integrate over all internal relative momenta.

A general amplitude without the external spinors then looks like

$$C_{abcd} \int \left[ \prod_{i=1}^{n} \frac{d^{d-1}\mathbf{p}_i}{(2\pi)^{d-1}} \right] iG_0(\mathbf{p}, \mathbf{p}_1, E) \delta V_1(\mathbf{p}_1, \mathbf{p}_2) iG_0(\mathbf{p}_2, \mathbf{p}_3, E) \dots \delta V_{n-1}(\mathbf{p}_{n-1}, \mathbf{p}_n) iG_0(\mathbf{p}_n, \mathbf{p}', E) \tag{15}$$

where $C_{abcd}$ sums up all colour factors. Including ultrasoft gluon interactions, this becomes more complicated. We then need the Coulomb Green function also for heavy quark-antiquark pairs in a colour octet state and all colour factors need to get adjusted.

# References


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