On-Shell EFTs III: Soft-Collinear Effective Theory

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1 Physical Picture of SCET

In the on-shell EFTs we investigated so far, the large scale was directly induced by the presence of large heavy quark masses. In the EFT we want to introduce now, the situation is different: we consider (nearly) massless particles that carry large energy-momentum. Such situations occur regularly in collider physics, where leptons or light hadrons are brought to collision at large centre-of-mass energy, resulting in production of jets of collinear particles. Another example are decays of heavy hadrons to light particles which carry large energy-momentum of order of the heavy hadron mass. These decays give rise to clusters of collinear particles in the direction of the decay (as can be seen in Figure 1) as well as ultrasoft radiation of massless particles, i.e. gluons in QCD.

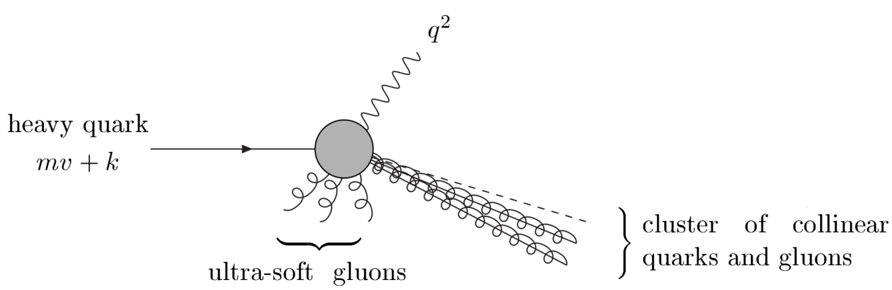

Figure 1: Heavy quark decaying into a single cluster of collinear and ultrasoft particles

The appropriate EFT for such situations is called soft-collinear effective theory (SCET). We will focus again on QCD and construct the associated SCET. It describes the interaction of very energetic quarks with ultrasoft and collinear gluons. The high scale of the external energies and momenta will be integrated out at the level of the SCET Lagrangian but as it is present in external states, the effective Lagrangian will be non-local, as for pNRQCD.

2 The SCET Lagrangian for QCD

In order to better understand this theory let's look at the decays of heavy hadrons. These are best studied in the rest frame of the heavy particle, in which it is useful to introduce light-cone coordinates. For this purpose we need to introduce a light cone direction, which will allow us to decompose any 4-vector along a collinear direction n_+^μ, an anti-collinear direction n_-^μ and the two components perpendicular to these two vectors $n_\perp^\mu$. They fulfil $n_+^2 = n_-^2 = 0$ and $n_+ n_- = 2$ [1]. Thus, any 4-vector A_μ can be decomposed in the following way[2]

$$A^\mu = \underbrace{(n_- \cdot A)}_{A_-} \frac{n_+^\mu}{2} + \underbrace{(n_+ \cdot A)}_{A_+} \frac{n_-^\mu}{2} + A_\perp^\mu \equiv A_+^\mu + A_-^\mu + A_\perp^\mu, \tag{1}$$

Be aware not to confuse the scalar quantities $A_\pm$ with the related 4-vectors $A_\pm^\mu = A_\mp \frac{n_\pm^\mu}{2}$. From this composition of 4-vectors, one then finds

$$A \cdot B = \frac{1}{2} A_+ B_- + \frac{1}{2} A_- B_+ + A_\perp \cdot B_\perp \tag{2}$$

implying

$$A^2 = A_+ A_- + A_\perp^2. \tag{3}$$

The components of a 4-vector with respect to the reference vectors are often written using the notation

$$A = (A_-, A_+, A_\perp). \tag{4}$$

As mentioned above, heavy hadron decays decays give rise to collinear particles. We first want to investigate how the momentum p of these particles behaves. For this purpose, let us decompose it using the newly introduces light-cone coordinates

$$p^\mu = \underbrace{(n_- p)}_{p_-} \frac{n_+^\mu}{2} + \underbrace{(n_+ p)}_{p_+} \frac{n_-^\mu}{2} + p_\perp^\mu \tag{5}$$

For large energies E of the collinear particles (or equivalently for large m), the momentum components are widely separated, with $p_+ \sim E \sim m$, while p_- and $p_\perp$ are small. As a result we can introduce the quantity $\lambda \sim p_\perp / p_+$, which will become our expansion parameter. This allows us to see how the components of a collinear momentum scale with λ

$$p_+ = n_+ \cdot p \sim m\lambda^0, \qquad p_\perp \sim m\lambda, \qquad p_- = n_- \cdot p \sim m\lambda^2 \tag{6}$$

Using these scalings it is easy to see that $p^2 \sim m^2\lambda^2$. We notice, that the invariant mass of the final state is unchanged if we add a particle with a momentum whose components scale like $m\lambda^2$. Hence the kinematic situation also allows for such particles, called ultrasoft, to be included in our theory. Assume now we also want to include particles with momentum k whose components scale like $m\lambda$ (soft particles). Then, the invariant mass of the cluster scales like $(p+k)^2 \sim m^2\lambda^2$ which contradicts our kinematical assumptions. Therefore the theory can only contain collinear and ultrasoft modes.

[1] One convenient choice is $n_+^\mu = (1, 0, 0, -1)$, $n_-^\mu = (1, 0, 0, 1)$.

[2] For the reference vectors of the previous footnote: $n_+ \cdot A = A^0 + A^3$, $n_- \cdot A = A^0 - A^3$ and $A_\perp^\mu = (0, A^1, A^2, 0)$.

To sum up, the following momentum regions are relevant to the theory

- hard (h): $p \sim (1, 1, 1)E$ [3]
- collinear (c): $p \sim (\lambda^2, 1, \lambda)E$
- anti-collinear ($\bar{c}$): $p \sim (1, \lambda^2, \lambda)E$
- ultrasoft (us): $p \sim (\lambda^2, \lambda^2, \lambda^2)E$

2.1 Fields and power counting

The SCET Lagrangian is constructed in such a way that each of the four regions found above is matched by individual SCET diagrams order by order in the expansion parameter λ. To achieve this, the full QCD gluon and quark fields have to be split into a collinear and an ultrasoft part

$$\begin{cases} A_\mu(x) \to A^c_\mu(x) + A^{us}_\mu(x) \\ \psi(x) \to \psi^c(x) + \psi^{us}(x) \end{cases} \tag{7}$$

which carry the appropriate momentum. For now, we neglect the anti-collinear part[4] and a hard field is not necessary as the hard contributions are integrated out. A collinear quark has two large (ξ) and two small (η) components which are separated by splitting the collinear quark field even further according to

$$\psi^c(x) = \xi(x) + \eta(x), \tag{8}$$

where the two components are given by

$$\xi(x) = P_+\psi^c(x) = \frac{\not{n}_-\not{n}_+}{4}\psi^c(x), \qquad \eta(x) = P_-\psi^c(x) = \frac{\not{n}_+\not{n}_-}{4}\psi^c(x). \tag{9}$$

The newly introduced quantities $P_\pm$ are projection operators since they satisfy

$$P_+ + P_- = 1 \qquad P_+^2 = P_+ \qquad P_-^2 = P_- \tag{10}$$

As a consequence of $n_\pm^2 = 0$, it is easy to see that

$$\not{n}_-\xi(x) = 0 \qquad \not{n}_+\eta(x) = 0 \tag{11}$$

We are now interested in the way different components of these fields scale with λ. In order to get the scaling behaviour, we look at two-point correlation functions. We start by analysing the large field component of the collinear quark $\xi(x)$

$$\begin{aligned} \langle\Omega|T\{\xi(x)\bar{\xi}(y)\}|\Omega\rangle &= \frac{\not{n}_-\not{n}_+}{4}\langle\Omega|T\{\psi^c(x)\bar{\psi}^c(y)\}|\Omega\rangle\frac{\not{n}_+\not{n}_-}{4} = \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot(x-y)}\frac{\not{n}_-\not{n}_+}{4}\not{p}\frac{\not{n}_+\not{n}_-}{4} \end{aligned} \tag{12}$$

[3]For heavy hadron decays, this scale is the mass m of the heavy hadron.

[4]The anti-collinear Lagrangian is basically a copy of the collinear Lagrangian with a few minor changes. Anti-collinear-collinear interactions appear not at leading order in λ, because of energy-momentum conservation.

We make use of $n_\pm^2 = 0$ and the following identity

$$\frac{\not{n}_-\not{n}_+}{4}\not{p}\frac{\not{n}_+\not{n}_-}{4} = \frac{\not{n}_-\not{n}_+}{4}\left[n_+\cdot p\frac{\not{n}_-}{2} + n_-\cdot p\frac{\not{n}_+}{2} + \not{p}_\perp\right]\frac{\not{n}_+\not{n}_-}{4} = n_+\cdot p\frac{\not{n}_-}{2} \sim \lambda^0 \tag{13}$$

to get the scaling for $\xi(x)$, knowing that for a collinear momentum p we have $p^2 \sim \lambda^2$

$$\langle\Omega|T\{\xi(x)\bar{\xi}(y)\}|\Omega\rangle \sim \lambda^4\frac{1}{\lambda^2}\lambda^0 = \lambda^2 \tag{14}$$

which implies that $\xi(x) \sim \lambda$. For the small field component of the collinear quark $\eta(x)$, we follow a similar logic

$$\begin{aligned}\langle\Omega|T\{\eta(x)\bar{\eta}(y)\}|\Omega\rangle &= \frac{\not{n}_+\not{n}_-}{4}\langle\Omega|T\{\psi^c(x)\bar{\psi}^c(y)\}|\Omega\rangle\frac{\not{n}_-\not{n}_+}{4} = \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot(x-y)}\underbrace{\frac{\not{n}_+\not{n}_-}{4}\not{p}\frac{\not{n}_-\not{n}_+}{4}}_{n_-\cdot p\frac{\not{n}_+}{2}} \sim \lambda^4\frac{1}{\lambda^2}\lambda^2 = \lambda^4\end{aligned} \tag{15}$$

which yields $\eta(x) \sim \lambda^2$. Notice that the small field collinear component is suppresed by a power of λ wrt the large field collinear component $\xi(x)$.
Next we can see the scaling for the ultrasoft quark field, for which $p \sim \lambda^2$

$$\langle\Omega|T\{\psi^{us}(x)\bar{\psi}^{us}(y)\}|\Omega\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i\not{p}}{p^2+i\epsilon}e^{-ip\cdot(x-y)} \sim (\lambda^2)^4\lambda^2\frac{1}{\lambda^4} = \lambda^6 \tag{16}$$

and thus $\psi^{us}(x) \sim \lambda^3$. Now, we turn our attention to the gluon fields, for which the two-point correlation function is

$$\langle\Omega|T\{A_\mu(x)A_\nu(y)\}|\Omega\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot(x-y)}\left[-g_{\mu\nu} + \xi\frac{p_\mu p_\nu}{p^2}\right] \tag{17}$$

From equation (17) it can be seen that the gluon field A_μ scales as momentum p. Therefore we have

- ultrasoft gluons $A_\mu^{us}(x) \sim p_\mu^{us} \sim \lambda^2$
- collinear gluons $A_\mu^c(x) \sim p_\mu^c$: $\quad n_+A^c \sim \lambda^0, \quad n_-A^c \sim \lambda^2, \quad A_\perp^c \sim \lambda$

Note that the only component of the collinear gluon wrt which the ultrasoft gluon is not suppressed is n_-A^c.

2.2 Construction of the effective Lagrangian

2.2.1 Collinear Lagrangian

We start by looking at the collinear quark Lagragian $\mathcal{L}_{\psi_c} = \bar{\psi}^c i\not{D}\psi^c$. The key observation is that the collinear small field component η is suppressed by one power of λ wrt to the large field component ξ. As a result, the field $\eta(x)$ can be integrated out. The Lagrangian $\mathcal{L}_c$ can be written in the following way

$$\mathcal{L}_{\psi_c} = \bar{\psi}^c i \not{D} \psi^c = (\bar{\xi} + \bar{\eta}) i \left[n_+ \cdot D \frac{\not{n}_-}{2} + n_- \cdot D \frac{\not{n}_+}{2} + \not{D}_\perp \right] (\xi + \eta) =$$

$$= \bar{\xi} \frac{\not{n}_+}{2} i n_- \cdot D \xi + \bar{\eta} \frac{\not{n}_-}{2} i n_+ \cdot D \eta + \bar{\xi} i \not{D}_\perp \eta + \bar{\eta} i \not{D}_\perp \xi \tag{18}$$

where we used $\not{n}\_\xi = \bar{\xi}\not{n}_- = 0$, $\not{n}_+\eta = \bar{\eta}\not{n}_+ = 0$, $\bar{\xi}(\bar{\eta})\not{D}_\perp \xi(\eta) = 0$.[5] The covariant derivative is defied as $D_\mu = \partial_\mu - ig(A^c_\mu + A^{us}_\mu)$.

In order to integrate out η we need to compute the EOM for this field, which gives

$$\not{D}_\perp \xi = -\frac{\not{n}_-}{2} n_+ \cdot D \eta \tag{19}$$

with the formal solution

$$\eta = -\frac{\not{n}_+}{2 n_+ \cdot D} \not{D}_\perp \xi \tag{20}$$

In a similar way we can also obtain $\bar{\eta}$, which we then plug back into the collinear Lagrangian to get

$$\mathcal{L}_{\psi_c} = \bar{\xi} \frac{\not{n}_+}{2} i n_- \cdot D \xi + \bar{\xi} i \not{D}_\perp \frac{1}{i n_+ \cdot D} i \not{D}_\perp \frac{\not{n}_+}{2} \xi \tag{21}$$

At leading order for collinear gluons, the Lagragian is simply just a copy of QCD, where the fields now are replaced by A^c_μ, such that the collinear Lagrangian can be written

$$\mathcal{L}_c = \bar{\xi} \frac{\not{n}_+}{2} i n_- \cdot D \xi + \bar{\xi} i \not{D}_\perp \frac{1}{i n_+ \cdot D} i \not{D}_\perp \frac{\not{n}_+}{2} \xi - \frac{1}{4} F^{c,a}_{\mu\nu} F^{\mu\nu,c,a} \tag{22}$$

Note that this is not a purely collinear Lagrangian since the ultrasoft gluons are hidden in the covariant derivate D_μ.

2.2.2 Addition of ultrasoft quarks

The purely ultrasoft Lagrangian is much easier to get since it is simply a copy of QCD

$$\mathcal{L}_{us} = \underbrace{\bar{\psi}^{us} i \not{D}_{us} \psi^{us}}_{\mathcal{L}_{\psi_{us}}} - \frac{1}{4} F^{us,a}_{\mu\nu} F^{\mu\nu,us,a} \tag{23}$$

However, there are two important observations we need to make:

1. Note that in the kinetic term of the ultrasoft quarks $\mathcal{L}_{\psi_{us}}$ the covariant derivative is iD_{us}, which contains only ultrasoft gluons. An interaction term between a collinear particle (gluon) and two ultrasoft particles (quarks) is not allowed by kinematics and as a result a term like $\bar{\psi}^{us} i \not{D}_c \psi^{us}$ is forbidden from the Lagrangian.

2. The field strength tenson $F^{\mu\nu,us,a}$ only contains ultrasoft gluons now.

[5] $\bar{\xi}\not{D}_\perp \xi = \bar{\xi} P_- \not{D}_\perp P_+ \xi = \bar{\xi} \not{D}_\perp P_- P_+ \xi = 0$

Besides these interactions, we can also have terms that mix collinear and ultrasoft quarks. In order to analyse these, we first need to introduce a new quantity, called the Wilson line W, defined as

$$W(x) = P \exp\left(\int_{-\infty}^{0} dsn_+ A(x + sn_+)\right) \tag{24}$$

where $A = A_c + A_{us}$ and P is the path-ordering which as the name suggests orders all the terms in the exponential according to their arguments, e.g., $P\left[A(x)A(x+sn_+)\right] = A(x)A(x+sn_+)$. The Wilson line is a unitary operator $WW^\dagger = W^\dagger W = 1$.

Similarly to the definition of W we can also introduce ultrasoft/collinear Wilson lines which only contain the ultrasoft/collinear part of the gluon fields

$$Z(x) = P \exp\left(\int_{-\infty}^{0} dsn_+ A_{us}(x + sn_+)\right) \tag{25}$$

$$W_c(x) = P \exp\left(\int_{-\infty}^{0} dsn_+ A_c(x + sn_+)\right) \tag{26}$$

These Wilson lines and the identities following from them will allow us to rewrite the order Lagrangian in a nicer form and simplify the systematic expansion in λ.
To get the ultrasoft-collinear quark interactions we need to use the full decomposition of the quark fields $\psi = \psi_{us} + \xi + \eta$ and keep all terms

$$\mathcal{L}_\psi = \mathcal{L}_{\psi_c} + \mathcal{L}_{\psi_{us}} + \bar{\xi} g \not{A}_c \psi_{us} + \bar{\eta} g \not{A}_c \psi_{us} + \bar{\psi}_{us} g \not{A}_c \xi + \bar{\psi}_{us} g \not{A}_c \eta \tag{27}$$

where $\mathcal{L}_{\psi_c}$ and $\mathcal{L}_{\psi_{us}}$ are the purely collinear and ultrasoft quark Lagrangians we found earlier. We can follow the same procedure as before and integrate oute the η field. The EOM will be slightly different now due to the new interaction terms. The quark Lagrangian can now be schematically written as

$$\mathcal{L}_\psi = \mathcal{L}_{\psi_c} + \mathcal{L}_{\psi_{us}} + \mathcal{L}_{\xi\psi_{us}} \tag{28}$$

where $\mathcal{L}_{\xi\psi_{us}}$ is given by

$$\mathcal{L}_{\xi\psi_{us}} = \bar{\xi} i \not{D}_\perp W Z^\dagger \psi_{us} + \bar{\psi}_{us} Z W^\dagger i \not{D}_\perp \xi + \mathcal{O}(\lambda^3) \tag{29}$$

Note that the presence of the Wilson lines makes the mixed quark terms in the Lagrangian gauge invariant under collinear and ultrasoft gauge transformations since under these transformations the Wilson lines behave as

1. Collinear gauge transformations: $W \to U_c W$ and $Z \to Z$
2. Ultrasoft gauge transformations: $W \to U_{us} W$ and $Z \to U_{us} Z$

For the result in equation (29), we neglected terms of order λ^3 and dropped interactions which are not kinematically possible. The whole soft-collinear effective Lagrangian up to order $O(\lambda^3)$ is

$$\begin{aligned}\mathcal{L}_{SCET} =& \bar{\xi}\frac{\not{n}_+}{2} in_- \cdot D\xi + \bar{\xi} i\not{D}_\perp \frac{1}{in_+ \cdot D} i\not{D}_\perp \frac{\not{n}_+}{2}\xi + \bar{\psi}^{us} i\not{D}_{us}\psi^{us} + \bar{\xi} i\not{D}_\perp W Z^\dagger \psi_{us} + \\ &+ \bar{\psi}_{us} Z W^\dagger i\not{D}_\perp \xi - \frac{1}{4} F^{c,a}_{\mu\nu} F^{\mu\nu,c,a} - \frac{1}{4} F^{us,a}_{\mu\nu} F^{\mu\nu,us,a} + O(\lambda^3)\end{aligned} \tag{30}$$

2.3 Expansion in λ

In this section we want to expand the Lagrangian from equation (30) in definite powers of λ. This is needed since in typical applications of SCET we compute scattering matrix elements to a certain power in the expansion parameter and thus we need the Feynman rules order by order in λ. For simplicity we will derive the leading order contribution and only state the next to leading order result.

In order to structure the Lagrangian in definite powers of λ, we need a new tool: multipole expansion. The interactions we considered can be of three types: collinear-collinear, ultrasoft-ultrasoft, collinear-ultrasoft. For this last type of interaction terms the sum of the momenta of the two fields scales as $p^\mu = p^\mu_{us} + p^\mu_c \sim (\lambda^2, 1, \lambda)E$. As a result, the components of the position x scale as

$$x \sim \left(1, \frac{1}{\lambda^2}, \frac{1}{\lambda}\right)\frac{1}{E} \tag{31}$$

This allows us to Taylor expand the ultrasoft fields around the point $x^\mu_- = (x\cdot n_+)n^\mu_-/2$

$$\phi_{us}(x) = \phi_{us}(x_-) + \underbrace{x_\perp \cdot \partial_\perp \phi_{us}(x_-)}_{O(\lambda)} + \underbrace{x_+ \cdot \partial_+ \phi_{us}(x_-)}_{O(\lambda^2)} + \ldots \tag{32}$$

Intuitively, this expansion is done because the ultrasoft fields vary more slowly in the perpendicular and n_+ directions than the collinear fields. From equation (32) we see that the ultrasoft field entering the mixed interactions terms produces interactions of different powers of λ, therefore the multipole expansion in equation (32) allows us to organize the mixed terms in the Lagrangian order by order in λ.
Multipole expansion is not needed for the collinear fields since all terms in the expansion are of the same order

$$\phi_c(x) = \phi_c(x_-) + \underbrace{x_\perp \cdot \partial_\perp \phi_c(x_-)}_{O(1)} + \underbrace{x_+ \cdot \partial_+ \phi_c(x_-)}_{O(1)} + \ldots \tag{33}$$

Furthermore, it is also not needed for purely ultrasoft interactions for which the sum of the momenta of the fields scales as $p^\mu \sim (\lambda^2, \lambda^2, \lambda^2)E$ and the position scales as $x \sim \left(\frac{1}{\lambda^2}, \frac{1}{\lambda^2}, \frac{1}{\lambda^2}\right)\frac{1}{E}$ such that

$$\phi_{us}(x) = \phi_{us}(x_-) + \underbrace{x_\perp \cdot \partial_\perp \phi_{us}(x_-)}_{O(1)} + \underbrace{x_+ \cdot \partial_+ \phi_{us}(x_-)}_{O(1)} + \ldots \tag{34}$$

Lastly, we can write the total covariant derivative as $D_\perp = D_{\perp c} - igA_{\perp us}$ and notice that the second term is suppressed by one power of λ wrt to the first one. We also notice that the mixed quark terms do no contribute to leading order since $\psi_{us} \sim \lambda^3$, making them suppressed wrt to the collinear quark interactions. Therefore, the leading order Lagrangian takes the form

$$\mathcal{L}^0_{SCET} = \bar{\xi}\left(in_- D + i\not{D}_{\perp c}\frac{1}{in_+ D_c} i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi + \bar{\psi}^{us} i\not{D}_{us}\psi^{us} - \frac{1}{4}F^{c,a}_{\mu\nu}F^{\mu\nu,c,a} - \frac{1}{4}F^{us,a}_{\mu\nu}F^{\mu\nu,us,a} \tag{35}$$

It is important to point out that there are collinear-ultrasoft interactions that enter at LO, namely through the term $\bar{\xi} in_- D\frac{\not{n}_+}{2}\xi$. Therefore, the ultrasoft gluon fields from the covariant derivative need to be evaluated at x_- now, as a result of the multipole expansion: $A_{us}(x) \to A_{us}(x_-)$. More precisely, all ultrasoft fields need to be understood as evaluated at position x_-.

The next to leading order Lagrangian has the form

$$\mathcal{L}^1_{SCET} = \mathcal{L}^1_\xi + \mathcal{L}^1_{\xi\psi_{us}} \tag{36}$$

where the two contributions are given by

- $\mathcal{L}^1_\xi = \bar{\xi}\left(x^\mu_\perp n^\nu_- W_c g F^{us}_{\mu\nu} W^\dagger_c\right)\frac{\not{n}_+}{2}\xi$
- $\mathcal{L}^1_{\xi\psi_{us}} = \bar{\xi} i\not{D}_{\perp c} W_c \psi_{us} + \bar{\psi}_{us} W^\dagger_c i\not{D}_{\perp c}\xi$

The first Lagrangian, $\mathcal{L}^1_\xi$, is the equivalent of the chromo-electric operator $\boldsymbol{x}\cdot\boldsymbol{E}$ from NRQCD. The second NLO term, $\mathcal{L}^1_{\xi\psi_{us}}$, comes from equation (30) after expanding the product of Wilson lines $WZ^\dagger$ and keeping only the first order in λ, W_c. It is important to note once again that the presence of the Wilson lines in the NLO interactions makes them invariant under collinear gauge transformations

$$\mathcal{L}^1_\xi \to \bar{\xi} U^\dagger_c\left(x^\mu_\perp n^\nu_- U_c W_c g F^{us}_{\mu\nu} W^\dagger_c U^\dagger_c\right)\frac{\not{n}_+}{2} U_c \xi = \mathcal{L}^1_\xi \tag{37}$$

$$\mathcal{L}^1_{\xi\psi_{us}} \to \bar{\xi} U^\dagger_c i U_c \not{D}_{\perp c} U^\dagger_c U_c W_c \psi_{us} + \bar{\psi}_{us} W^\dagger_c U c^\dagger U_c i\not{D}_{\perp c} U^\dagger_c U_c \xi = \mathcal{L}^1_{\xi\psi_{us}} \tag{38}$$

2.4 Soft decoupling transformation

There exists an interesting transformation for the leading order collinear Lagrangian

$$\mathcal{L}^0_c = \bar{\xi}\left(in_- D + i\not{D}_{\perp c}\frac{1}{in_+ D_c} i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi - \frac{1}{4} F^{c,a}_{\mu\nu} F^{\mu\nu,c,a}. \tag{39}$$

We redefine the fields in the following way

$$\begin{aligned} \xi(x) &\to S_-(x_-)\xi^{(0)}(x) \\ A^\mu_c(x) &\to S_-(x_-) A^{(0)\mu}_c(x) S^\dagger_-(x_-) \end{aligned} \tag{40}$$

where we have introduced the ultrasoft Wilson line given by

$$S_-(x) = P\exp\left(\int_{-\infty}^0 ds n_- A_{us}(x + s n_-)\right) \tag{41}$$

Applying this transformation to the covariant derivative $D_\mu = \partial_\mu - ig(A^c_\mu + A^{us}_\mu)$ we find

$$\begin{aligned} in_- D\xi(x) &\to in_- D' S_-(x_-)\xi^{(0)}(x) \\ &= \left(in_-\partial + n_- S_-(x_-) g A^{(0)\mu}_c(x) S^\dagger_-(x_-) + g n_- A_{us}(x_-)\right) S_-(x_-)\xi^{(0)}(x) \\ &= \left(in_-\partial_- S_-(x_-) + S_-(x_-) in_-\partial + S_-(x_-) g n_- A^{(0)\mu}_c(x) + g n_- A_{us}(x_-) S_-(x_-)\right)\xi^{(0)}(x) \\ &= \left(in_- D_- S_-(x_-) + S_-(x_-) in_-\partial + S_-(x_-) g n_- A^{(0)\mu}_c(x)\right)\xi^{(0)}(x) \\ &= S_-(x_-)\left(in_-\partial + g n_- A^{(0)\mu}_c(x)\right)\xi^{(0)}(x) \equiv S_-(x_-) in_- D^{(0)}_c \xi^{(0)}(x) \end{aligned}$$

where from the third to the fourth line we have used the definition of the covariant derivative acting on the soft Wilson line and from the fourth to the fifth line we have used that the covariant derivative along a Wilson line vanishes.
Noticing that

$$\frac{\partial}{\partial x^\alpha} S_-(x_-) = \frac{\partial x_-^\beta}{\partial x^\alpha}\frac{\partial}{\partial x_-^\beta} S_-(x_-) = \frac{(n_+)_\alpha}{2} n_- \cdot \partial_- S_-(x_-) \tag{42}$$

it is easy to prove that

$$in_+ \cdot D_c \to S_-(x_-) in_+ \cdot D_c^{(0)} S_-^\dagger(x_-), \qquad iD_{c,\perp} \to S_-(x_-) in_+ \cdot iD_{c,\perp}^{(0)} S_-^\dagger(x_-) \tag{43}$$

where the subscript (0) indicates that one should replace $A_c \to A_c^{(0)}$. Using all this, one then finds that the collinear Lagrangian transforms under this redefinition of fields as

$$\mathcal{L}_c \to \bar{\xi}^{(0)} \left(\frac{n_+}{2} in_- D_c^{(0)} \xi^{(0)} + i\not{D}_{\perp c}^{(0)} \frac{1}{in_+ D_c^{(0)}} i\not{D}_{\perp c}^{(0)} \frac{\not{n}_+}{2} \right) \xi^{(0)} - \frac{1}{4} F_{\mu\nu}^{(0),c,a} F^{(0),\mu\nu,c,a}. \tag{44}$$

It takes the same form as before, but the ultrasoft gluon interaction disappears. The transformation we just performed is called *decoupling transformation* because it decouples the ultrasoft gluon from the collinear Lagrangian at leading order. However at subleading order these interactions are still present. The soft decoupling transformation is important to prove factorization theorems but only implies factorization of the Lagrangian at leading order.

References

[1] M. Beneke, A.P. Chapovsky, M. Diehl, and T. Feldmann. Soft collinear effective theory and heavy to light currents beyond leading power. *Nucl. Phys. B*, 643:431–476, 2002, hep-ph/0206152.

[2] Timothy Cohen. As Scales Become Separated: Lectures on Effective Field Theory. *PoS*, TASI2018:011, 2019, hep-ph/1903.03622.

[3] T. Becher, A. Broggio, and A. Ferroglia. Introduction to soft-collinear effective theory. *Lecture Notes in Physics*, 2015.

[4] M. Beneke and Th. Feldmann. Multipole-expanded soft-collinear effective theory with non-abelian gauge symmetry. *Physics Letters B*, 553(3-4):267–276, Feb 2003.