On-shell EFTs II: NRQCD

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1 Non-relativistic EFTs

Many particle physics systems in nature, e.g. the typical example of the Hydrogen atom, can be well described using non-relativistic quantum mechanics (QM), although the fundamental framework to describe particle interactions in nature is actually provided by relativistic QFTs. It is therefore natural to expect, that the QFT description of such systems reproduces the QM one[1] and also reveals corrections to it. But how to recover QM from QFT? The way to do this systematically is through the use of the concept of non-relativistic EFTs (NREFTs).
This field of EFTs focuses on the description of systems where all (massive) constituents move with non-relativistic velocities, which e.g. applies for a variety of bound state systems. As the weak gauge bosons are massive and predominantly responsible for decays, (somewhat) stable bound states in nature rely on strong or electromagnetic interactions. This lead to the emergence of non-relativistic QED (NRQED) in the 1980s and non-relativistic QCD (NRQCD) in the 1990s, which are effective field theories of the respective full QFTs in a non-relativistic setting. Both also admit a so called potential formulation (pNRQED or pNRQCD respectively) well applicable to bound state dynamics. Their Lagrangians contain spatially non-local, but instantaneous interactions, which establish the connection to the potential description of interactions in QM in a very accessible way.
NRQED has a number of well-established successes, e.g. the description of QED bound states such as positronium (e^-e^+) or muonium (μ^-e^+). It is also applied in atomic and molecular physics to study QED corrections to bound state energy levels of atoms or molecules. Recalling e.g. the fine structure of the Hydrogen atom from undergraduate QM, NRQED provides a systematic way to calculate the Lamb shift and Darwin term corrections.
Applications of NRQCD mostly evolve around heavy quarkonium which are QCD bound states formed by a heavy quark and a corresponding antiquark, usually symbolically denoted by $Q\bar{Q}$. Concretely, these are charmonium (called J/Ψ meson) and bottomonium (called Υ meson). Toponium does not really exist as a narrow resonance in the mass spectrum, since top (anti)quarks decay too quickly into weak gauge bosons for the bound state to form. Additionally to the description of heavy quarkonium states, especially more recent applications include also $Q\bar{Q}$ production near threshold (twice the heavy quark mass) and their decays.
Furthermore, NREFT concepts are also applied in the context of modern QFTs for dark matter, as experimental data hints at dark matter being a weakly coupled non-relativistic quantum system.
Unfortunately, limited time will not allow us to go into more detail on any of these. Our focus will be to understand conceptual elements of NREFTs using the example of NRQCD as it represents the more general case compared to NRQED. Applying the NREFTs to do calculations should afterwards be (in principle) straight forward.

[1] If not, we are in conceptual trouble.

2 Physical picture of non-relativistic QCD

The physical system of concern is heavy quarkonium, symbolically $Q\bar{Q}$, which is observed to be a bound state, even very much hydrogen-like. Consequently, in the center-of-mass (c.o.m.) frame of the system, the heavy quark and antiquark move with non-relativistic velocities and are almost on shell. This motivates the introduction of a non-relativistic "heavy quark 3-velocity" $\mathbf{v}$ such that the 4-momenta of the heavy quark and antiquark, p and $\bar{p}$, in the c.o.m. frame read

$$p = \begin{pmatrix} m_Q + E \\ m_Q\mathbf{v} \end{pmatrix}, \bar{p} = \begin{pmatrix} m_Q + E \\ -m_Q\mathbf{v} \end{pmatrix}, \tag{1}$$

where E is the "non-relativistic energy" of the heavy quark and antiquark. As they are almost on shell, we necessarily have $E \sim m_Q v^2$ with $v = |\mathbf{v}| \ll 1$. As in HQET, in the EFT description the rest energy m_Q will be substracted from the total relativistic heavy (anti)quark energy. For convenience, NRQCD is typically formulated explicitly in the c.o.m. frame of reference, making use of the small parameter v and consequently breaking Lorentz symmetry at the level of the Lagrangian. Notice that heavy quark non-relativistic energy and momentum scale different with v. This will either induce intricacies in the power counting of the effective theory or lead to spatially non-local operators in the Lagrangian as will become clearer soon.

2.1 Energy scales of $Q\bar{Q}$ systems

In NRQCD, four relevant and hierarchically organized energy regions are distinguished. For a non-relativistic energy l^0 and momentum $\mathbf{l}$, these can be labelled in the following way:

$$hard\ (h) : l^0 \sim m_Q, \mathbf{l} \sim m_Q$$

$$soft\ (s) : l^0 \sim m_Q v, \mathbf{l} \sim m_Q v$$

$$potential\ (p) : l^0 \sim m_Q v^2, \mathbf{l} \sim m_Q v$$

$$ultrasoft\ (us) : l^0 \sim m_Q v^2, \mathbf{l} \sim m_Q v^2$$

Consequently, external heavy (anti)quarks are said to be potential and they are the only particles that can be potential when on shell[2]. In contrast only particles with negligible masses compared to m_Q[3] can be ultrasoft when on shell[4].

To understand the significance of these different energy regions further, lets draw again a parallel to the Hydrogen atom in QM to get some intuition. As you might have done yourself in your undergraduate studies, calculating the energy required to ionize the atom in the interaction with an external electromagnetic field turns out to be of the ultrasoft order related to the fact that this is also the order of the splitting between different energy eigenvalues of the atom. Similar results are found for $Q\bar{Q}$ systems. It turns out that the ultrasoft scale is the typical scale of splittings between excitations of the system and a transfer of soft order energy to the system will kick the

[2] $(m + m_Q v^2)^2 - (m_Q v)^2 \sim m^2$ can only hold for $m \sim m_Q v$ or higher and this is typically only fulfilled for the heavy quarks themselves. The exception here is the charm quark in the case of bottomonium.

[3] Basically all particles but the heavy quarks.

[4] $(m_Q + m_Q v^2)^2 - (m_Q v^2)^2 - m_Q^2 \sim m_Q^2 v^2$ which is a non-negligible amount by which the heavy quark would be off shell.

heavy quarks off-shell and consequently destroy the bound state. Note that this implies that any other external states than the two heavy quarks that might appear in the bound state dynamics, e.g. gluons, have to be ultrasoft. But how does the soft region then impact bound state dynamics as it can't appear in external states? It enters at loop level. To see this, consider the QCD one loop diagram shown in Figure 1 with all external heavy quarks potential and on shell. The appearing

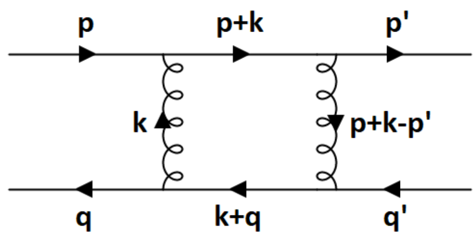

Figure 1: Feynman diagram to demonstrate soft loop momentum contributions

loop momentum integral integral has the form

$$\int d^4k \frac{\cdots}{k^2[(p+k)^2 - m_Q^2](p+k-p')^2[(k+q)^2 - m_Q^2]}. \tag{2}$$

Evaluating the k^0 integral, we will find e.g. a contribution from the pole $k^0 = \pm|\mathbf{k}|$. Now we can close the contour such that it contains the pole with the plus sign and apply the residue theorem. We find for the contribution of this pole to the integral the expression

$$\int d^3\mathbf{k} \frac{\cdots}{16|\mathbf{k}|(p\cdot k)(q\cdot k)[m_Q^2 + (p-p')\cdot k - p\cdot p']} \tag{3}$$

where $k^0 = |\mathbf{k}|$ is understood. Decomposing the zero component of the external momenta into rest mass and non-relativistic energy, e.g. for p as $p^0 = m_Q + E_p$, we can expand all terms in powers of v and find the leading contribution to the integral to be

$$\int d^3\mathbf{k} \frac{\cdots}{16m_Q^2 k^3[\mathbf{p}\cdot\mathbf{p'} - m_Q(E_p + E_{p'}) - (\mathbf{p}-\mathbf{p'})\cdot\mathbf{k}]} \tag{4}$$

where now $k = |\mathbf{k}|$. For soft k this contribution is clearly relevant and different from the other regions, as for hard loop momenta, only the last term in the bracket is leading order, while in the ultrasoft region only the first two are.

The last possible scale that can appear in $Q\bar{Q}$ systems is Λ_{QCD}, the energy region that separates perturbative from non-perturbative QCD. Whether non-perturbative effects of QCD can become relevant depends on how Λ_{QCD} compares to the scales $m_Q v, m_Q v^2$. We will not address any implications of possible non-perturbativity further and assume, that we are in the weak coupling regime $\Lambda_{QCD} \ll m_Q v^2$.

Having now established the energy regions relevant to $Q\bar{Q}$ systems as well as their hierarchy, there is in general now a sequence of EFTs which can be constructed from full QCD to obtain low energy descriptions of $Q\bar{Q}$, depending on which energy scales are integrated out. What is typically done is shown in Figure 2. Q represents the heavy (anti)quark field while g stands for any of the nearly massless fields. The round brackets contain the modes of these fields that are still present in the respective theory. μ is the scale up to which the theory is predictive.

$$\mathcal{L}_{\mathrm{QCD}}\left[Q(h,s,p),\ g(h,s,p,us)\right] \qquad \mu > m$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{NRQCD}}\left[Q(s,p),\ g(s,p,us)\right] \qquad mv < \mu < m$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{PNRQCD}}\left[Q(p),\ g(us)\right] \qquad \mu < mv$$

Figure 2: The two step sequence of EFTs construction

In a first step, all hard modes of QCD are integrated out, resulting in NRQCD, where all interactions are local. The second step consists of integrating out all soft modes and potential massless modes. The new effective theory, called potential NRQCD, contains only potential quarks and ultrasoft gluons and describes the physics of the $Q\bar{Q}$ systems below the soft mv scale. The resulting Lagrangian is spatially non-local because the three-momenta of potential heavy quarks are of the same order as the soft and potential modes of the nearly massless fields, that were previously integrated out. The name potential is in no way coincidental. As it will turn out, the matching coefficients of non-local operators that describe heavy quark heavy antiquark interaction without gluon emission in pNRQCD will have the valid interpretation of a QM potentials. More on this will follow later.

2.2 Power counting

In NRQCD, the power counting of terms in the Lagrangian is different from usual EFTs. The relevance of operators now depends on how they scale with v instead of $1/m_Q$. That powers of m_Q don't matter can be seen on dimensional grounds: all energy regions defined in the subsection before include exactly one power of m_Q and this is the only dimensionful scale defining the energy regions. Hence any operator will scale will m_Q according to its mass dimension such that all possible terms in the Lagrangian have to be of order m_Q^4 in the end. A difference in relevance then comes from powers of the small parameter v. Still, higher dimensional operators are often less relevant then lower dimensional ones as they bring at least powers of v equal to their dimension[5].
Consequently, power counting in NRQCD is less straightforward than usual as the scaling of an operator with v is ambiguous and depends on the considered energy region, e.g. a soft gluon field scales different with v than an ultrasoft one. In pNRQCD, this problem will be no longer present as the energy regions of fields are then uniquely specified.
For a given Feynman diagram with specification of the participating energy scales, the scaling with v can be uniquely determined. Therefore use the following set of rules derived from the non-relativistic energy and momentum scaling specified in the previous subsection:
The integration measure d^4l scales as

- hard modes: $d^4l \sim v^0$,
- soft modes: $d^4l \sim v^4$,
- potential modes: $d^4l \sim v^5$,

[5]Reminder: hard modes without powers of v are integrated out in NRQCD.

- ultrasoft modes: $d^4l \sim v^8$.

In Feynman gauge, a gluon propagator with momentum $(l^0, \mathbf{l})$ at leading order (LO) scales as

$$\frac{1}{l^2} = \begin{cases} \frac{1}{l^2} \sim v^0, & \text{for hard modes} \\ \frac{1}{l^2} \sim v^{-2}, & \text{for soft modes} \\ \frac{-1}{\mathbf{l}^2} \sim v^{-2}, & \text{for potential modes} \\ \frac{1}{l^2} \sim v^{-4}, & \text{for ultrasoft modes} \end{cases} \tag{5}$$

The denominator[6] of a heavy quark propagator with four momentum $q + l = (m_Q + l^0, \mathbf{l})$ goes like

$$\frac{1}{(q+l)^2 - m_Q^2} = \frac{1}{2m_Q l^0 + l^2} \approx \begin{cases} \frac{1}{2m_Q l^0 + l^2} \sim v^0, & \text{for hard modes } l \\ \frac{1}{2m_Q l^0} \sim v^{-1}, & \text{for soft modes } l \\ \frac{1}{2m_Q l^0 - \mathbf{l}^2} \sim v^{-2}, & \text{for potential modes } l \end{cases} \tag{6}$$

The rules for the hard region are needed in the matching procedure for full QCD calculations involving hard loop momenta.
Another subtlety of power counting in NRQCD is that typically $v \sim \alpha_s$, which can be motivated again by looking at QM results: Heavy quarkonium is observed to be a bound state, hence from a non-relativistic quantum mechanics perspective, there has to be an attractive potential responsible for this. In the QFT language, this potential corresponds to gluon exchange, which is proportional to at least two powers of the strong coupling, $g_s^2 \sim \alpha_s$, and this has to be a relevant effect. Going even one step further, as gluon exchange between quarks in QCD is at leading order very much similar to photon exchange between charged particles in QED, we can assume that this potential is at leading order Coulomb-like[7]. Comparing then the non-relativistic energy $\sim m_Q v^2$ of the $Q\bar{Q}$ system to the energy eigenvalues of the hydrogen atom

$$E_n \sim m_e \frac{\alpha^2}{n^2} \tag{7}$$

we can conclude $v \sim \alpha_s$[8]. Consequently, quantum loop effects have to be taken into account simultaneously with the inclusion of new operators that contribute at higher orders in v when calculating corrections to the LO terms. The full expansion of NRQCD is then simultaneously in v and α_s while α_s/v is of order 1.

3 The NRQCD Lagrangian

3.1 Construction of the Lagrangian

We follow the general strategy for the construction of an EFT in order to obtain the NRQCD Lagrangian but stay with a non-relativistic description in the c.o.m. frame. The steps involved are the following:

1. Identification of the fields (degrees of freedom) that describe the low energy physics: the heavy quark and antiquark, which we will describe by non-relativistic Pauli 2-spinor fields ψ and χ, the gauge fields A_μ and the light quarks. Thereby ψ and χ carry only the non-relativistic energy of the respective particles.

[6] The numerator $\not{q} + \not{l} + m_Q$ always goes like v^0.
[7] This will become clearer when we will talk about potential NRQCD later.
[8] This is also consistent with the strong interaction being perturbative at scales of the heavy quark masses.

2. Identification of the QCD symmetries that have to be carried over in the EFT: $SU(3)$ gauge symmetry, charge conjugation, parity and rotational symmetry[9]. Additionally, we demand the EFT to be invariant under separate $U(1)$ transformation of the heavy quark and anti-quark fields as their number should be separately conserved, because heavy quark-antiquark annihilation processes happen at the scale m_Q that is integrated out.

3. Write the most general Lagrangian compatible with the symmetries, up to a certain accuracy level: this is more delicate than in the usual construction of EFTs because of the more intricate power counting. As argued in the chapter before, an expansion in the dimension of composite field operators is nevertheless reasonable.

4. Each operator in the effective Lagrangian will be multiplied by a coefficient, which should be determined by matching the EFT to full QCD.

The resulting effective Lagrangian can be structured as follows

$$\mathcal{L}_{NRQCD} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\psi\chi} + \mathcal{L}_g + \mathcal{L}_{light} \tag{8}$$

where $\mathcal{L}_g$ contains purely gluon fields, $\mathcal{L}_{light}$ contains the light quarks and $\mathcal{L}_{\psi\chi}$ describes heavy quark-antiquark interactions.
The heavy quark Lagrangian is

$$\mathcal{L}_\psi = \psi^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2m_Q} \right) \psi - \frac{d_1 g_s}{2m_Q} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B}\, \psi + O\left(\frac{1}{m_Q^2} \right). \tag{9}$$

which to order $1/m_Q$ is just a copy of the HQET Lagrangian in the rest frame $v = (1, 0, 0, 0)$. The chromo-magnetic field $\mathbf{B}$ is defined such that

$$\boldsymbol{\sigma} \cdot \mathbf{B} = -\frac{1}{2} \sigma^{ij} G^{ij}. \tag{10}$$

The heavy antiquark Lagrangian $\mathcal{L}_\chi$ follows from the heavy quark Lagrangian by $\mathcal{L}_\psi = -\mathcal{L}_\chi$ and $\psi \to \chi, iD^0 \to -iD^0$. The gluon Lagrangian takes the form

$$\mathcal{L}_g = -\frac{d_4}{4} G^A_{\mu\nu} G^{\mu\nu\, A} + \frac{d_5}{m_Q^2} G^A_{\mu\nu} D^2 G^{\mu\nu\, A} + \frac{d_6}{m_Q^2} g_s f^{ABC} G^A_{\mu\nu} G^{\alpha\, B\mu} G^{C\nu\alpha} + O\left(\frac{1}{m_Q^4} \right). \tag{11}$$

It originates from integrating out heavy quark loops with non-hard gluon lines attached to it. The light quark sector $\mathcal{L}_{light}$ is the same as in QCD. The LO of the heavy quark-antiquark interaction Lagrangian consists of four fermion interactions. They read

$$\begin{aligned} \mathcal{L}_{\psi\chi} = {} & \frac{d_{ss}}{m_Q^2} \psi^\dagger \psi \chi^\dagger \chi - \frac{d_{sv}}{8m_Q^2} \psi^\dagger [\sigma^i, \sigma^j] \psi \chi^\dagger [\sigma^i, \sigma^j] \chi + \frac{d_{vs}}{m_Q^2} \psi^\dagger T^A \psi \chi^\dagger T^A \chi - \\ & - \frac{d_{vv}}{8m_Q^2} \psi^\dagger T^A [\sigma^i, \sigma^j] \psi \chi^\dagger T^A [\sigma^i, \sigma^j] \chi + (\psi \longleftrightarrow \chi) + O\left(\frac{1}{m_Q^3} \right) \end{aligned} \tag{12}$$

where $\psi \longleftrightarrow \chi$ means exchanging only the "fields to the right" to get "annihilation order" $\chi^\dagger \psi \psi^\dagger \chi$ from the "scattering order" $\psi^\dagger \psi \chi^\dagger \chi$. In four space-time dimensions, they are related by a Fierz transformation, but if we regulate our theory in dimensional regularization, they are not. Further, we also neglected interaction terms between heavy and light quarks, such as $\chi^\dagger \psi \bar{q} q$, as these only

[9] The still intact subgroup of Lorentz symmetry in the non-relativistic description in the c.o.m. frame.

contribute to higher order effects in v than the terms shown above, because light quarks don't appear as external states in heavy quarkonium.
Operators that vanish on the leading order equations of motion (or equivalently can be eliminated by field redefinitions) were excluded from $\mathcal{L}_{NRQCD}$, hence NRQCD formulated as above reproduces only on-shell QCD Green's functions at low energy. Consequently, the Wilson coefficients of NRQCD should be determined by matching on-shell Green's functions of full QCD and NRQCD.
The term bilinear in the heavy quarks field in the heavy quark Lagrangian (9) can be used to calculate its propagator. This is straight forward and the momentum space Feynman rule reads

$$\frac{i}{k^0 - \frac{\mathbf{k}^2}{2m_Q} + i\epsilon}, \tag{13}$$

where $k = (k^0, \mathbf{k})$ is the heavy quark's non-relativistic energy and momentum in the c.o.m. frame, which is the residual momentum carried by the non-relativistic spinor field ψ. Its velocity-scaling agrees with the rules deduced in subsection 2.2 as expected. For the heavy anti-quark propagator, the sign in between the energy and momentum terms changes to a plus sign.

3.2 Velocity scaling of fields and operators

In this subsection we want to derive the leading order scaling of different operators with v (of course depending on the respective momentum region) in order get a grasp on their relative importance. We can get the velocity scaling of the heavy quark fields, using the power counting method described in subsection 2.2. In NRQCD, heavy quarks can be potential or soft. We get at LO

$$\langle\Omega|T\{\psi^\dagger(x)\psi(y)\}|\Omega\rangle \approx \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^0 - \frac{\mathbf{p}^2}{2m_Q} + i\epsilon} e^{-ip(x-y)} \sim \begin{cases} v^4 \cdot v^{-1} = v^3 \text{ in the soft region} \\ v^5 \cdot v^{-2} = v^3 \text{ in the potential region} \end{cases}$$

Hence for both soft and potential modes, the heavy (anti)quark fields scale as $v^{3/2}$. An analogous procedure reveals that the gluon field scales as v (soft), $v^{3/2}$ (potential) and v^2 (ultrasoft). Derivatives on fields scale according to the energy (for time derivatives) or momentum (for spacial derivatives) carried by the field. Keep in mind: these rules only reflect the leading order scaling and don't account for subleading order contributions.
Recalling also that $g_s \sim v^{1/2}$, we can now analyse the leading order scaling of the terms appearing in equation (9), depending on the modes of the heavy quark:

- potential heavy quarks: The terms containing only derivatives in $\psi^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2m_Q}\right)\psi$ both scale as v^5, hence, contrary to HQET, both contribute at LO. The interaction terms containing gluons are at least of order v^6 for all in NRQCD possible gluon modes[10] with the exception of the $g_s\psi^\dagger\psi A^0$ interaction, which scales as v^5 for potential gluons such that it has to be treated non-pertubatively in this region. The chromo-magnetic interaction scales at least as v^6 and never contributes at LO, hence at LO NRQCD has heavy quark spin symmetry.
- soft heavy quarks: The derivative term in $\psi^\dagger iD_0\psi$ now scales as v^4 while the derivative term in $\mathbf{D}^2$ term still scales as v^5. It can therefor be treated as a perturbation and at leading order in an expansion in v, the heavy quark propagator then simplifies to the HQET propagator in the rest frame. All interaction terms with gluon fields only contribute at higher order than the leading term, independently of the gluon mode considered. They can all be treated perturbatively.

[10]Note that soft gluon modes are not relevant in $\psi\psi A$-type vertices with potential heavy quarks as they violate energy conservation.

The four quark operators in equation (12) are generated either by scattering with hard gluon exchange or quark-antiquark annihilation. Naively, these interactions seem to be of order v^6. For scattering with hard gluon exchange, the coefficient functions in front are all at least of order $\alpha_s^2 \sim v^2$ though as this can only appear at one loop order in QCD, making these terms in total at least $\sim v^8$. The quark-antiquark annihilation terms can be found at tree level in QCD for colour octet states of the $Q\bar{Q}$ pair[11]. The coefficients would then still be at least order $\alpha_s \sim v$, but as the $Q\bar{Q}$-system in nature has to be in a color singlet state because of confinement, all terms in annihilation order also only contribute at higher order.
For the gluon Lagrangian, with respect to the LO term $-\frac{d_4}{4} G^A_{\mu\nu} G^{\mu\nu A}$, the second term $\frac{d_5}{m_Q^2} G^A_{\mu\nu} D^2 G^{\mu\nu A}$ contains an extra $\mathbf{D}^2$ and the coefficient has to be $d_5 \sim \alpha_s \sim v$. Therefore, this operators scales with at least three extra powers of v compared to the leading term. The third term in the Lagrangian is clearly even higher order.

3.3 Matching between QCD and NRQCD

While heavy (anti)quarks are relativistic 4-component Dirac spinor fields in full QCD, we decided to describe them in NRQCD by separate 2-component Pauli spinor fields, that also carry only the non-relativistic energy of the particle. In order to perform matching calculations, their relation has to be clarified.
Denoting the external non-relativistic heavy quark and antiquark 2-spinors in NRQCD by $\xi(s)$ and $\eta(s)$ respectively, the external heavy-quark 4-spinors of QCD in the Dirac basis of the γ-matrices can be written as

$$u(p,s) = \frac{1}{\sqrt{E+m_Q}} \begin{pmatrix} (E+m_Q)\,\xi(s) \\ \boldsymbol{\sigma}\cdot\mathbf{p}\,\xi(s) \end{pmatrix},\; v(p,s) = \frac{1}{\sqrt{E+m_Q}} \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{p}\,\eta(s) \\ (E+m_Q)\,\eta(s) \end{pmatrix}. \tag{14}$$

for an on-shell heavy-quark 4-momentum $p = (E, \mathbf{p})$ with $E = \sqrt{m_Q^2 + \mathbf{p}^2}$. The 2-spinors are normalized according to

$$\xi^\dagger \xi = \eta^\dagger \eta = 1. \tag{15}$$

Hence during the matching procedure, in QCD calculation the momentum dependence hidden in the Dirac structure needs to be extracted in order to relate consistently to the NRQCD 2-spinors. Furthermore, one has to take into account that the normalization of relativistic QCD heavy quark states and non-relativistic NRQCD heavy quark states differ by a factor $\sqrt{2E}$.
Let's see this at work for the following example: In the on-shell renormalization scheme, to all orders in perturbation theory, the heavy quark part of the amputated on-shell two-point function of the heavy quark field is in QCD given by

$$(-i)\,\bar{u}(p,s)(\not{p} - m_Q)u(p,s) = ... = (-i)\,2E\,\xi^\dagger(s)\xi(s)\,\left(E - \sqrt{m_Q^2 + \mathbf{p}^2}\right), \tag{16}$$

where we plugged in the above expression for $u(p,s)$ and skipped the steps of multiplying out the Dirac structure. Expanding the square root, we get

$$(-i)\,\bar{u}(p,s)(\not{p} - m_Q)u(p,s) = (-i)\,2E\,\xi^\dagger(s)\xi(s)\,\left(E - m_Q - \frac{\mathbf{p}^2}{2m_Q} + ...\right) \tag{17}$$

In the effective theory, the factor $2E$ result from the normalization of the heavy quark states, the polarization vector product $\xi^\dagger(s)\xi(s)$ follows directly and from the heavy quark Lagrangian bilinear in the heavy quark field ψ, we get $i\partial^0 \to E - m_Q$, $i\vec{\partial} \to \mathbf{p}$, hence this checks and the Wilson coefficients of these terms are 1 to all orders in perturbation theory.

[11]For singlet states, the tree level diagrams vanish due to $tr(T^A) = 0$.

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