On-Shell EFTs III: Applications of Soft-Collinear Effective Theory

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1 The off-shell Sudakov form factor

1.1 The strategy of Regions

A key conceptual aspect of SCET, which is also employed in NRQCD, is the strategy of regions. It is a technique that allows to carry out asymptotic expansions of loop integrals by splitting the integration in different regions and expanding the integrands appropriately in each one of them.
A simple example to see this at work is the integral

$$I = \frac{1}{2\pi i}\int d^2k \frac{1}{(k^2-m^2)(k^2-M^2)} = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln(\frac{M}{m})}{M^2-m^2} \tag{1}$$

appearing in the one-loop self energy of a 2-dimensional theory of e.g. two scalar fields ϕ, Φ with respective masses m^2, M^2 interacting via $\phi^2\Phi$. Assuming the hierarchy $m^2 \ll M^2$, the result can be asymptotically expanded as

$$I = \frac{\ln(\frac{M}{m})}{M^2}\left(1+\frac{m^2}{M^2}+...\right). \tag{2}$$

The strategy of regions gives a prescription on how to recover this asymptotic result when expanding first and then performing the integral: First, one needs to identify all regions of the integrand that lead to singularities. Second, one expands the integrand in these regions, regulates the integrals properly and integrates them over the full phase space[1]. Third, one adds up the results of all the different regions.
Let's see that this works at the example of I above. First, the relevant regions to this integral are $k \sim m \ll M$ and $m \ll M \sim k$. Second, expanding the integrands in these regions we find the leading contributions to be

$$I_1 = \int_0^\infty dk \frac{k^{1-\epsilon}}{(k^2+m^2)M^2} = \frac{1}{M^2}\left(\frac{1}{\epsilon} - \ln m + O(\epsilon)\right) \tag{3}$$

[1] The reason that a regulator is necessary is that the expansion of the integral was asymptotic when performed after the integration: the result was not analytic in the limit around which we expanded. Hence there have to occur divergences in the integral upon expanding before performing the integration.

$$I_2 = \int_0^\infty dk \frac{k^{1-\epsilon}}{k^2(k^2+M^2)} = \frac{1}{M^2}\left(-\frac{1}{\epsilon} + \ln M + O(\epsilon)\right) \tag{4}$$

The integrals where regulated using dimensional regularisation. Third, we add both up and recover the leading order result for I from above:

$$I_1 + I_2 = \frac{\ln(\frac{M}{m})}{M^2}. \tag{5}$$

The fact that we integrated in both regions over the full phase space deserves a comment. The procedure is intuitively better justified, if we introduce a separation scale Λ, $m \ll \Lambda \ll M$, and integrate for I_1 over the interval $(0, \Lambda)$ and for I_2 over (Λ, ∞). As it turns out the "zero-bin" contributions added to the integral by sending $\Lambda \to \infty$ for I_1 and $\Lambda \to 0$ or I_2 cancel against each other - a feature of the strategy of regions that is non-trivial and leads to subtleties in more involved situations. Finally, it has to be pointed out that up to date, there is no proof that this procedure always works, but many situations are well understood.

1.2 The Sudakov problem in full QCD

One well established success of SCET is the resummation of leading double logarithms, so called Sudakov logarithms. We will examine this to one loop order at the example of the QCD matrix element

$$\int d^4x \int d^4y \; e^{-ip\cdot x + il\cdot y} \langle 0|T\{\psi(x)J^\mu(0)\bar{\psi}(y)\}|0\rangle\Big|_{amp.} \tag{6}$$

of the external vector current operator

$$J^\mu = \bar{\psi}\gamma^\mu\psi \tag{7}$$

where ψ is a massless quark field. In the kinematic situation where the external momenta are slightly off-shell in a space-like way and carry large energies,

$$-p^2, -l^2 \ll -(p-l)^2 \equiv Q^2, \tag{8}$$

this problem can be tackled by the SCET formalism we developed above. We define the small parameter λ such that

$$\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2}, \tag{9}$$

where $P^2 \equiv -p^2$ and $L^2 \equiv -l^2$. Next we go to the frame where $\vec{Q} = 0$ and introduce the two light-like reference vectors n_+^μ and n_-^μ such that l approximately points in n_+^μ and p approximately points in n_-^μ direction. From $p^2 \sim \lambda^2 Q^2$, we can then conclude that either $p^\mu \sim (0, 1, \lambda)Q$ or $p^\mu \sim (\lambda^2, 1, 0)Q$ or $p^\mu \sim (\lambda^n, 1, \lambda)Q$ with $n \geq 2$ or $p^\mu \sim (\lambda^2, 1, \lambda^n)Q$ with $n \geq 1$ and analogously for l with $Q \equiv \sqrt{Q^2}$. Hence the most general case is to assume a collinear scaling for p and an anti-collinear one for l:

$$p \sim (\lambda^2, 1, \lambda)Q, \;\; l \sim (1, \lambda^2, \lambda)Q. \tag{10}$$

Before we match the above vector current matrix element in SCET, let's look at it in full QCD. The matching at tree level turns out to be trivial, so let's directly examine what happens at one loop level. There is only one contributing diagram, shown in Figure 1.

In the evaluation of the corresponding amplitude, we ignore all colour structures and complications arising due to spin, because to see all the interesting features of the strategy of regions, only

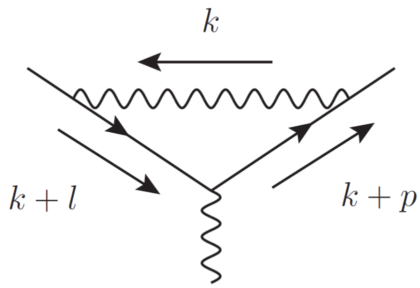

Figure 1: One loop Feynman diagram in QCD. The straight lines are quark lines, the horizontal curly line is a gluon line and the vertical curly line represents the current operator.

the singularity structure contained in the denominator is essential. The resulting loop integral to evaluate then reads

$$I = \pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{i}{(k^2 + i\epsilon)[(k+l)^2 + i\epsilon][(k+p)^2 + i\epsilon]}. \tag{11}$$

It is already written as usual in dimensional regularisation although, as for the simple example from above, the integral is finite before it is split into the different singular regions. Adapting a scaling $k \sim (\lambda^a, \lambda^b, \lambda^c)Q$ for the loop momentum and expanding the integrand for all possible configurations (a, b, c), one then finds that only the following regions lead to non-vanishing contributions:

- Hard region (h): $k \sim (1, 1, 1)Q$.
- Region collinear to p (c): $k \sim (\lambda^2, 1, \lambda)Q$.
- Region anti-collinear to p $(\bar{c})$: $k \sim (1, \lambda^2, \lambda)Q$.
- Ultrasoft region (us): $k \sim (\lambda^2, \lambda^2, \lambda^2)!$.

E.g. soft contributions $k \sim (\lambda, \lambda, \lambda)Q$ or so called Glauber contributions $k \sim (\lambda^2, \lambda^2, \lambda)Q$ indeed vanish. This also shows that this problem can indeed be tackled by the SCET we developed above, but we need to also include anti-collinear fields in the theory. Upon expanding to leading order in λ, we obtain

$$I_h = \pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{i}{(k^2 + i\epsilon)[k^2 + 2k_- \cdot l_+ + i\epsilon][k^2 + 2k_+ \cdot p_- + i\epsilon]}, \tag{12}$$

$$I_c = \pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{i}{(k^2 + i\epsilon)[2k_- \cdot l_+ + i\epsilon][(k+p)^2 + i\epsilon]}, \tag{13}$$

$$I_{\bar{c}} = \pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{i}{(k^2 + i\epsilon)[(k+l)^2 + i\epsilon][2k_+ \cdot p_- + i\epsilon]}, \tag{14}$$

$$I_{us} = \pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{i}{(k^2 + i\epsilon)[2k_- \cdot l_+ + l^2 + i\epsilon][2k_+ \cdot p_- + p^2 + i\epsilon]} \tag{15}$$

and performing the integrals using standard techniques, the results read

$$I_h = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}+O(\epsilon,\lambda)\right), \tag{16}$$

$$I_c = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}+O(\epsilon,\lambda)\right), \tag{17}$$

$$I_{\bar{c}} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}+O(\epsilon,\lambda)\right), \tag{18}$$

$$I_{us} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2P^2}+\frac{\pi^2}{6}+O(\epsilon,\lambda)\right). \tag{19}$$

They contain double poles originating from collinear and soft IR divergences in the hard region and UV divergences in the other three regions. After summing up the four contributions, all poles cancel each other and the limit $\epsilon \to 0$ can be trivially performed to obtain the result

$$I = I_h + I_c + I_{\bar{c}} + I_{us} = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{L^2}+\frac{\pi}{3}+O(\lambda)\right). \tag{20}$$

One can check that it indeed coincides with the result that would be obtained by first performing the integral and then expanding in λ. The structure of a product of two potentially large logarithms at leading order is known as the Sudakov problem. Their appearance is closely related to the fact that the individual regions exhibited double poles. For the rest of this section, we will sketch how these double logs can be resummed in SCET.

1.3 The vector current matrix element in SCET

As already noted above, we will also have to include anti-collinear fields in the SCET Lagrangian in order to match the QCD vector current operator in SCET. They are defined as the collinear ones with exchange of the reference vectors: $n_+^\mu \leftrightarrow n_-^\mu$ (implying $x_+ \leftrightarrow x_-$). The anti-collinear Lagrangian is then just a copy of the collinear one with the same exchange of the reference vectors. Collinear-anti-collinear interaction don't appear in the theory, because the sum of a collinear and anti-collinear momenta gives a hard momentum, which we intergated out in SCET. Schematically,

$$\mathcal{L} = \mathcal{L}(c,us) + \mathcal{L}(\bar{c},us) + \mathcal{L}(us) \xRightarrow{SDT} \mathcal{L}(c) + \mathcal{L}(\bar{c}) + \mathcal{L}(us) + O(\lambda) \tag{21}$$

and after the soft decoupling transformation, all regions are separated at leading order, which is called factorisation.

Let's now construct the leading order SCET operator corresponding to the vector current operator $J^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$ step by step. As we are interested in evaluating the matrix element of the operator in between an initial anti-collinear and a final collinear state, we first make the replacement

$$\bar{\psi}(x)\gamma^\mu\psi(x) \to \bar{\xi}_c(x)\gamma^\mu\xi_{\bar{c}}(x). \tag{22}$$

The properties $\not{n}_-\xi_c = \not{n}_+\xi_{\bar{c}} = 0$ imply

$$\bar{\xi}_c(x)\gamma^\mu\xi_{\bar{c}}(x) = \bar{\xi}_c(x)\gamma_\perp^\mu\xi_{\bar{c}}(x) \tag{23}$$

hence it is enough to sandwich the orthogonal components of the gamma matrices with respect to the reference vectors.
As collinear fields carry momenta of order of the hard scale in n_-^μ direction, acting with any number of derivatives on the field and projecting on the n_+^μ direction[2], $(n_+\cdot\partial)^m\xi_c(x),\ m\in\mathbb{N}$, is not power-suppressed with respect to field $\xi_c(x)$ itself. For the matching of an effective theory to the full theory to work out at any order in the expansion parameter, all operators compatible with the symmetries of the theory have to be included. To achieve this, we make the replacement

$$\xi_c(x)\to\int ds\ B(s)\sum_{m=0}^{\infty}\frac{s^m}{m!}(n_+\cdot\partial)^m\xi_c(x)=\int ds\ B(s)\xi_c(x+sn_+). \tag{24}$$

where $B(s)$ plays the role of the Wilson coefficient[3]. An analogous consideration for the anti-collinear field then leads to

$$\bar{\xi}_c(x)\gamma_\perp^\mu\xi_{\bar{c}}(x)\to\int dsdt\ C_V(s,t)\bar{\xi}_c(x+sn_+)\gamma_\perp^\mu\xi_{\bar{c}}(x+tn_-). \tag{25}$$

In a last step, we want to restore gauge invariance under independent collinear and anti-collinear gauge transformations. Gauge invariance has been spoiled in the above procedure by adding the terms containing any number of ordinary derivatives, but also by the initial replacement of the QCD fields by a collinear and an anti-collinear field, because they transform only under their respective gauge transformations. This goal is achieved by the introduction of two Wilson lines, a collinear one W_c and an anti-collinear one $W_{\bar{c}}$ (see Appendix A equation (57)), just as we saw last time for the $\mathcal{L}_{\xi\psi_{us}}$ part of the SCET Lagrangian. The SCET external vector current operator then reads

$$J^\mu_{SCET}=\int dsdt\ C_V(s,t)(\bar{\xi}_cW_c)(x+sn_+)\gamma_\perp^\mu(W_{\bar{c}}^\dagger\xi_{\bar{c}})(x+tn_-). \tag{26}$$

If the soft decoupling transformation (see Appendix A equation (59)) is performed, it can be shown to act on $\chi_{c/\bar{c}}=W^\dagger_{c/\bar{c}}\xi_{c/\bar{c}}$ as

$$\chi_c(x)\to S_-(x_-)\chi_c^{(0)}(x),\qquad \chi_{\bar{c}}(x)\to S_+(x_+)\chi_{\bar{c}}^{(0)}(x) \tag{27}$$

with the soft Wilson lines defined as in Appendix A equation (58). This implies for the vector current the transformation

$$J^\mu_{SCET}\to\int dsdt\ C_V(s,t)\bar{\chi}_c^{(0)}(x+sn_+)S_-^\dagger(x_-)S_+(x_+)\gamma_\perp^\mu\chi_{\bar{c}}^{(0)}(x+tn_-)$$

and multipole-expanding to obtain leading order[4] gives

$$J^\mu_{SCET}=\int dsdt\ C_V(s,t)\bar{\chi}_c^{(0)}(x_++x_\perp+sn_+)S_-^\dagger(0)S_+(0)\gamma_\perp^\mu\chi_{\bar{c}}^{(0)}(x_-+x_\perp+tn_-). \tag{28}$$

[2]Remember: $n_\pm^2=0$ while $n_+\cdot n_-=2$.

[3]This replacement is based on the assumption that there exists a function $B(s)$ such that $C_m=\int dsB(s)\frac{s^m}{m!}$ for C_m the Wilson coefficient of the term containing m derivatives. It is proven by showing that the matching procedure works out even when we do this and B(s) plays the role of the Wilson coefficient that is to be determined. Furthermore, this replacement also shows that the inclusion of terms containing any number of derivatives is equivalent to allowing non-locality of the operator and consequently demonstrating why non-locality is a consequence of fields carrying momenta that are of order of the high energy scale and hence not power-suppressed.

[4]As x is conjugate to a sum of a collinear and an anti-collinear momentum, it scales as $x\sim(1,1,1/\lambda)$.

Observe that contrary to the SCET Lagrangian, the ultrasoft contributions do not decouple, even if we look at the matrix element only at leading order in λ. Diagrammatically speaking, the soft decoupling transformation moved the leading order collinear-ultrasoft interaction vertices into the (external) sources of the theory. From now on, we will drop the subscript (0).
The SCET analogue to the QCD matrix element we examined in the last section is the so called "off-shell Sudakov form factor", defined as

$$F(Q^2, L^2, P^2) \equiv \int d^4x \int d^4y \; e^{-ip\cdot x + il\cdot y} \langle 0|T\{\xi_c(x) J^{\mu}_{SCET}(0) \bar{\xi}_{\bar{c}}(y)\}|0\rangle\Big|_{amp.}. \tag{29}$$

Based on the ultrasoft decoupling at the level of the Lagrangian, the total SCET Fock space factorises at leading order to a direct product of the subspaces of collinear, anti-collinear and ultrasoft states such that all field operators can be restricted to only act non-trivially in the respective subspaces. This makes it possible to prove a factorisation theorem, valid at leading order in SCET, that allows to factorise $F(Q^2, L^2, P^2)$ in a collinear, anti-collinear and ultrasoft part and the Wilson coefficient:

$$F(Q^2, L^2, P^2) = \tilde{C}_V(Q^2, \mu^2) \; \mathcal{J}_{\bar{c}}(L^2, \mu^2) \; \gamma^{\mu}_{\perp} \; \mathcal{J}_c(P^2, \mu^2) \; \mathcal{S}(\Lambda_s^2, \mu^2) \tag{30}$$

where the colliner/anti-collinear functions $\mathcal{J}_c, \mathcal{J}_{\bar{c}}$ are defined as

$$\mathcal{J}_c(P^2, \mu^2) \equiv \int d^4x \; e^{-ip\cdot x} \; \langle 0|T\{\xi_c(x)\bar{\chi}_c(0)\}|0\rangle\Big|_{amp.}, \tag{31}$$

$$\mathcal{J}_{\bar{c}}(L^2, \mu^2) \equiv \int d^4y \; e^{il\cdot y} \; \langle 0|T\{\chi_{\bar{c}}(0)\bar{\xi}_{\bar{c}}(y)\}|0\rangle\Big|_{amp.}, \tag{32}$$

and the ultrasoft function $\mathcal{S}$, depending on the ultrasoft scale $\Lambda_s^2 = L^2P^2/Q^2$, is

$$\mathcal{S}(\Lambda_s^2, \mu^2) \equiv \langle 0|T\{S_-^{\dagger}(0)S_+(0)\}|0\rangle. \tag{33}$$

This decomposition included a shift $x \to x - sn_+, y \to y - tn_-$ in the d^4x, d^4y integrals respectively and the use of translation invariance in the correlation functions. Furthermore, the vacuum states have to be interpreted as those of the respective subspaces and we have indicated that the individual factors will depend on the renormalization scale μ. The factor $\tilde{C}_V(Q^2, \mu^2)$ is the Fourier transform of $C_V(s, t, \mu^2)$ according to the directions of non-locality of the fields in the current:

$$\tilde{C}_V(Q^2, \mu^2) = \int dsdt \; e^{isn_+\cdot p} e^{-itn_-\cdot l} C_V(s, t, \mu^2). \tag{34}$$

From symmetry arguments[5], it can even only depend on the product $(n_+ \cdot p)(n_- \cdot l) = p_+ \; l_-$ and as $Q^2 = 2l \cdot p = p_+ \; l_- + O(\lambda)$, to leading order the dependency is $\tilde{C}_V(Q^2, \mu^2)$.
The factorisation is graphically summarised in Figure 1.3.

[5] The introduction of $n_\pm$ breaks down Lorentz invariance to rotations in the plane where $n_\pm$ have zero-components. Similarly to HQET, we get reparametrization symmetries in exchange. The one which we use here to argue is called RPI-III: the SCET Lagrangian is invariant under $n_+ \to an_+$ and $n_- \to \frac{1}{a}n_-$ for some number a. It corresponds to Lorentz boosts in the spatial direction in which $n_\pm$ have non-zero component.

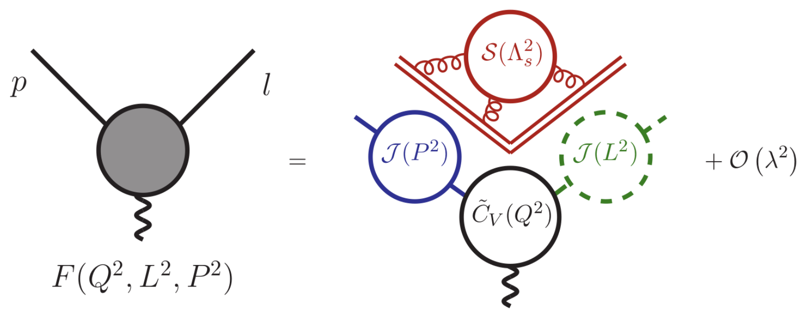

Figure 2: Diagrammatic representation of the Sudakov form factor in QCD showing the factorization of the different scales in the problem.

1.4 Matching calculation for the Sudakov form factor

We now want to determine $\tilde{C}_V(Q^2, \mu)$ via matching the off-shell Sudakov form factor $F(Q^2, L^2, P^2)$ to the corresponding QCD matrix element:

$$\begin{aligned}&\int d^4x \int d^4y\; e^{-ip\cdot x+il\cdot y}\langle 0|T\{\psi(x)\bar{\psi}(0)\gamma^\mu\psi(0)\bar{\psi}(y)\}|0\rangle\Big|_{amp.} = \\ &\tilde{C}_V(Q^2,\mu)\int d^4x \int d^4y\; e^{-ip\cdot x+il\cdot y}\langle 0|T\{\xi_c(x)\bar{\chi}_c(0)S_-^\dagger(0)S_+(0)\gamma_\perp^\mu\chi_{\bar{c}}(0)\bar{\xi}_{\bar{c}}(y)\}|0\rangle\Big|_{amp.}.\end{aligned} \tag{35}$$

The matching at tree level simply yields $\tilde{C}_V^{tree}(Q^2) = 1 \leftrightarrow C_V^{tree}(s,t) = \delta(s)\delta(t)$. Moving on to up to one-loop level, the contributing SCET diagrams are shown in Figure 1.4, while the QCD diagrams are the tree diagram plus the one-loop diagram from Figure 1.

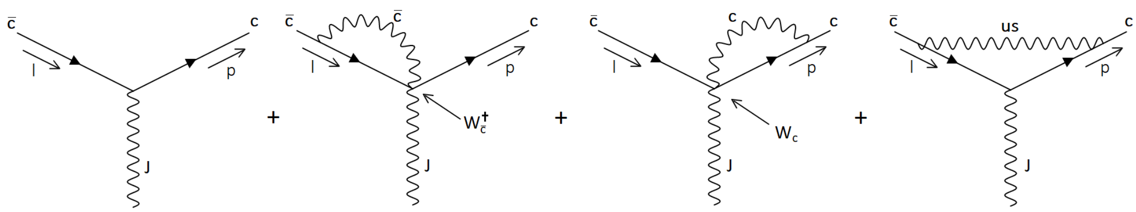

Figure 3: One loop level SCET diagrams contributing to the matching of the off-shell Sudakov form factor in SCET. In the second and third diagram, the (anti-)collinear gluon lines couple to the (anti-) collinear Wilson lines contained in the external current operator.

Doing the full matching calculation in dimensional regularisation and not only the simplified one as we did in subsection 1.2, one finds the result

$$\tilde{C}_{V,bare}^{1\text{loop}}(Q^2,\epsilon) = 1 + \frac{\alpha_s^0}{4\pi}C_F\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{\pi^2}{6} + O(\epsilon)\right)\left(\frac{e^{\gamma_E}Q^2}{4\pi}\right)^{-\epsilon} + O(\alpha_s^2), \tag{36}$$

where α_s^0 is the bare strong coupling constant and γ_E the Euler-Mascheroni constant. The poles originate from the IR divergences of the hard region, whose contributions are absorbed in the

Wilson coefficients of the effective theory. The three SCET one-loop graphs cancel exactly against the QCD contributions from the collinear, anti-collinear and ultrasoft regions[6]. Next we introduce the $\overline{\text{MS}}$ renormalized coupling α_s as usual,

$$Z_\alpha \alpha_s(\mu)\mu^{2\epsilon} = e^{-\epsilon\gamma_E}(4\pi)^\epsilon \alpha_s^0 \tag{37}$$

with $Z_\alpha = 1 + O(\alpha_s)$, and multiplicative operator renormalization for the external current via a factor Z_C that absorbs all remaining divergences such that

$$\tilde{C}_V(Q^2,\mu) = \lim_{\epsilon\to 0} Z_C^{-1}(Q^2,\mu,\epsilon)\tilde{C}_{V,bare}(Q^2,\mu,\epsilon) \tag{38}$$

is well-defined. Explicitly, this means

$$Z_C(Q^2,\mu,\epsilon) = 1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{Q^2}{\mu^2} - \frac{3}{\epsilon}\right) + O(\alpha_s^2) \tag{39}$$

Plugging in, the renormalized Wilson coefficient $\tilde{C}(Q^2,\mu)$ is

$$\tilde{C}_V(Q^2,\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-\ln^2\frac{Q^2}{\mu^2} + 3\ln\frac{Q^2}{\mu^2} + \frac{\pi^2}{6} - 8\right) + O(\alpha_s^2). \tag{40}$$

It contains leading and potentially large Sudakov logarithms as expected, when we want to set $\mu^2 \sim P^2, L^2$, the external low-energy scales of the problem.

1.5 RGE and resummation of Sudakov logarithms

In this concluding subsection, we will sketch how the leading Sudakov logarithms as well as the subleading single logarithms can be resummed using RGE techniques and how the entire Sudakov form factor can be run to a common scale without the appearance of large logarithms.
We start with the resummation of the logarithms in $\tilde{C}_V(Q^2,\mu)$. Its RGE can be calculated to take the somewhat unusual form

$$\frac{d}{d\ln\mu}\tilde{C}_V(Q^2,\mu) = \left[C_F\gamma_{cusp}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]\tilde{C}_V(Q^2,\mu), \tag{41}$$

where γ_{cusp} is the so called cusp anomalous dimension[7] and γ_V the anomalous dimension associated with the function $\tilde{C}_V(Q^2,\mu)$. By plugging in the result we found, we obtain

$$\gamma_{cusp}(\alpha_s) = 4\frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2), \qquad \gamma_V(\alpha_s) = -6C_F\frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2) \tag{42}$$

at one loop accuracy. To resum the potentially large logarithms, we need to solve the RGE. This can be done by separation of variables and yields the solution

$$\tilde{C}_V(Q^2,\mu) = U(\mu_h,\mu)\tilde{C}_V(Q^2,\mu_h), \tag{43}$$

[6]This is by construction: SCET is constructed in such a way that individual SCET diagrams match the different contributions to the full theory loop integral expanded via the strategy of regions with the exception of the hard region, which is absorbed in the Wilson coefficients. Hence we could have determined $\tilde{C}_V(Q^2)$ directly by matching it onto the hard regions of the QCD graph, corresponding to setting $L^2 = P^2 = 0$ at the start of the calculation.

[7]Its appearance is related to the ultrasoft operator $S_-^\dagger(0)S_+(0)$ requiring renormalization. The name "cusp" originates from the fact that this operator can be seen as one Wilson line running along sn_+ direction from $s=-\infty$ to $s=0$, making a cusp and then running from $s=0$ to $s=\infty$ along sn_- direction.

where the evolution matrix after some rewriting reads

$$U(\mu_h,\mu) = \exp\left[2C_F S(\mu_h,\mu) - A_{\gamma_V}(\mu_h,\mu)\right]\left(\frac{Q^2}{\mu_h^2}\right)^{-C_F A_{\gamma_{cusp}}(\mu_h,\mu)} \tag{44}$$

with

$$S(\mu_h,\mu) = -\int\limits_{\alpha_s(\mu_h)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_{cusp}(\alpha)}{\beta(\alpha)} \int\limits_{\alpha_s(\mu_h)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}, \tag{45}$$

$$A_{\gamma_i}(\mu_h,\mu) = -\int\limits_{\alpha_s(\mu_h)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_i(\alpha)}{\beta(\alpha)}. \tag{46}$$

In these expressions

$$\beta(\alpha_s(\mu)) = \frac{d\alpha_s(\mu)}{d\ln\mu} = -2\alpha_s\beta_0\frac{\alpha_s}{4\pi} + O(\alpha_s^3) \tag{47}$$

is the QCD beta-function. If we solve equation (44) perturbatively via plugging in the one-loop results for $\beta(\alpha_s(\mu)), \gamma_{cusp}(\alpha_s), \gamma_V(\alpha_s)$ and choose $\mu_h \sim Q$, we will achieve resummation of the Sudakov logarithms as well as the single logarithms in favor of ratios $\alpha_s(\mu)/\alpha_s(\mu_h)$ and the result will be valid for any value of μ for which $\alpha_s(\mu)$ is sufficiently small and the theory perturbative. Thereby, S is responsible for the resummation of the Sudakov logarithms, which we can see when we take the explicit result

$$S(\mu_h,\mu) = -\frac{4\pi}{\beta_0^2}\left[\frac{1}{\alpha_s(\mu_h)}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_h)} + \frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_h)}\right], \tag{48}$$

insert the one-loop solution to the RGE for α_s in equation (47),

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_h)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_h)\ln\frac{\mu}{\mu_h}}, \tag{49}$$

and then expand in $\alpha_s(\mu_h)$ (using $\ln(1+x) = x - \frac{1}{2}x^2 + O(x^3)$ for $|x| \ll 1$):

$$S(\mu_h,\mu) = \frac{4\pi}{\beta_0^2}\left[\frac{1}{\alpha_s(\mu_h)}\ln\left(1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_h)\ln\frac{\mu}{\mu_h}\right) - \frac{\beta_0}{2\pi}\ln\frac{\mu}{\mu_h}\right] = \tag{50}$$

$$= -\frac{\alpha_s(\mu_h)}{2\pi}\ln^2\frac{\mu}{\mu_h} + O(\alpha_s^2). \tag{51}$$

Now that we have achieved resummation of leading logarithms for the Wilson coefficient function $\tilde{C}_V(Q^2,\mu)$ containing the hard contributions, we want to sketch the resummation for the entire off-shell Sudakov form factor. It is necessary, because each factor appearing in the factorisation of $F(Q^2,L^2,P^2)$ in equation (30) has its own characteristic scale appearing in the logarithms. Hence to resum all leading logarithms contained in the form factor, one needs to solve the individual RGEs for each factor at its own characteristic scale in order to trade the logarithms for ratios of the strong coupling constant at different scales. When this is done, the resummation of all leading logarithms

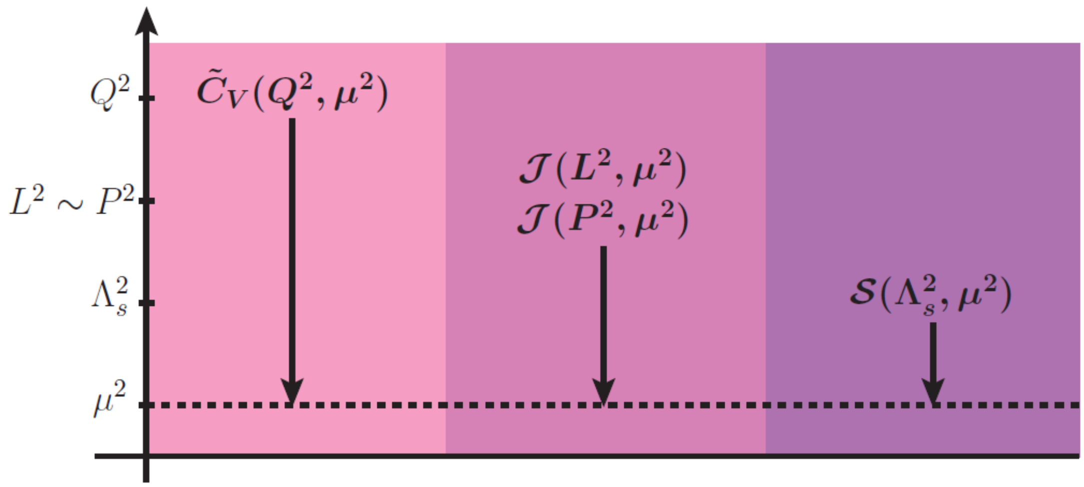

Figure 4: Schematic representation of the scale separation and the calculational procedure in the resummation of $F(Q^2, L^2, P^2)$.

in $F(Q^2, L^2, P^2)$ is complete and all its factors can be run to some common scale μ. This procedure is graphically summarized in Figure 1.5.

Concretely, the other RGEs read

$$\frac{d}{d\ln\mu}\mathcal{J}_c(P^2,\mu) = -\left[C_F\gamma_{cusp}(\alpha_s)\ln\frac{P^2}{\mu^2} + \gamma_J(\alpha_s)\right]\mathcal{J}_c(P^2,\mu), \tag{52}$$

$$\frac{d}{d\ln\mu}\mathcal{S}_c(\Lambda_s^2,\mu) = \left[C_F\gamma_{cusp}(\alpha_s)\ln\frac{\Lambda_s^2}{\mu^2} + \gamma_S(\alpha_s)\right]\mathcal{S}_c(\Lambda_s^2,\mu), \tag{53}$$

and the one for $\mathcal{J}_{\bar{c}}$ can be obtained by $P^2 \to L^2$ in the one for $\mathcal{J}_c$. They have the exact same structure as the RGE for $\tilde{C}_V(Q^2,\mu)$ and γ_{cusp} turns out to be the same for $\tilde{C}_V, \mathcal{J}, \mathcal{S}$. The fact that $F(Q^2, L^2, P^2)$ is independent of μ leads to a constraint connecting all the different anomalous dimensions appearing:

$$0 = \frac{d}{d\ln\mu}\left[\tilde{C}_V(Q^2,\mu^2)\ \mathcal{J}_{\bar{c}}(L^2,\mu^2)\ \gamma_\perp^\mu\ \mathcal{J}_c(P^2,\mu^2)\ \mathcal{S}(\Lambda_s^2,\mu^2)\right] \tag{54}$$

which one finds to be equivalent to

$$0 = \gamma_V - 2\gamma_J + \gamma_S. \tag{55}$$

2 Applications of SCET

SCET is the effective theory describing very energetic particles, so in principle it should be possible to describe any process in high-energy physics in SCET. We want to discuss three examples here. The first one is event shapes in hadronic collider physics, the second one is the discussion of the

Sudakov problem in the context of electroweak physics and the last maybe a bit counterintuitive application is in low-energy heavy quark physics.

2.1 Event Shapes

One way to describe high-energy particle collisions are event-shape variables. They describe events according to simple geometric properties of the final-state hadron momenta. Event-shape variables are designed in such a way that they are not sensitive to hadronization effects and can be computed in perturbation theory. One event-shape variable that is often used at e^+e^- collitder is the thrust T which is defined as

$$T = \frac{1}{P_{tot}} \max_{\vec{n}_T} \sum_i |\vec{n}_T \cdot \vec{p}_i| \tag{56}$$

where the so-called thrust unit vector $\vec{n}_T$ point along the axis of biggest momentum flow in the event and the thrust is the ratio of the momentum along this direction over the total momentum $P_{tot} = \sum_i |\vec{p}_i|$. This observable has been measured to high precision, e.g. by the LEP experiments at CERN. Typical events in the measurement have large thrust close to $T = 1$ which is not surprising: the lowest order perturbation theoy consists of a back-to-back $q\bar{q}$ pair with $T = 1$. Most events therefore have two narrow jets formed by the $q\bar{q}$ pair and subsequent soft and collinear radiation. The typical mass of the jets is $M_J \sim Q^2(1-T)$ and gets perturbative corrections of order $M_J/Q^2 \sim (1-T)$. The 2-jet region can be analyzed in SCET and factorization theorems can be derived for the cross section similar to the ones we found for the vector current. Using SCET RG methods the thrust distribution can be resummed which has been done to N^3LL accuracy in the literature. This is two orders higher in logarithmic accuracy than what has been achieved with other methods.

2.2 Electroweak Sudakov Logarithms

Another application of SCET is in high-energy collisions at energies exceeding the masses of electroweak bosons where large electroweak Sudakov logarithms arise. These logarithms can become numerically large when e.g. performing calculations at LHC energies but can be resummed with the help of SCET techniques. Examples are single Z, W, γ production at large transverse momentum or the analysis of electroweak effects in Higgs production via vector boson fusion.

2.3 Heavy Quark Physics

SCET is surprisingly also applicable in low-energy processes like B decays. The effective theory becomes relevant in the heavy-quark limit $m_q \to \infty$ because the energy of the light decay products is of the order of the b-quark mass. In the example of B mesons SCET is applicable in exclusive decays such as $B \to \pi\pi$ where the energy of the pions is large compared to their mass $E_\pi \approx M_B/2 \gg M_\pi$. But also in inclusive decays such as $B \to X_u l\nu$ with a hadronic jet of small invariant mass m_X which again allows us to perform the usual expansion in the invariant mass of the jet over the jet energy. This leads to factorization theorems of the form hard times jet times soft function.
SCET was first proposed in the context of inclusive B decays, as an alternative method to sum Sudakov logarithms in a certain kinematic region. The effective theory made it possible to not only study the leading power factorization theorem that had already been derived with other methods earlier. But to also study the power corrections which affect the rate of inclusive B decays.

Similarly to our discussion of semi-leptonic decays in HQET, one can use SCET methods to determine $|V_{ub}|$ very precisely from $B \to X_u l\nu$ decays. Here, SCET is used to calculate the decay rate in a systematic expansion and makes it possible to factor the rate into hard, jet and soft physics. This allows for a very precise determination of $|V_{ub}|$.

A Wilson lines and soft decoupling transformation

Collinear/ anti-collinear Wilson lines:

$$W_c(x) = P\exp\left(ig\int_{-\infty}^{0} ds n_+ A_c(x+sn_+)\right), \tag{57}$$
$$W_{\bar{c}}(x) = P\exp\left(ig\int_{-\infty}^{0} ds n_- A_{\bar{c}}(x+sn_-)\right).$$

Soft Wilson lines:

$$S_-(x) = P\exp\left(ig\int_{-\infty}^{0} ds n_- A_{us}(x+sn_-)\right), \tag{58}$$
$$S_+(x) = P\exp\left(ig\int_{-\infty}^{0} ds n_+ A_{us}(x+sn_+)\right),$$

Soft decoupling transformation for collinear and anti-collinear fields:

$$\begin{aligned} \xi(x) &\to S_+(x_+)S_-(x_-)\xi^{(0)}(x), \\ A_c^\mu(x) &\to S_-(x_-)A_c^{(0)\mu}(x)S_-^\dagger(x_-), \\ A_{\bar{c}}^\mu(x) &\to S_+(x_+)A_{\bar{c}}^{(0)\mu}(x)S_+^\dagger(x_+). \end{aligned} \tag{59}$$

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