On-Shell EFTs I: Heavy Quark Effective Theory

Jonathan Kley, Anca Preda, Fabian Wagner

December 8, 2020

1 A Brief Summary of the Main Concepts of EFTs

Let us briefly review the basic idea and concepts of EFTs. The typical situation where EFT concepts can be applied is when there is a QFT with a large separation of scales $\Lambda_L \ll \Lambda_H$. In this case, the heavy physics decouple at energies of order of the low scale Λ_L and we can write down an effective theory excluding the heavy degrees of freedom associated with the large scale Λ_H. We can use this effective theory in two ways

1.) *Top-down approach:* When we already know the full theory, we can use the EFT to make some (approximate) symmetries of the theory more apparent or use tools of EFTs to improve the convergence of perturbation theory.

2.) *Bottom-up approach:* When we know that our theory is valid at low energies but shows small deviations in measurements and therefore cannot be valid at all energies, we can write down all operators that are allowed by symmetry and use the EFT obtained in this way to find the correct direction for a UV completion by casting the experimental deviations in the coefficients of our effective operators.

1.1 How to Construct an EFT

There are several steps going into the construction of an EFT. As already mentioned, the system we want to describe has to have a large separation of scales $\Lambda_L \ll \Lambda_H$. Then, we have to find all symmetries and degrees of freedom that describe the system below our factorisation scale μ ($\Lambda_L < \mu < \Lambda_H$) of our EFT and construct all operators that are allowed by the symmetries of the system. Assuming a natural scaling of our operators (of mass-dimension n) $\langle \mathcal{O}_n \rangle \sim \Lambda_L^n$ and their coefficients $C_n \sim 1$ in the Lagrangian we can write down a Lagrangian summing over the mass dimension of the operators

$$\mathcal{L} = \sum_{n=0}^{\infty} C_n(\mu, \Lambda_H) \frac{\mathcal{O}_n(\mu, \Lambda_L)}{\Lambda_H^{n-4}} \tag{1}$$

where the C_n, the so-called Wilson coefficients, model the correct non-analytic IR behaviour of the theory for the operators $\mathcal{O}_n$ made of the low-energy degrees of freedom. The Wilson coefficients

can be related to a more fundamental theory by matching or obtained by fitting to data.
We effectively expanded our Lagrangian in the parameter

$$\lambda = \frac{\Lambda_L}{\Lambda_H} \tag{2}$$

which is called the *power counting* of the EFT. In calculations we can choose our desired accuracy in λ and the power counting tells us which operators from the Lagrangian we have to keep in order to obtain this accuracy.

1.2 Matching

The process of matching is often also referred to as integrating out the heavy particles of a system. During the matching process we relate the Wilson coefficients of our EFT to a more fundamental theory or fit them to data. After doing this the theory will correctly model the behaviour of our full theory at low energies without including the heavy degrees of freedom. During the matching procedure we usually pick out one process for which we calculate the renormalized on-shell amplitude $i\mathcal{M}_{\text{full/EFT}}$ in the EFT and the full theory and equate them at a matching scale Λ_M

$$i\mathcal{M}_{\text{full}}(\Lambda_M) = i\mathcal{M}_{\text{EFT}}(\Lambda_M) \tag{3}$$

For a given accuracy in the power counting of our EFT, we have then obtained a theory with a finite number of parameters which assures the predictive power of the EFT.
If we are only interested in the tree-level accuracy of our theory, we can also integrate out the heavy particles by using the classical equations of motion of our theory. E.g., for a theory with a light particle ϕ and a heavy particle Φ, we can calculate the equations of motion for our heavy field $\frac{\delta S}{\delta \Phi}(\phi) = 0$ [1] and formally solve them $\Phi = f(\phi)$. Plugging this back into the Lagrangian of the system gives us the tree-level matched Lagrangian of the EFT

$$\mathcal{L}_{\text{EFT}}^{\text{tree}} = \mathcal{L}(\phi, \Phi = f(\phi)) \tag{4}$$

1.3 The Renormalization Group Equation

The Lagrangian in eqn. (1) seems to be non-renormalizable because it contains operators with negative mass dimension and we therefore have to keep adding operators of higher mass dimension to cancel all UV divergences in the theory. That is where the power counting of the theory comes into play. For a given accuracy in terms of the power counting λ^n, we have a finite number of terms in our Lagrangian. Now, if we e.g. want to calculate up to dimension 6 in the mass dimension and insert two of these dimension 6 operators in an amplitude, we would have to add a dimension 8 operator to cancel this divergence. However, this is already below our accuracy, so we can neglect it. So, an EFT is renormalizable order by order in its power counting expansion.
We can then derive the RGE of our theory. For a set of operators mixing under renormalization we have $\mathcal{O}_i^{(0)} = Z_{ij}\mathcal{O}_j$. Unlike the bare operators, the renormalized operators depend on the subtraction scale via

$$\mu \frac{d}{d\mu}\mathcal{O}_i = \left(\mu \frac{d}{d\mu} Z_{ij}^{-1}\right) \mathcal{O}_j^{(0)} \equiv -\gamma_{ij}\mathcal{O}_j \tag{5}$$

[1]where $S = \int d^4x \mathcal{L}(\phi, \Phi)$ is the classical action of our system

where we have defined the anomalous dimensions of the operators

$$\gamma_{ik} = Z_{ij}^{-1} \mu \frac{dZ_{jk}}{d\mu} \tag{6}$$

With this we can also derive an anomalous dimension for the couplings of our operators. Our EFT Lagrangian must be independent of the subtraction point μ

$$0 = \mu \frac{d}{d\mu} (C_i O_i) = \left(\mu \frac{d}{d\mu} C_i \right) O_i + C_i \left(\mu \frac{d}{d\mu} O_i \right) = \left(\mu \frac{d}{d\mu} C_i \right) O_i - C_i \gamma_{ij} O_j \tag{7}$$

where we have used the definition of the anomalous dimension in the last step. We then get

$$\mu \frac{d}{d\mu} C_i = \gamma_{ji} C_j \tag{8}$$

Solving this differential equation we can calculate the scale dependence of the Wilson coefficients. Alongside the usual matching procedure, this can be used to resum terms like $\frac{g^2}{16\pi^2} \log \frac{\Lambda_H^2}{\Lambda_L^2}$ which can appear in the fundamental theory.

1.4 On-Shell EFTs

There are certain classes of theories where a large separation of scales is given but the heavy d.o.f.s cannot be completely integrated out because they still appear in the external states of the theory. This can happen in strongly coupled theories where the heavy degrees of freedom appear together with light degrees of freedom in bound states, like a meson with a light and heavy quark. Even though we cannot completely integrate out the particle associated with the large scale from the theory, we can still integrate out the heavy scale and have the heavy degree of freedom appear in external states. We can do this for very heavy but also massless energetic particles for which we can decompose the momentum in

$$p^\mu = m v^\mu + k^\mu \qquad \text{or} \qquad p^\mu = E n^\mu + k^\mu \tag{9}$$

where k is a soft fluctuation around the large scale m, E. These kind of theories are called on-shell EFTs and we will explore them further in the following, starting with the aforementioned example and the corresponding EFT, the Heavy Quark Effective Theory (HQET).

2 Motivation of HQET

The quarks of the Standard Model fall into two categories: light quarks q (up, down, strange) for which $m_q \ll \Lambda_{QCD}$ and heavy quarks Q (charm, bottom, top)[2] with $m_Q \gg \Lambda_{QCD}$. The scale $\Lambda_{QCD} \sim 200 MeV$ separates the regions of large and small effective QCD coupling constant α_s, thus for heavy quarks the strong interaction can be treated perturbatively.
HQET is an EFT of QCD which describes interactions in hadrons containing a heavy quark Q and other light constituents. The typical momenta exchanged between Q and these constituents via soft

[2] Of particular interest to us will be the charm and bottom quarks, since the top quark decays to quickly to form bound systems with light quarks.

gluons are of order Λ_{QCD}. Since $\Delta v = \frac{\Delta p}{m_Q}$, the velocity of the heavy quark v is almost unchanged by these strong interactions and becomes a conserved quantity in the heavy quark limit $m_Q \to \infty$. Furthermore the soft degrees of freedom can only probe scales of $l \sim \Lambda_{QCD}$ and therefore cannot resolve the quantum numbers of the heavy quark.
HQET is constructed in such a way that it reproduces QCD predictions at energies below some separation scale μ chosen such that $\Lambda_{QCD} \ll \mu \ll m_Q$. This is illustrated in Figure 1. In contrary to typical EFTs, in HQET the heavy quark can not be completely removed from the theory as it still appears as an external state. However, its spinor contains components that can be integrated out, as we will see in the following.

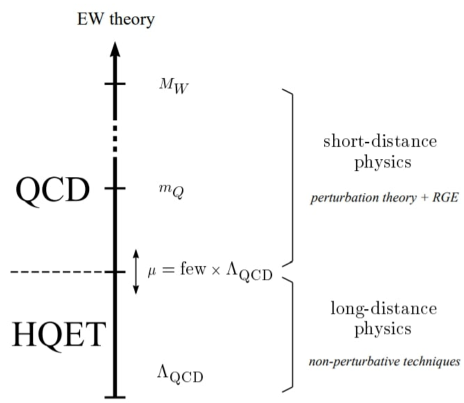

Figure 1: Energy scales for the full theory (QCD) and the effective theory (HQET)

3 The HQET Lagrangian and Its Features

3.1 HQET Lagrangian

One important observation in the construction of the HQET Lagrangian is that the heavy quark bound inside the hadron moves more or less with the hadron's velocity v and it is almost on shell. Thus the momentum of the off-shell heavy quark can be written as

$$p_Q^\mu = m_Q v^\mu + k^\mu, \tag{10}$$

where $v^2 = 1$ and $k \sim \Lambda_{QCD}$ is the residual momentum, much smaller than m_Q. With this decomposition the fermion propagator of a heavy quark becomes

$$\frac{i}{\not{p} - m_Q} = \frac{i(\not{p} + m_Q)}{p^2 - m_Q^2} = i\frac{m_Q\not{v} + \not{k} + m_Q}{m_Q^2 + 2m_Q v \cdot k + k^2 - m_Q^2} = \frac{1+\not{v}}{2}\frac{i}{v \cdot k} + \mathcal{O}(1/m_Q) \tag{11}$$

and the heavy-quark-gluon vertex is modified to

$$\frac{i}{\not p - m_Q}\left(-igT^a\gamma^\mu\right)\frac{i}{\not p' - m_Q} = \frac{i}{v\cdot k}\frac{1+\not v}{2}\left(-igT^a\gamma^\mu\right)\frac{1+\not v}{2}\frac{i}{v\cdot k'} + \mathcal{O}(1/m_Q) =$$
$$= \frac{1+\not v}{2}\frac{i}{v\cdot k}\left(-igT^a v^\mu\right)\frac{1+\not v}{2}\frac{i}{v\cdot k'} + \mathcal{O}(1/m_Q) \tag{12}$$

where we used $\frac{1+\not v}{2}\gamma^\mu\frac{1+\not v}{2} = \frac{1+\not v}{2}v^\mu\frac{1+\not v}{2}$ which can be easily proven with the Clifford algebra and the fact that $\frac{1+\not v}{2}$ is a projector. This motivates the following decomposition of the heavy quark field $Q(x)$ into a large and small component field

$$h_v(x) = e^{im_Q v\cdot x}P_+Q(x), \tag{13}$$

$$H_v(x) = e^{im_Q v\cdot x}P_-Q(x). \tag{14}$$

which satisfy $\not v h_v = h_v$ and $\not v H_v = -H_v$. The projection operators $P_\pm$ are defined as $P_\pm = \frac{1\pm\not v}{2}$. The exponential factor in the definition of h_v and H_v subtracts an amount $m_Q v$ from the heavy quark momentum such that they only carry the residual momentum k. In the rest frame ($v^\mu = (1,\vec{0})$), h_v corresponds to the upper two components of Q, while H_v corresponds to the lower components. The field h_v annihilates a heavy quark whereas H_v creates a heavy antiquark. The heavy quark field can then be written as

$$Q(x) = e^{-im_Q v\cdot x}\left[h_v(x) + H_v(x)\right]. \tag{15}$$

With this we can rewrite the QCD Lagrangian for a heavy quark

$$\mathcal{L}_\mathcal{Q} = \bar{Q}(i\not D - m_Q)Q. \tag{16}$$

In terms of the component fields, equation (16) yields

$$\mathcal{L}_\mathcal{Q} = \bar{h}_v iv\cdot D\, h_v - \bar{H}_v(iv\cdot D + 2m_Q)H_v + \bar{h}_v i\not D_\perp H_v + \bar{H}_v i\not D_\perp h_v, \tag{17}$$

where $D_\perp^\mu = D^\mu - v^\mu\, v\cdot D$ is orthogonal to the heavy quark velocity: $v\cdot D_\perp = 0$. We also used $\not v h_v = h_v$ and $\not v H_v = -H_v$ which immediately follows from the definition of h_v and H_v. Looking at the Lagrangian we can see that h_v describes massless degrees of freedom, whereas H_v describes massive ones. The third and fourth terms correspond to pair creation and annihilation of heavy quarks and antiquarks.

At tree level, the massive H_v can be integrated out via the equations of motion (EOM)[3].The EOM for H_v is straightforward

$$(iv\cdot D + 2m_Q)H_v = i\not D_\perp\, h_v, \tag{18}$$

and the formal solution to this reads

$$H_v = \frac{1}{iv\cdot D + 2m_Q}i\not D_\perp\, h_v. \tag{19}$$

Inserting equation (19) back into equation (17) results in the following non local effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \bar{h}_v iv\cdot D\, h_v + \bar{h}_v i\not D_\perp\frac{1}{iv\cdot D + 2m_Q}i\not D_\perp\, h_v. \tag{20}$$

[3]Another way to see why we should integrate out the H_v field is that it corresponds to the antiquark field. For the description of a hadron containing only one heavy quark at energies below its mass, we don't need the anti-quark as it only appears in pair creation/annihilation processes which can't be resolved at the low scale.

In momentum space, the derivative acting on h_v produces powers of the residual momentum k, which is much smaller than m_Q. Moreover, the gluon field A^μ is in HQET implicitly reinterpreted as a soft gluon field A^μ_{soft} which can only carry momenta of order Λ_{QCD}. Consequently, the second term can be expanded to obtain

$$\mathcal{L}_{\text{eff}} = \bar{h}_v iv \cdot D\, h_v + \frac{1}{2m_Q} \sum_{n=0}^{\infty} \bar{h}_v i\not{D}_\perp \left(-\frac{iv \cdot D}{2m_Q} \right)^n i\not{D}_\perp\, h_v. \tag{21}$$

The first term in the sum can be rewritten using

$$i\not{D}_\perp\, i\not{D}_\perp = \left[(iD_\perp)^2 + \frac{g_s}{2} \sigma_{\mu\nu} G^{\mu\nu} \right], \tag{22}$$

such that the Lagrangian becomes

$$\mathcal{L}_{\text{eff}} = \bar{h}_v iv \cdot D\, h_v + \frac{1}{2m_Q} \bar{h}_v (iD_\perp)^2 h_v + \frac{g_s}{4m_Q} \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v + O\left(\frac{1}{m_Q^2} \right). \tag{23}$$

The operators at order $1/m_Q$ are called the gauge-covariant extension of the kinetic energy arising from the residual motion of the heavy quark and the chromo-magnetic interaction, which describes the coupling of the heavy quark spin to the gluon field. In the heavy quark limit $m_Q \to \infty$ we arrive at

$$\mathcal{L}_0 = \mathcal{L}_{\text{eff}} \Big|_{m_Q \to \infty} = \bar{h}_v iv \cdot D\, h_v. \tag{24}$$

We can also eliminate H_v from the quark field $Q(x) = e^{-im_Q v \cdot x} \left[h_v(x) + H_v(x) \right]$ by using the equation of motion

$$Q(x) = e^{-im_Q v \cdot x} \left[h_v(x) + \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv \cdot D}{2m_Q} \right)^n i\not{D}_\perp\, h_v \right] \tag{25}$$

which allows us to write down all currents of the form $\bar{q}\Gamma Q$ (where Γ denotes some Lorentz structure) which we e.g. need to describe the decays of mesons in HQET.

One thing that is also worth being noted is the different renormalization of states in HQET. The usual normalization of a hadron state $|H(p)\rangle$ in QCD is

$$\langle H(p') | H(p) \rangle = 2E_{\boldsymbol{p}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'}) \tag{26}$$

In HQET our states are labeled by a velocity v and the residual momentum k so its convenient to chose the normalization

$$\langle H(v', k') | H(v, k) \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k'}) \tag{27}$$

which means for the states for $k = 0$: $|H(p)\rangle = \sqrt{m_H} \left(|H(v)\rangle + \mathcal{O}(1/m_Q) \right)$.

3.2 Heavy Quark Spin-Flavour Symmetry

Similar to the $SU(3)_L \times SU(3)_R$ chiral symmetry in QCD, which is an approximate symmetry taken in the $m_q \to 0$ limit of the light quarks (u, d, s) the above Lagrangian also has a symmetry

but here in the $m_Q \to \infty$ limit for the heavy quarks (c, b, t). This symmetry is called spin-flavour symmetry which unlike chiral symmetry is not a symmetry of QCD but only of the effective theory we constructed.
We can now generalise equation (24) to N_h flavours which simply gives

$$\mathcal{L}_0^{N_h} = \sum_{f=1}^{N_h} \bar{h}_v^f iv \cdot D\, h_v^f. \tag{28}$$

This Lagrangian obviously has a $U(N_h)$ flavour symmetry. Combined with the SU(2) spin symmetry this Lagrangian also has at leading order for each flavour, we arrive at the $U(2N_h)$ heavy quark spin-flavour symmetry. This symmetry is only exact in the heavy quark limit and is broken by adding $1/m_Q$ corrections to the leading order Lagrangian. The flavour symmetry is broken by terms like $1/m_{Q_i} - 1/m_{Q_j}$, where indices i, j label two different flavours. The spin symmetry is explicitly broken by the dipole operator which couples the quark spin to the gluon field.

3.3 Reparametrization Invariance

The HQET Lagrangian admits at order $1/m_Q$ an additional symmetry resulting from the fact that the decomposition of the heavy-quark momentum,

$$p_Q^\mu = m_Q v^\mu + k^\mu, \tag{29}$$

is not unique. A small change in the heavy quark velocity of order Λ_{QCD}/m_Q can be absorbed by the opposite change in the residual momentum k

$$v \to v' = v + \frac{\epsilon}{m_Q},\ k \to k' = k - \epsilon \tag{30}$$

where ϵ is 4-vector of order Λ_{QCD} that satisfies $v \cdot \epsilon = 0$ such that $v'^2 = 1$ up to terms of order $(\Lambda_{QCD}/m_Q)^2$. Demanding that $\not{v} h_v = h_v$ still holds at order (Λ_{QCD}/m_Q) after the reparametrization, also the heavy quark field h_v has to transform such that

$$\not{v}' h'_{v'} = h'_{v'} \tag{31}$$

holds for $h'_{v'} = h_v + \delta h_v$ the transformed heavy-quark field at order (Λ_{QCD}/m_Q). Plugging in the expression for v', we find

$$h_v + \delta h_v = \left(\not{v} + \frac{\not{\epsilon}}{m_Q}\right)(h_v + \delta h_v) = h_v + \frac{\not{\epsilon}}{m_Q} h_v + \not{v}\delta h_v + \mathcal{O}(1/m_Q^2) \tag{32}$$

Then a suitable choice for δh_v satisfying $\not{v}\delta h_v = -\delta h_v$ is

$$\delta h_v = \frac{\not{\epsilon}}{2m_Q} h_v. \tag{33}$$

Also adding the change in the residual momentum, we find the reparametrization transformation

$$v \to v + \frac{\epsilon}{m_Q},\ h_v \to e^{i\epsilon\cdot x}\left(1 + \frac{\not{\epsilon}}{2m_Q}\right) h_v. \tag{34}$$

The leading order HQET Lagrangian (24) transforms as

$$\mathcal{L}_0 \to \mathcal{L}_0 + \frac{1}{m_Q}\bar{h}_v i\epsilon \cdot D\, h_v + O\left(\frac{1}{m_Q^2}\right), \tag{35}$$

while at the sub-leading order, the chromo-magnetic interaction is invariant up to order $1/m_Q^2$ and the gauge-covariant extension of the kinetic energy changes as

$$\mathcal{L}_1 \to \mathcal{L}_1 - \frac{1}{m_Q}\bar{h}_v i\epsilon \cdot D\, h_v + O\left(\frac{1}{m_Q^2}\right), \tag{36}$$

hence at order $1/m_Q$ the HQET Lagrangian is reparametrization invariant. This also has one further very important implication which will be useful when we renormalize the first power correction to the Lagrangian. Reparametrization invariance is only valid if the leading order Lagrangian and the covariant extension of the kinetic energy have the same coefficient. This is a non-trivial result because it connects terms in the Lagrangian with different power countings.

4 Renormalization of HQET

The HQET Lagrangian we derived so far is only valid at tree-level. If we also want to include radiative corrections we have to renormalize both the full and the effective theory and do the matching at loop order. The quantities appearing in the effective Lagrangian are bare quantities - denoted by superscript 0. The renormalized heavy quark field is

$$h_v = \frac{1}{\sqrt{Z_h}} h_v^{(0)}. \tag{37}$$

with ϵ defined by $d = 4 - \epsilon$ and μ is the usual dimensionful parameter of dimensional regularization. We can do the same for the light fields and the coupling in the theory

$$q = \frac{1}{\sqrt{Z_q}} q^{(0)} \qquad A_\mu = \frac{1}{\sqrt{Z_A}} A_\mu^{(0)} \qquad g = \frac{1}{Z_g}\mu^{-\epsilon/2} g^{(0)} \tag{38}$$

In the effective theory, all loops including only heavy quarks vanish. A heavy quark loop would look like

$$\frac{i}{v\cdot q + i\epsilon}\frac{i}{v\cdot(k-q)+i\epsilon} \tag{39}$$

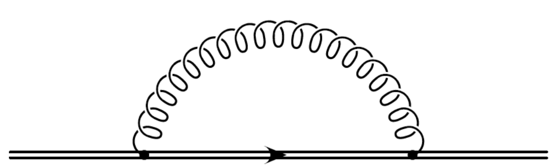

Figure 2: One loop correction to the heavy quark propagator

where q is the loop momentum. However, this only has two poles in k^0, both below the real k^0 axis. Therefore, the contour integral can be closed in the upper half plane and it is zero.
Physically this can be seen by remembering that the field h_v annihilates a heavy quark, while the field H_v which would create the corresponding heavy antiquark has been integrated out. As a consequence, the heavy quarks do not influence the renormalization of the light constituents, the gluon or of the coupling constant and they are the same as in full QCD below the mass scale of the heavy quark we integrate out.
We can rewrite the effective Lagrangian in terms of renormalized quantities

$$\mathcal{L}_{\text{eff}} = \bar{h}_v iv \cdot D\, h_v + (Z_h - 1)\bar{h}_v iv \cdot D\, h_v \tag{40}$$

Now we are ready to calculate the renormalization constant for the heavy quark field from the heavy quark self energy diagram in Figure 2. The diagram in Feynman gauge reads

$$\int \frac{d^dq}{(2\pi)^d} \left(-igT^A\mu^{\epsilon/2}v_\mu\right) \frac{i}{v\cdot(q+k)} \left(-igT^A\mu^{\epsilon/2}v_\nu\right) \frac{-ig^{\mu\nu}}{q^2} = ... = -\frac{ig^2}{3\pi^2\epsilon} v\cdot k + \text{finite} \tag{41}$$

with q the loop momentum and k the external residual momentum[4]. The loop integral is both UV and IR divergent. We skipped all of the computational steps where we regulate the IR divergence (done by the introduction of gluon mass that is set to zero at the and of the calculation) and do the usual Wick rotation and Feynman parameterization to solve the integral. Then we can add the counter term to extract the UV divergence and calculate the wave function renormalization constant. In the $\overline{\text{MS}}$ scheme we get

$$Z_h = 1 + \frac{g^2}{3\pi^2\epsilon}, \tag{42}$$

while in the on shell scheme (corresponding to setting $k = 0$) the above integral is scaleless and trivially vanishes in dimensional regularization such that

$$Z_h^{OS} = 1 \tag{43}$$

and this will hold to all orders in perturbation theory.

Detour: From QCD to HQET on the Integral Level
We can also evaluate the integral we just calculated starting from the full integral in QCD. Ignoring constants we get

$$\int \frac{d^dq}{(2\pi)^d}\gamma^\mu \frac{\not{q}+\not{p}+m_Q}{(q+p)^2-m_Q^2}\gamma_\mu\frac{1}{q^2} = \int \frac{d^dq}{(2\pi)^d}\gamma^\mu \frac{\not{q}+m_Q(1+\not{v})}{q^2+2m_Qv\cdot(q+k)+2q\cdot k}\gamma_\mu\frac{1}{q^2} \tag{44}$$

where we used $p = m_Qv + k$ and $k \ll m_Q$. We can expand this integral in two regions, for $q^2 \ll m_Q^2$ and $q^2 \gg m_Q^2$. Let us start with the first case, the soft gluon region. The integrand then reads

$$\gamma^\mu \frac{m_Q(1+\not{v})}{2m_Qv\cdot(q+k)}\gamma_\mu\frac{1}{q^2} = (P_+ + P_-)\gamma^\mu P_+\gamma_\mu(P_+ + P_-)\frac{1}{v\cdot(q+k)q^2} = P_+\frac{1}{v\cdot(q+k)q^2} \tag{45}$$

[4]Reminder: we subtracted in the definition of the heavy quark field h_v an amount m_Qv from its momentum.

where we used $P_\pm \gamma^\mu P_\pm = \pm v^\mu P_\pm$ and the other combinations vanish. We can ignore the P_+ in the last line because trivially, $P_+ h_v = h_v$. This is the integrand we get from the HQET Feynman rules. Now, let us have a look at the other limit, which is the hard gluon region. Here, the integrand is

$$\gamma^\mu \frac{\not{q} + m_Q(1+\not{v})}{q^2\left(1 + \left[2m_Q v\cdot(q+k) + 2q\cdot k\right]/q^2\right)} \gamma_\mu \frac{1}{q^2} = \tag{46}$$

$$= \gamma^\mu \frac{1}{q^4}\left[\not{q} + m_Q(1+\not{v})\right]\left[1 - \frac{2m_Q v\cdot(q+k) + 2q\cdot k}{q^2}\right]\gamma_\mu + O\left(\frac{1}{q^6}\right) = \tag{47}$$

$$= \frac{1}{q^4}\gamma^\mu \left[m_Q(1+\not{v}) - 2\frac{(m_Q v + k)\cdot q}{q^2}\not{q}\right]\gamma_\mu + O\left(\frac{1}{q^6}\right) = \tag{48}$$

$$= \frac{2}{q^4}\left[m_Q(2-\not{v}) + 2\frac{(m_Q v + k)\cdot q}{q^2}\not{q}\right] + O\left(\frac{1}{q^6}\right) \tag{49}$$

where we threw away odd terms in q in the second to last step and used the Clifford algebra in the last step. The second term gives a $1/\epsilon$ pole and contributes to the heavy quark field renormalization in full QCD. This shows that the heavy quark field is differently renormalized in HQET and full QCD.

5 1-Loop Matching of HQET

The effective HQET Lagrangian

$$\mathcal{L}_{\text{eff}} = \bar{h}_v iv\cdot D\, h_v + \frac{1}{2m_Q}\bar{h}_v(iD_\perp)^2 h_v + \frac{g_s}{4m_Q}\bar{h}_v\sigma_{\mu\nu}G^{\mu\nu}h_v + O\left(\frac{1}{m_Q^2}\right). \tag{50}$$

was derived in Section 3 via the use of the EOM for the small component field H_v. It gives the Wilson coefficients of the HQET operators only at tree level. To determine the corrections coming from quantum loops, a matching calculation between full QCD and HQET is required.
In general, a proper matching calculation in the construction of an EFT requires that the EFT Lagrangian contains all operators compatible with the demanded symmetries[5]. Let's check that this is the case for (50) before we start calculating the loop corrections.

5.1 Completeness of the (Sub-)Leading HQET Operator Basis

Our degrees of freedom in HQET are described by the heavy quark field h_v and the soft gluon field A_μ which are our building blocks to construct the EFT. The gluon only couples to the quark fields via the covariant derivative and we can also write down the gluon field strength in term of the covariant derivative $G^{\mu\nu} = (ig_s)^{-1}[D^\mu, D^\nu]$. So we only have to include the covariant derivative to include the gluon in our theory. Furthermore the dual field strength tensor is forbidden by invariance of our Lagrangian under parity transformations.
Because of Lorentz invariance the heavy quark field always has to appear with its conjugate which together has mass dimension $[\bar{h}_v h_v] = 3$. The two vectors we can contract the covariant derivative

[5]For the case of HQET, we require Lorentz invariance, gauge invariance, parity invariance, reparametrization invariance and at leading order spin-flavour symmetry.

with are γ^μ and v^μ. Both contractions turn out to be equivalent

$$\bar{h}_v i\not{D} h_v = \frac{1}{2}\bar{h}_v i(\not{v}\not{D} + \not{D}\not{v})h_v = \bar{h}_v(iv \cdot D)h_v. \tag{51}$$

The only other renormalizable term we can write down is a residual mass term

$$\mathcal{L}_{\delta m} = -\delta m\, \bar{h}_v h_v \tag{52}$$

In our definition of the momentum decomposition we chose the heavy quark mass to be the physical mass and therefor $\delta m = 0$. If we chose for example the $\overline{\text{MS}}$ mass, the residual momentum k would have to correct for the additional piece of order $\alpha_s m_Q$ which would spoil our power counting of $k \sim \frac{\Lambda_{QCD}}{m_Q}$. As we have seen above, in the on-shell scheme the integral contributing to a mass correction is scaleless, so δm can also not be reintroduced by quantum corrections.
This covers all possible terms for renormalizable terms in our Lagrangian. Let us now go to the subleading order in our power expansion. Here we can use the EOM to eliminate redundant terms from our Lagrangian. The leading order EOM obtained by the variation of (50) with respect to $\bar{h}_v$ is

$$(v \cdot D)\, h_v = 0. \tag{53}$$

It restricts the set of possible operators at sub-leading order further as operators that vanish by the EOM do not contribute to on-shell Green's functions. Consequently it justifies the replacement $D \to D_\perp^\mu = D^\mu - v^\mu\, v \cdot D$ in operators at sub-leading order.
At sub-leading order, the only contributing operator structure of mass dimension 5 is $\bar{h}_v \not{D}_\perp \not{D}_\perp h_v$ as exploiting once again $\not{v} h_v = h_v$ we can show that e.g. the following operator doesn't contribute

$$\bar{h}_v v \cdot D \not{D} h_v = \frac{1}{2}\bar{h}_v v \cdot D(\not{v}\not{D} + \not{D}\not{v})h_v = \bar{h}_v(v \cdot D)^2 h_v. \tag{54}$$

which is zero by the equations of motion. Using the identity we also used earlier this gives the same Lagrangian we already found by integrating out H_v at tree level using the equations of motion. Now we can add general coefficients in front of these operators and see how quantum corrections change the tree-level Wilson coefficients.

5.2 Radiative Corrections to the Subleading Wilson Coefficients

We split the Lagrangian (50) we got from the tree level matching into three parts

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mag}}, \tag{55}$$

where $\mathcal{L}_0$ is the leading order term as in equation (24) and

$$\mathcal{L}_{\text{kin}} = \frac{1}{2m_Q}\bar{h}_v(iD_\perp)^2 h_v \equiv \frac{1}{m_Q}O_{\text{kin}}, \tag{56}$$

$$\mathcal{L}_{\text{mag}} = \frac{g_s}{4m_Q}\bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v \equiv \frac{1}{m_Q}O_{\text{mag}}. \tag{57}$$

The soft degrees of freedom will change the EFT at loop-level which we have to account for by introducing general coefficients in front of the operators

$$\mathcal{L}_{\text{kin}} = \frac{C_{\text{kin}}(\mu)}{m_Q}O_{\text{kin}}, \qquad \mathcal{L}_{\text{mag}} = \frac{C_{\text{mag}}(\mu)}{m_Q}O_{\text{mag}}. \tag{58}$$

Now, the power of reparametrization invariance comes into play. For the Lagrangian with general Wilson coefficients the transformations up to $\mathcal{O}(1/m_Q^2)$ are

$$\mathcal{L}_0 \to \mathcal{L}_0 + \frac{1}{m_Q}\bar{h}_v(i\epsilon\cdot D)h_v \tag{59}$$

$$\mathcal{L}_{\text{kin}} \to \mathcal{L}_{\text{kin}} - \frac{C_{\text{kin}}}{m_Q}\bar{h}_v(i\epsilon\cdot D)h_v, \qquad \mathcal{L}_{\text{mag}} \to \mathcal{L}_{\text{mag}}. \tag{60}$$

If we want this to be invariant we need

$$C_{\text{kin}} = 1 \tag{61}$$

This is true to all orders in perturbation theory and shows the power of reparametrization invariance which relates the Wilson coefficients of operators with different power counting. Had we constructed the Lagrangian up to order $1/m_Q^2$ and required our Lagrangian to also be invariant under reparametrization invariance to this order, we would have found non-trivial relations between Wilson coefficients up to order $1/m_Q^2$.

The only the Wilson coefficient getting non-trivial loop corrections is the one of the chromo-magnetic interaction. For this, we will now do the matching of this Wilson coefficient to QCD. The matrix element under consideration is the Green's function of two heavy quarks in a background gluon field A. We write the amplitude as

$$i\mathcal{M} = i\mu^\epsilon g_s \epsilon_\mu^{*a}\bar{u}_h\Gamma^\mu T^a u_h \tag{62}$$

where the two heavy quarks are taken on-shell and we continue working at precision $1/m_Q$. Before we can start with the calculation we first have to figure out which part of the full QCD quark spinor we have to keep in HQET. This is easily done by looking at the expansion of the quark field $Q(x) = e^{-im_Q v\cdot x}\left(1 + \frac{i\not{D}}{2m_Q} + \mathcal{O}(1/m_Q^2)\right)h_v$ giving

$$u_Q(p_Q,s) = \left(1 + \frac{\not{k}}{2m_Q}\right)u_h(v,s) + O\left(1/m_Q^2\right) \tag{63}$$

In full QCD, the contributing diagrams are given in Figure 3. The first diagram gives the tree level

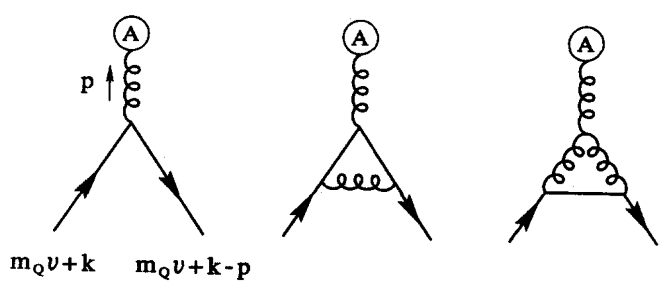

Figure 3: Diagrams needed for the computation of the heavy quark-gluon vertex in QCD

contribution

$$\Gamma^\mu_{\mathrm{QCD},0} = \left(1 + \frac{\not{k} - \not{p}}{2m_Q}\right)\gamma^\mu\left(1 + \frac{\not{k}}{2m_Q}\right) + O\left(\frac{1}{m_Q^2}\right) = \\ = v^\mu + \frac{(2k-p)^\mu}{2m_Q} + \frac{[\gamma^\mu, \not{p}]}{4m_Q} + O\left(\frac{1}{m_Q^2}\right) \tag{64}$$

where we used that γ^μ can be replaced by v^μ in between HQET spinors[6]. Using dimensional regularization for IR and UV divergencies[7], the one loop contribution from the second and third diagram is in the $\overline{\mathrm{MS}}$ scheme

$$\Gamma^\mu_{\mathrm{QCD},1} = ... = -\frac{[\gamma^\mu, \not{p}]}{4m_Q}\frac{3\alpha_s}{2\pi}\left(\log\frac{m_Q}{\mu} - \frac{13}{9}\right) + O\left(\frac{1}{m_Q^2}\right). \tag{65}$$

In HQET, we only have to calculate at tree level as all heavy quark loops vanish in dimensional regularization. The three operators in the effective Lagrangian up to mass dimension 5 give

$$\Gamma^\mu_{\mathrm{HQET}} = v^\mu + \frac{(2k-p)^\mu}{2m_Q} + C_{\mathrm{mag}}(\mu)\frac{[\gamma^\mu, \not{p}]}{4m_Q} + O\left(\frac{1}{m_Q^2}\right). \tag{66}$$

The second term comes from $\mathcal{O}_{\mathrm{kin}}$, showing parameterization invariance at work. By equating the two on-shell amplitudes we get

$$\Gamma^\mu_{\mathrm{HQET}} = \Gamma^\mu_{\mathrm{QCD},0} + \Gamma^\mu_{\mathrm{QCD},1} \tag{67}$$

By comparison we obtain the one loop result for the chromomagnetic interaction

$$C_{\mathrm{mag}}(\mu) = 1 - \frac{3\alpha_s}{2\pi}\left(\log\frac{m_Q}{\mu} - \frac{13}{9}\right) + O\left(\frac{1}{m_Q^2}\right). \tag{68}$$

It contains the logarithm $\log\frac{m_Q}{\mu}$ which becomes large for $\mu \ll m_Q$ which is exactly the energy regime one is interested in in HQET. Here, we can use the powerful tool of the RGE to exponentiate this large logarithm. The chromo-magnetic operator doesn't mix under renormalization as it is the only operator that gets renormalized at order $1/m_Q$. Thus equation (8) becomes

$$\mu\frac{d}{d\mu}C_{\mathrm{mag}}(\mu) = \gamma_{\mathrm{mag}}C_{\mathrm{mag}}(\mu). \tag{69}$$

Expanding the anomalous dimension γ_{mag} in the renormalized coupling constant

$$\gamma_{\mathrm{mag}} = \gamma_0\frac{\alpha_s}{4\pi} + O\left(\alpha_s^2\right) \tag{70}$$

and plugging in our result (68), we obtain $\gamma_0 = 6$. With this result we can solve (69) by using the chain rule

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g_s)\frac{\partial}{\partial g_s}, \tag{71}$$

[6]Remember $\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$ from section 3.

[7]The three-gluon vertex cannot be regulated by a gluon mass in a gauge-invariant way.

where $\beta(g_s) = \mu \frac{dg_s}{d\mu} = -\beta_0 \frac{g_s^3}{16\pi^2} + \mathcal{O}(g_s^5)$ is the QCD beta function of the running strong coupling g_s. This yields the formal solution

$$C_{\text{mag}}(\mu) = U(m_Q, \mu) C_{\text{mag}}(m_Q) \tag{72}$$

where

$$U(m_Q, \mu) = exp\left(\int\limits_{g_s(m_Q)}^{g_s(\mu)} dg_s' \frac{\gamma_{\text{mag}}(g_s')}{\beta(g_s')} \right) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{\frac{\gamma_0}{2\beta_0}} + \mathcal{O}(\alpha_s) \tag{73}$$

and the second equality is a result of a perturbative solution of the integral. Equation (68) yields for $\mu = m_Q$ the result

$$C_{\text{mag}}(m_Q) = 1 + \frac{13\alpha_s}{6\pi}. \tag{74}$$

Hence we find for the Wilson coefficient of the chromo-magnetic interaction the one-loop result

$$C_{\text{mag}}(\mu) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{\frac{3}{\beta_0}} \left(1 + \frac{13\alpha_s}{6\pi} \right) \tag{75}$$

and the large logarithm has been resummed.

6 Applications of HQET

6.1 Hadron Mass Splitting

One of the more straightforward applications of HQET are its implications on the mass splitting of hadrons which lie in the same spin multiplet, e.g. the B and B^* meson. We start with the $1/m_Q^0$ order Lagrangian

$$\mathcal{L}_0 = \bar{h}_v iv \cdot D h_v + \bar{q} i \not{D} q - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} = -\mathcal{H}_0 \tag{76}$$

for the heavy quarks and the massless QCD Lagrangian for the light quark flavours and the gluons. Then, at leading order in the heavy quark mass expansion we have [8]

$$m_H - m_Q = \bar{\Lambda} + \mathcal{O}\left(\frac{1}{m_Q} \right) \tag{77}$$

where the heavy quark spin-flavour symmetry assures that all hadrons in the same spin multiplet have the same mass as their heaviest quark. $\bar{\Lambda}$ is the non-perturbative correction coming from $\mathcal{H}_0$ given by

$$\bar{\Lambda} \equiv \frac{\langle H^{(Q)} | \mathcal{H}_0 | H^{(Q)} \rangle}{\langle H^{(Q)} | H^{(Q)} \rangle} \tag{78}$$

where $|H^{(Q)}\rangle$ is the hadron state in the effective theory at rest and the 1/2 accounts for the normalization of the hadron states. Since $\bar{\Lambda}$ describes the influence of the soft degrees of freedom, it will be the same for all particles in the same spin multiplet. We also assumed an SU(3) flavour

[8] The hadron mass in the effective theory is $m_H - m_Q$ because we subtracted m_Q from all energies in the field definition 15.

symmetry for the light quarks, such that $\bar{\Lambda}_{u,d,s} = \bar{\Lambda}$.
Next, we want to calculate corrections to the hadron masses coming from the order $1/m_Q$ Lagrangian

$$\mathcal{L}_1 = -\bar{h}_v \frac{D_\perp^2}{2m_Q} h_v + gC_{\rm mag}(\mu)\bar{h}_v \frac{\sigma_{\mu\nu}G^{\mu\nu}}{4m_Q} h_v = -\mathcal{H}_1 \tag{79}$$

From this we can again define two non-perturbative matrix elements

$$2\lambda_1 = -\frac{\langle H^{(Q)}|\bar{h}_v D_\perp^2 h_v|H^{(Q)}\rangle}{\langle H^{(Q)}|H^{(Q)}\rangle} \tag{80}$$

$$4\left(\boldsymbol{S_Q}\cdot\boldsymbol{S_l}\right)\lambda_2(m_Q) = -C_{\rm mag}(\mu)\frac{\langle H^{(Q)}|\bar{h}_v g\sigma_{\mu\nu}G^{\mu\nu} h_v|H^{(Q)}\rangle}{\langle H^{(Q)}|H^{(Q)}\rangle} \tag{81}$$

where λ_1 does not depend on m_Q, while λ_2 depends on m_Q through the scale dependence of $C_{\rm mag}(\mu)$. Similarly to the fine structure splitting in atoms, where the magnetic fields couples the angular momentum of the electron to its spin via its dipole interaction, here the chromomagnetic field of the heavy quark couples the spin of the light system in the hadron to the heavy quark via the chromomagnetic dipole interaction. This gives rise to the spin structure $\boldsymbol{S_Q}\cdot\boldsymbol{S_l}$ in the definition of λ_2. Using $\boldsymbol{S_Q}\cdot\boldsymbol{S_l} = \left(\boldsymbol{J}^2 - \boldsymbol{S_Q}^2 - \boldsymbol{S_l}^2\right)/2$, we find [9]

$$m_{B,D} = m_{b,c} + \bar{\Lambda} - \frac{\lambda_1}{m_{b,c}} - \frac{3\lambda_2(m_{b,c})}{m_{b,c}} \tag{82}$$

$$m_{B^*,D^*} = m_{b,c} + \bar{\Lambda} - \frac{\lambda_1}{m_{b,c}} + \frac{\lambda_2(m_{b,c})}{m_{b,c}} \tag{83}$$

$$m_{\Lambda_b} = m_b + \bar{\Lambda}_\Lambda - \frac{\lambda_{\Lambda,1}}{m_b} \tag{84}$$

$$m_{\Sigma_b} = m_b + \bar{\Lambda}_\Sigma - \frac{\lambda_{\Sigma,1}}{m_b} - \frac{4\lambda_{\Sigma,2}(m_b)}{m_b} \tag{85}$$

With the expressions from above we find that the dipole operator is responsible for the mass splitting in the $B-B^*$ system. With the observed value of the mass splitting we find $\lambda_2(m_b) = 0.12$ GeV^2. We also find

$$0.49\ {\rm GeV}^2 \simeq m_{B^*}^2 - m_B^2 \simeq 8\lambda_2 \simeq m_{D^*}^2 - m_D^2 \simeq 0.55\ {\rm GeV}^2 \tag{86}$$

which is approximately the same at order $1/m_{b,c}$ ignoring the weak dependence of λ_2 on the heavy quark mass. It seems like we haven't gained a lot by deriving these relations for the masses, however the matrix elements which appear in the masses of the hadrons also appear in other observables. So, after we fix the values of $\lambda_{1,2}$ with the mass differences as we did above, we can use them to make predictions for other observables in the theory.
We can go even one step further by noticing that the quadratic difference in the meson masses only depends on the matrix element of the dipole interaction. Since we calculated the one-loop correction to this in the last section we can also calculate the ratio $R = \frac{m_{B^*}^2 - m_B^2}{m_{D^*}^2 - m_D^2}$ of the quadratic difference in the quark masses which is also known from experiment. With what we just calculated we get

$$R = \frac{m_{B^*}^2 - m_B^2}{m_{D^*}^2 - m_D^2} = \frac{C_{\rm mag}(m_b/\mu)}{C_{\rm mag}e(m_c/\mu)} + \mathcal{O}(1/m_Q) \tag{87}$$

[9]In the parton model we can identify: $|B\rangle = |d\bar{b}\rangle$, $|D\rangle = |c\bar{u}\rangle$, $|\Lambda_b\rangle = |udb\rangle$, $|\Sigma_b\rangle = |udb\rangle$. $|\Lambda_b\rangle$ has $I(J^P) = 0(1/2^+)$, while $|\Sigma_b\rangle$ $I(J^P) = 1(1/2^+)$.

for $N=3$ colours and $n_f=4$ light flavours

$$R=\left(\frac{\alpha_S(m_b)}{\alpha_S(m_c)}\right)^{\frac{9}{25}}\left(1-\frac{7921}{3750}\frac{\alpha_S(m_c)-\alpha_S(m_b)}{\pi}\right)+\mathcal{O}(1/m_Q) \tag{88}$$

where the second term in the bracket corresponds to the two-loop correction of the anomalous dimension of the magnetic dipole operator.
For $\alpha_S(m_b)=0.22$ and $\alpha_S(m_c)=0.36$, we get $R_{\text{tree}}=1$, $R_{1-\text{loop}}\simeq 0.84$ and $R_{2-\text{loop}}\simeq 0.76$ which is close to the experimental value $R_{exp}=0.89\pm 0.01$. It seems like our prediction gets worse and worse if we go to higher loop order. But at this precision we also have to include higher order in our power expansion to get a complete result. If we estimate the subleading correction as $\Lambda_{QCD}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)$ we find again agreement with the experimental result.

6.2 Semi-leptonic Decays (if time allows)

Probably the most important application of HQET are semi-leptonic decays of hadrons. We will only briefly sketch the calculations here, because they are very lengthy and focus on the decay $\bar{B}\to D^{(*)}e\bar{\nu}_e$. We can describe these decays with the electroweak Hamiltonian

$$\mathcal{H}_W=\frac{4G_F}{\sqrt{2}}V_{cb}\left(\bar{c}\gamma_\mu P_L b\right)\left(\bar{e}\gamma^\mu P_L\nu_e\right) \tag{89}$$

Neglecting higher order electroweak corrections the matrix elements factor into hadronic and leptonic matrix elements. We can write down the most general decomposition of the hadronic matrix element in terms of form factors in the HQET normalization

$$\begin{aligned}
\frac{\langle D(p')|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(p)\rangle}{\sqrt{m_B m_D}}&=h_+(w)\left(v_\mu+v'_\mu\right)+h_-(w)\left(v_\mu-v'_\mu\right)\\
\frac{\langle D^*(p',\epsilon)|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(p)\rangle}{\sqrt{m_B m_D}}&=h_V(w)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu}v'^\alpha v^\beta\\
\frac{\langle D^*(p',\epsilon)|\bar{c}_{v'}\gamma_\mu\gamma_5 b_v|\bar{B}(p)\rangle}{\sqrt{m_B m_D}}&=-ih_{A_1}(w)(1+w)\epsilon^*_\mu+ih_{A_2}(w)(1+w)(\epsilon^*\cdot v)v_\mu+\\
&\quad+ih_{A_3}(w)(1+w)(\epsilon^*\cdot v)v'_\mu
\end{aligned} \tag{90}$$

The form factors are functions of the variable $w=v\cdot v'$ With this we can compute the differential cross section for the meson decays

$$\frac{d\Gamma}{dw}(\bar{B}\to D^{(*)}e\bar{\nu}_e)=\frac{G_F^2|V_{cb}|m_B^5}{48\pi^3}K^{(*)}(w,r^{(*)})\mathcal{F}_{D^{(*)}}(w)^2 \tag{91}$$

where $K^{(*)}(w)$ is a function of w and $r^{(*)}=\frac{m_{D^{(*)}}}{m_B}$ and $\mathcal{F}_{D^{(*)}}$ is a function of the form factors.
We can do the same now in HQET using the heavy quark symmetry. After some work, we can relate all bispinors using the relation $\not{v}H_v^{(f)}=H_v^{(f)}$ giving

$$\bar{c}_{v'}\Gamma b_v=-\xi(w)H_{v'}^{(c)}\Gamma H_v^{(b)} \tag{92}$$

where the $H_v^{(i)}$ are the meson fields with heavy quark flavour i and velocity v and $\xi(w)$ is the so-called Isgur-Wise function. Then, the form factors for the matrix elements simplify drastically

$$\begin{aligned}\langle D(v')|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(v)\rangle &= \xi(w)\left(v_\mu + v'_\mu\right)\\ \langle D^*(v',\epsilon)|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(v)\rangle &= \xi(w)\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu}v'^\alpha v^\beta \\ \langle D^*(v',\epsilon)|\bar{c}_{v'}\gamma_\mu\gamma_5 b_v|\bar{B}(v)\rangle &= -i\xi(w)\left((1+w)\epsilon^*_\mu - (\epsilon^*\cdot v)v'_\mu\right)\end{aligned} \tag{93}$$

Comparing this to what we found for the calculations in the electroweak theory we find

$$h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w) \tag{94}$$

$$h_-(w) = h_{A_2}(w) = 0 \tag{95}$$

which implies

$$\mathcal{F}_D(w) = \mathcal{F}_{D^*}(w) = \xi(w) \tag{96}$$

It turns out that $1/m_Q$ corrections are absent in these calculations in the zero recoil limit $w \to 1$, so we can test how well the heavy quark symmetry works. In figure 4 the ratio $\mathcal{F}_{\mathcal{D}^*}/\mathcal{F}_{\mathcal{D}}$ is plotted as it was measured by the ALEPH collaboration. There we can see that in the zero recoil limit our predictions using heavy quark symmetry are correct. The huge uncertainties for $w \to 1$ come from the fact that it is not possible to make measurements at that kinematic point so the results have to be extrapolated.

Physically this is easy to understand. In the heavy quark limit we cannot differentiate between the b and c quark. Therefore in the zero recoil limit where the velocity of the heavy quark is unchanged effectively nothing happens during the decay. Therefore the ratio must be one.

Another application of this is that it allows us to extract the value of $|V_{cb}|$ of the CKM matrix very precisely which is otherwise really hard to determine.

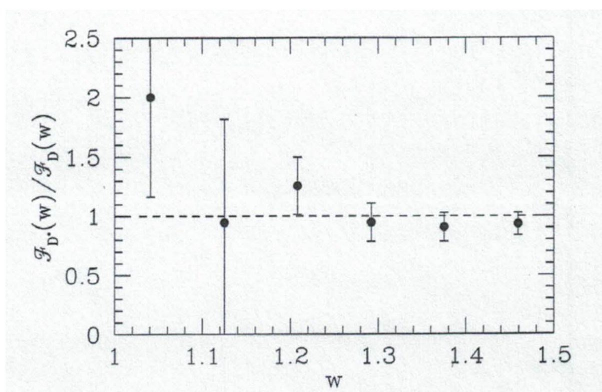

Figure 4: The ratio $\mathcal{F}_{\mathcal{D}^*}/\mathcal{F}_{\mathcal{D}}$ as measured by the ALEPH collaboration.

References

[1] Timothy Cohen. As Scales Become Separated: Lectures on Effective Field Theory. *PoS*, TASI2018:011, 2019, hep-ph/1903.03622.

[2] Aneesh V. Manohar. Introduction to Effective Field Theories. *Les Houches Lect. Notes*, 108, 2020, hep-ph/1804.05863.

[3] Antonio Vairo. Lecture on Effective Field Theories, TUM, SS 2020. https://www.ph.tum.de/academics/org/cc/mh/PH2245.

[4] Matthias Neubert. B decays and the heavy-quark expansion. *Advanced Series on Directions in High Energy Physics*, page 239–293, Jun 1998.

[5] Aneesh V. Manohar and Mark B. Wise. *Heavy Quark Physics*. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press, 2000.

[6] G. Amoros, M. Beneke, and M. Neubert. Two loop anomalous dimension of the chromomagnetic moment of a heavy quark. *Phys. Lett. B*, 401:81–90, 1997, hep-ph/9701375.