# EXACT RENORMALIZATION GROUP AND ASYMPTOTIC SAFETY II

Anja Stuhlfauth, Juan Sebastián Valbuena Bermúdez, Mauricio Valencia Villegas

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## 1 Bosonization

It is a common feature of strongly interacting field theories that macroscopic degrees of freedom can be very different from microscopic degrees of freedom. For an efficient description of the physics, it is advisable to take this transition from microscopic to macroscopic degrees of freedom into account. For instance in QCD, quarks and gluons represent the microscopic degrees of freedom, whereas macroscopic degrees of freedom are mesons and baryons. The latter are bound states of quarks and gluons. Take for example the pseudo-scalar mesons (pions, kaons, . . . ) which carry bifermionic quantum numbers,

$$\phi \sim \overline{\psi}\psi.$$

This type of fermionic pairing occurs in many systems (e.g. in superconductors through the bosonization of Cooper pairs). Generically, a strong fermionic (self-)interaction is required for this pairing. The transition from fermionic to bosonic degrees of freedom is know as *bosonization.* Below we introduce this concept by considering the NJL model as a first example, and then we discuss it in the context of QCD with the help ERG.

### 1.1 Partial Bosonization

Following [1], lets consider the Nambu–Jona-Lasinio (NJL) model [5] as a first example of Bosonization. The (fermionic) action is given by

$$S_{\mathrm{F}} = \int \mathrm{d}^4x \left\{ \overline{\psi} i \partial\!\!\!/ \psi + \frac{1}{2}\lambda \left[ \left(\overline{\psi}\psi\right)^2 - \left(\overline{\psi}\gamma_5\psi\right)^2 \right] \right\} . \tag{1}$$

where $\psi$ represents a Dirac fermion. The NJL model has a $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry given by the following transformations:

$$\psi \to \exp[i\alpha]\psi, \quad \overline{\psi} \to \overline{\psi}\exp[-i\alpha]; \tag{2}$$

$$\psi \to \exp[i\alpha\gamma_5]\psi, \; \overline{\psi} \to \overline{\psi}\exp[i\alpha\gamma_5]; \tag{3}$$

where $\alpha$ is an arbitrary constant phase. The symmetry transformation (3) corresponds to the chiral symmetry which, as in QCD, protects the fermion against acquiring a mass due to fluctuations.

Partial bosonization is obtained with the aid of the following mixed fermionic-bosonic model,

$$S_{\mathrm{FB}} = \int \mathrm{d}^4x \left\{ \overline{\psi} i \partial\!\!\!/ \psi + m^2\phi^*\phi + h\left[\overline{\psi}P_{\mathrm{L}}\phi\psi - \overline{\psi}P_{\mathrm{R}}\phi^*\psi\right] \right\} , \tag{4}$$

where $P_{\mathrm{L,R}} = \frac{1}{2}(1 \pm \gamma_5)$ are the projector operators onto the left- and right-handed components of the Dirac fermion, respectively.

The models 1 and 4 are equivalent (also at the quantum level), if

$$m^2 = \frac{h^2}{2\lambda} \tag{5}$$

To see this fact, lets start by noticing the equivalence between the generating functionals

$$Z_{\mathrm{FB}} \equiv \mathcal{N} Z_{\mathrm{F}}\big|_{\lambda=\frac{h^2}{2m^2}}, \tag{6}$$

where $\mathcal{N} = \int \mathcal{D}\phi \exp\left[-\int \phi^* m^2 \phi\right]$ is a normalisation factor. To prove equation 6, we start by noticing that

$$\int \mathcal{D}\phi \exp\left[-\int (\phi^* + \mathcal{O}_L) m^2 (\phi - \mathcal{O}_R)\right] = \mathcal{N} \exp\left[-\int m^2 \mathcal{O}_L \mathcal{O}_R\right], \tag{7}$$

for any $\phi$-independent operators $\mathcal{O}_L$ and $\mathcal{O}_R$. where $\int = \int \mathrm{d}x^4$. One can show identity 7 by using the *Hubbard–Stratonovich* transformation:

$$\exp\left[-\frac{a}{2}x^2\right] = \sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} \exp\left[-\frac{y^2}{2a} - ixy\right] dy, \tag{8}$$

Now, starting from the generating functional for the fermionic-bosonic model,

$$\begin{aligned}
Z_{\mathrm{FB}} &= \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\phi \exp\left[-S_{\mathrm{FB}}\right] \\
&= \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\phi \exp\left[-\int \left\{\overline{\psi} i \partial\!\!\!/ \psi + m^2 \phi^* \phi + h\left[\overline{\psi} P_{\mathrm{L}} \phi \psi - \overline{\psi} P_{\mathrm{R}} \phi^* \psi\right]\right\}\right] \\
&= \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left[-\int \overline{\psi} i \partial\!\!\!/ \psi\right] \\
&\quad \times \int \mathcal{D}\phi \exp\left[-\int \left\{\left(\phi^* + \frac{h}{m^2}\overline{\psi} P_{\mathrm{L}} \psi\right) m^2 \left(\phi - \frac{h}{m^2}\overline{\psi} P_{\mathrm{R}} \psi\right) - \frac{h^2}{m^2}\left(\overline{\psi} P_{\mathrm{L}} \psi\right)\left(\overline{\psi} P_{\mathrm{R}} \psi\right)\right\}\right] \\
&= \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left[-\int \left(\overline{\psi} i \partial\!\!\!/ \psi + \frac{h^2}{m^2}\left(\overline{\psi} P_{\mathrm{L}} \psi\right)\left(\overline{\psi} P_{\mathrm{R}} \psi\right)\right)\right] \\
&\quad \times \int \mathcal{D}\phi \exp\left[-\int \left(\phi^* + \frac{h}{m^2}\overline{\psi} P_{\mathrm{L}} \psi\right) m^2 \left(\phi - \frac{h}{m^2}\overline{\psi} P_{\mathrm{R}} \psi\right)\right],
\end{aligned}$$

where using the identity 7, with $\mathcal{O}_L = \frac{h}{m^2}\overline{\psi} P_{\mathrm{L}} \psi$ and $\mathcal{O}_R = \frac{h}{m^2}\overline{\psi} P_{\mathrm{R}} \psi$, and the Fierz identity [3] $4\left(\overline{\psi} P_{\mathrm{L}} \psi\right)\left(\overline{\psi} P_{\mathrm{R}} \psi\right) = \left(\overline{\psi}\psi\right)^2 - \left(\overline{\psi}\gamma_5\psi\right)^2$

$$\begin{aligned}
Z_{\mathrm{FB}} &= \mathcal{N} \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left[-\int \left(\overline{\psi} i \partial\!\!\!/ \psi + \frac{2h^2}{m^2}\left(\overline{\psi} P_{\mathrm{L}} \psi\right)\left(\overline{\psi} P_{\mathrm{R}} \psi\right)\right)\right] \\
&= \mathcal{N} \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left[-\int \left(\overline{\psi} i \partial\!\!\!/ \psi + \frac{h^2}{2m^2}\left[\left(\overline{\psi}\psi\right)^2 - \left(\overline{\psi}\gamma_5\psi\right)^2\right]\right)\right] \\
&\equiv \mathcal{N} \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left[S_{\mathrm{F}}|_{\lambda = \frac{h^2}{m^2}}\right] = \mathcal{N} Z_{\mathrm{F}}|_{\lambda = \frac{h^2}{m^2}},
\end{aligned}$$

notice that $\mathcal{N}$ can be absorbed into the normalisation of the remaining fermionic generating functional $Z_{\mathrm{F}}$. Also at the classical level, the equations of motion of $\phi$ shows explicitly the bosonization:

$$\phi = \frac{h}{m^2}\overline{\psi} P_{\mathrm{R}} \psi, \ \phi^* = -\frac{h}{m^2}\overline{\psi} P_{\mathrm{L}} \psi. \tag{9}$$

Lets now study the bosonic mean-field approximation for the model 4. Integrating over the the fermionic fields we get

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\phi \exp\left[-S_{\mathrm{FB}}\right] = \int \mathcal{D}\phi \exp\left[-S_{\mathrm{B}}\right]$$

where the fermionic Gaussian integral leads to the purely bosonic action

$$S_{\mathrm{B}} = \int \mathrm{d}^4x \left\{m^2 \phi^* \phi - \ln\det\left[i\partial\!\!\!/ + h(P_{\mathrm{L}}\phi - P_{\mathrm{R}}\phi^*)\right]\right\} \tag{10}$$

The mean-field theory now neglects bosonic fluctuations and assumes that the bosonic ground state corresponds to that of the classical bosonic action 10. Assuming that the ground state is homogeneous in space and time, $\phi =$ const., the determinant can be computed by standard means (see [6] and section IV.3 of [7]). For our discussion it is sufficient to know that for

$$\lambda > \frac{8\pi^2}{\Lambda^2} \equiv \lambda_{\mathrm{cr}},$$

with $\Lambda$ being the UV cutoff, the resulting effective potential $V_B(\phi\phi^*)$ has a nonzero minimum implying a nonzero vacuum expectation value

$$\langle\phi\rangle\neq 0.$$

$\langle\phi\rangle$ corresponds to a bifermionic condensate $\langle\overline{\psi}\psi\rangle$. In the QCD context it corresponds to a chiral condensate. The non zero vacuum expectation value generates a fermion mass term $\sim m_f\psi\gamma_5\psi$, where

$$m_f \sim \langle\phi\rangle,$$

and the $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry is spontaneously broken to $\mathrm{U}(1)_B$, corresponding to the conservation of the fermion (Baryon) number. The Goldstone theorem ensures the existence of a massless boson, corresponding to excitations of the phase of the field $\phi$ (the pion).

### 1.1.1 Partial Bosonization in QCD

In the QCD context, this scenario corresponds to the spontaneous break-down of chiral symmetry with the pseudo-scalar mesons (pions) as Goldstone bosons. For $\lambda < \lambda_{\mathrm{cr}}$, the vacuum expectation value of $\phi$ and the $\mathrm{U}(1)\times\mathrm{U}(1)$ remains unbroken. Although the NJL model and QCD show many similarities, in the latter there is no microscopic four-quark (or higher) self-interaction beyond criticality, $\lambda|_{\mathrm{QCD}} \to 0$, since four-fermion operators are RG irrelevant. The Euclidean microscopic action for vanishing current quark masses is given by

$$S_{\mathrm{F}} = \int \mathrm{d}x^D \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + i\overline{\psi}\not{D}\psi \right\}. \tag{11}$$

Four-quark operators in the effective action are generated by gluon exchange from fluctuations described by box diagrams as the shown in figure 1.

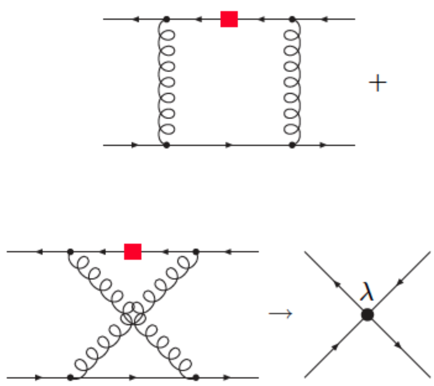


Figure 1: Examples of QCD Box diagrams with fundamental interactions that generate effective four fermion self-interactions $\lambda$. (Figure taken from [1])

At the lowest order, the resulting $\beta_\lambda$ function for the four-quark coupling reads

$$\partial_t\lambda \equiv \beta_\lambda = -c_\lambda \frac{1}{k^2} g^4 \tag{12}$$

where the coefficient $c_\lambda$ depends on the algebraic structure of the theory and the details of the IR regularisation. For QCD, with a SU(3) gauge group and one massless quark flavor $N_{\mathrm{f}} = 1$, and considering the same regulator used in the euclidean anharmonic oscillator

$$R_k(p) = (k^2 - p^2)\theta(k^2 - p^2)$$

one gets $c_\lambda = \frac{5}{12\pi^2}$.

Notice that $\beta_\lambda < 0$, and the four-quark self-interaction $\lambda$ is asymptotically free as expected. A naive extrapolation of equation 12 towards the IR predicts that $\lambda$ can become critical, $\lambda > \lambda_{\mathrm{cr}}$, at some IR scale $k_{\mathrm{cr}}$. If this holds true for the $S_{\mathrm{FB}}$ model, the fermion self-interaction $\lambda\left[\left(\overline{\psi}\psi\right)^2 - \left(\overline{\psi}\gamma_5\psi\right)^2\right]$

becomes strongly RG relevant at scales below $k_{\text{cr}}$ and we expect the system to end up in the symmetry-broken phase.

The conceptual problem of switching from the description in terms of quarks and gluons to the description involving boson fields is still present. Given the Hubbard-Stratonovich transformation in equation 8, as a first attempt to solve the problem, we apply partial bosonization at some scale $k_{\text{B}} < k_{\text{cr}}$. However, this procedure leads to a strong spurious dependence on the particular choice of $k_B$ in generic truncation. Diagrammatically, as shown in figure 2, $\lambda = 0$ in $S_{\text{FB}}$ after bosonization.

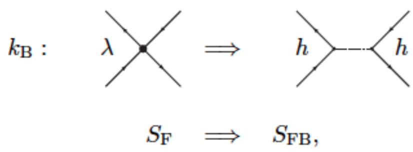


Figure 2: Partial bosonization at scale $k_{\text{B}}$. (Figure taken from [1])

Suppose one performs another RG step and integrate out another momentum shell $\Delta k$. In that case, new quark self-interactions are generated again in this RG step, as shown in figure 3. Thus, the field $\phi$ partially bosonizes all quarks self-interactions at the scale $k_B$, but not at $k_B - \Delta k$.

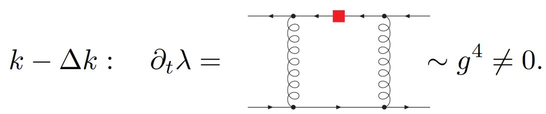


Figure 3: Box diagrams generate new quark self-interactions when integrating the momentum shell between momenta $k$ and $k - \Delta k$. (Figure taken from [1])

Incidentally, this problem does not only occur if gauge interactions are present. The same problem arises, e.g., in purely fermionic systems where the flow in many different channels with non-local structure generates fermions self-interactions. Since partial bosonization with a *local* bosonic interaction can never account for all four-fermion vertices, the remaining four-fermion structure will again generate the RG flow's full structure. If many couplings run fast, neglecting the newly generated terms can introduce large errors. However, keeping these terms seems to make partial bosonization redundant. In the following section, we discuss a solution to the dilemma mentioned above.

## 2 Scale-dependent field transformation

We have observed that different bosonizing fields are needed to compensate for the quark self-interactions at different scales. To solve the problem, we promote the bosonizing field $\phi$ to be scale-dependent, $\phi \to \phi_k$. We write the flow of $\phi$ as

$$\partial_t \phi_k = \mathcal{C}_k \left[ \phi, \psi, \overline{\psi}, \ldots \right], \tag{13}$$

where $\mathcal{C}_k$ is an *a priory* arbitrary functional of possibly all fields in the system. We choose $\mathcal{C}_k$ such that the resulting effective action $\Gamma_k[\phi_k]$ becomes simple and does not possess fermionic self-interactions. Here simplicity is an appropriate criterion for identifying the relevant degrees of freedom at the various scales [2]. The former idea is equivalent to finding a functional $\mathcal{C}_k$ which yields a flow of $\Gamma_k[\phi_k]$–taken at fixed $\phi_k$,

$$\boxed{\partial_t \Gamma_k[\phi_k]|_{\phi_k} = \partial_t \Gamma_k[\phi_k] - \int_q \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k(q)} \partial_t \phi_k(q),} \tag{14}$$

such that the flow of the fermion self-interaction vanishes for all $k$, $\partial_t \lambda|_{\phi_k} = 0$. Therefore, if $\lambda = 0$ holds at one scale $k$, $\lambda$ stays zero at all scales, and $\phi_k$ becomes the "perfect" boson at all scales. Note that in equation 14, we have suppressed further field dependencies on $\psi, \overline{\psi}, \ldots$. This procedure fixes the functional form of $\mathcal{C}_k$.

Following [1], lets formulate the flow equation for the effective action $\Gamma_k[\phi_k]$ with scale dependent field variables as follows. Consider the modified generating functional

$$Z_k[J] = \mathrm{e}^{W_k[J]} = \int \mathcal{D}\varphi \mathrm{e}^{-S[\varphi]-\frac{1}{2}\int \varphi_k R_k \varphi_k + \int J\varphi_k},$$

where a scale-dependent field $\varphi_k$ is coupled to the source and the regulator. This combination guarantees that the resulting flow equation has a one-loop structure [4]. Similar to equation 13, the scale dependence of $\varphi$ is given by

$$\partial_t \varphi_k = \tilde{\mathcal{C}}_k[\varphi]. \tag{15}$$

It is important to mention the difference between equations 13 and 15. The latter is formulated for $\varphi_k$, which is defined under the functional integral, while 13, holds for the fields conjugate to the source $J$,

$$\phi_k = \langle \varphi_k \rangle \equiv \frac{\delta W_k[J]}{\delta J}. \tag{16}$$

In general, $\mathcal{C}_k$ and $\tilde{\mathcal{C}}_k$ are not identical.

Now, the derivation of the flow of $W_k[J]$ is similar to the one Wetterich equation derivation: [1]

$$\begin{aligned}
\partial_t W_k[J] &= \frac{1}{Z_k[J]} \int D\varphi \left(-\partial_t \Delta S_k\right) e^{-S[\varphi]-\Delta S_k + \int J\varphi} \\
&= \frac{1}{Z_k[J]} \int D\varphi \; - \partial_t \left(\frac{1}{2}\int \varphi_k R_k \varphi_k - \int J\varphi_k\right) e^{-S[\varphi]-(\frac{1}{2}\int \varphi_k R_k\varphi_k - \int J\varphi_k + \int J\varphi)+\int J\varphi} \\
&= \frac{1}{Z_k[J]} \int D\varphi \left(-\frac{1}{2}\int \varphi_k(\partial_t R_k)\varphi_k - \int \varphi_k R_k \partial_t\varphi_k + \int J\partial_t\varphi_k\right) e^{-S[\varphi]-\frac{1}{2}\int \varphi_k R_k\varphi_k + \int J\varphi_k} \\
&= \int J\langle\partial_t\varphi_k\rangle - \frac{1}{2}\mathrm{Tr}\partial_t R_k G_k - \int \frac{\delta}{\delta J} R_k \langle \partial_t \varphi_k\rangle - \int \phi_k R_k \langle\partial_t\varphi_k\rangle - \frac{1}{2}\int \phi_k \partial_t R_k \phi_k.
\end{aligned}$$

Here, the modified propagator is defined as

$$\begin{aligned}
G_k(p) &= \frac{\delta^2 W_k}{\delta J \delta J}(p) \\
&= \langle \varphi_k(-p)\varphi_k(p)\rangle - \langle\varphi_k(-p)\rangle\langle\varphi_k(p)\rangle. \\
&= \langle \varphi_k(-p)\varphi_k(p)\rangle - \phi_k(-p)\phi_k(p).
\end{aligned}$$

We used also the relation:

$$\begin{aligned}
\langle \varphi_k \partial_k \varphi_k\rangle &= (\langle \varphi_k\partial_k\varphi_k\rangle - \langle\varphi_k\rangle\langle\partial_k\varphi_k\rangle) + \phi_k\langle\partial_k\varphi_k\rangle \\
&= \left(\frac{\delta}{\delta J} + \phi_k\right)\langle\partial_k\varphi_k\rangle
\end{aligned}$$

We define the *effective action* by means of a Legendre transformation, this time involving the scale-dependent field variables:

$$\Gamma_k[\phi_k] = \sup_J \left(\int J\phi_k - W_k[J]\right) - \frac{1}{2}\int \phi_k R_k \phi_k. \tag{17}$$

Finally the resulting flow of the effective action $\Gamma_k[\phi_k]$ is

$$\boxed{\partial_t \Gamma_k[\phi_k] = \frac{1}{2}\mathrm{Tr}\partial_t R_k G_k + \int \left(G_k \frac{\delta}{\delta\phi_k}\right) R_k \langle\partial_t\varphi_k\rangle + \int \frac{\delta\Gamma_k}{\delta\phi_k}\left(\partial_t\phi_k - \langle\partial_t\varphi_k\rangle\right),} \tag{18}$$

where, we have used the quantum equation of motion:

$$J_{\mathrm{sup}} = \frac{\delta\Gamma_k}{\delta\phi_k} + R_k\phi_k,$$

and its implication $\frac{\delta}{\delta J} = G_k \frac{\delta}{\delta\phi_k}$. The flow equation 14 simplifies as

[1] Note: Compared to the previous notes in sec 5.2, here we take $\Delta S_k := \frac{1}{2}\int \varphi_k R_k \varphi_k - \int J\varphi_k + \int J\varphi$.

$$\partial_t \Gamma_k[\phi_k]|_{\phi_k} = \partial_t \Gamma_k[\phi_k] - \int_q \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k(q)} \partial_t \phi_k(q), \tag{19}$$

$$= \frac{1}{2} \mathrm{Tr} \partial_t R_k G_k + \int \left( G_k \frac{\delta}{\delta \phi_k} \right) R_k \langle \partial_t \varphi_k \rangle - \int \frac{\delta \Gamma_k}{\delta \phi_k} \langle \partial_t \varphi_k \rangle, \tag{20}$$

$$= \frac{1}{2} \mathrm{Tr} \partial_t R_k G_k + \int \left( G_k \frac{\delta}{\delta \phi_k} \right) R_k \langle \tilde{\mathcal{C}}_k [\varphi] \rangle - \int \frac{\delta \Gamma_k}{\delta \phi_k} \langle \tilde{\mathcal{C}}_k [\varphi] \rangle, \tag{21}$$

For a given scale-dependent field transformation $\tilde{\mathcal{C}}_k$ we can successively work out $\langle \tilde{\mathcal{C}}_k \rangle$, $\phi_k$, and $\mathcal{C}_k$. Following this strategy, we would compute the flow of $\Gamma_k[\phi_k]$. Also, $\mathcal{C}_k$ would be a derived quantity, fixed implicitly by $\tilde{\mathcal{C}}_k$. By contrast, we can supplement the flow equation 21 with a bootstrap argument: we do not need to know the precise form of $\tilde{\mathcal{C}}_k$, because all we want to choose in the end is $\mathcal{C}_k$. In fact, $\tilde{\mathcal{C}}_k$ occurs only in expectation values. Therefore, we *assume* that a suitable $\tilde{\mathcal{C}}_k$ exists for the desired $\mathcal{C}_k$ such that

$$\langle \tilde{\mathcal{C}}_k \rangle = \mathcal{C}_k \tag{22}$$

Of course, this is a highly implicit construction and given the complicated structure of the mapping $\tilde{\mathcal{C}}_k \to \mathcal{C}_k$, the existence of a suitable $\tilde{\mathcal{C}}_k$ for an arbitrary $\mathcal{C}_k$ is generally not guaranteed or, at least, difficult to prove. Nevertheless, in practice, we will use the resulting flow equation together with a truncation. Thus, **it is reasonable to assume that Eq. 22 can at least be satisfied within the truncation order**. As a consequence, flow equation 21 simplifies,

$$\boxed{\partial_t \Gamma_k[\phi_k]|_{\phi_k} = \frac{1}{2} \mathrm{Tr} \partial_t R_k G_k + \int \left( G_k \frac{\delta}{\delta \phi_k} \right) R_k \partial_t \phi_k - \int \frac{\delta \Gamma_k}{\delta \phi_k} \partial_t \phi_k,} \tag{23}$$

The second term on the RHS of 23 takes care of fluctuation contributions to the renormalization flow of the operator insertion $\partial_t \phi_k$ in the functional integral. This term will generally be subdominant for not too large coupling, and we expect that this term does not induce strong modifications for couplings up to $\mathcal{O}(1)$ for the following reasons:

- it is of higher order in the coupling,
- $R_k$ insertions lead to weaker numerical coefficients than $\partial_t R_k$ insertions for standard regulators.

## 2.1 Scale-dependent field transformations for QCD: Rebosonization

Let us consider one-flavor QCD in a simple truncation. Apart from the standard kinetic terms for the quark and gluons, supplemented by wave function renormalization factors, we include a point-like four-quark self-interaction in the scalar–pseudo-scalar sector. Of course, in order to avoid any ambiguity with respect to possible Fierz rearrangements of the four-fermion interactions in the point-like limit, all possible linearly-independent four-fermion interactions, in principle, have to be included in the truncation. For simplicity, we confine ourselves here just to the scalar-pseudo-scalar channel, where chiral condensation is expected to occur. For the four-fermion interactions that will be generated by the flow, we use the Fierz decomposition as proposed in[2001 Wetterich Gies]

$$\Gamma_{\mathrm{F},k} = \int \mathrm{d}^4 x \left\{ \frac{Z_k}{4} F^a_{\mu\nu} F^a_{\mu\nu} + i Z_\psi \overline{\psi} \not{D} \psi + \frac{1}{2} \lambda_k \left[ \left( \overline{\psi} \psi \right)^2 - \left( \overline{\psi} \gamma_5 \psi \right)^2 \right] \right\} . \tag{24}$$

Compare eq. 24 and 1, plus 11 and 4

The initial condition of the four-quark operator in the UV is given by $\lambda\_k \to \Lambda \to 0$. As a first step towards rebosonization, we include a complex mesonic scalar field in the truncation on equal footing,

$$\Gamma_k = \Gamma_{\mathrm{F},k} + \int \mathrm{d}^4 x \left( Z_\phi \partial_\mu \phi^*_k \partial_\mu \phi_k + V(\phi^*_k \phi_k) + h_k \left[ \overline{\psi} P_{\mathrm{L}} \phi_k \psi - \overline{\psi} P_{\mathrm{R}} \phi^*_k \psi \right] \right) , \tag{25}$$

where $V(\phi^*_k \phi_k) = m^2_k \phi^*_k \phi_k + \mathcal{O}((\phi^*_k \phi_k)^2)$.

The initial conditions for the scalar field at $k \to \Lambda$ need to be chosen such that the scalar has no observable effect on the QCD sector whatsoever. This is easily done by demanding that t

- The Yukawa interaction with the quarks vanishes $h_{k\to\Lambda} \to 0$.
- We also choose a large scalar mass ensuring a fast decoupling of the scalar $m^2_{k\to\Lambda} \to \mathcal{O}(\Lambda^2)$.
- $Z_\phi|_{k\to\Lambda} \to 0$ which makes the scalar non-dynamical at the UV scale.

Solving the flow with these initial conditions, the scalars rapidly decouple and only the standard QCD flow remains, revealing the purely formal character of this first step towards rebosonization.

As a second step, we now use the freedom to perform scale-dependent field transformations, as suggested in Eq. 13. We promote the field $\phi$ to be scale dependent, and choose the functional $\mathcal{C}_k$ characterizing this scale dependence to be of the form 9

$$\partial_t \phi_k = \overline{\psi} P_{\mathrm{R}} \psi \partial_t \alpha_k,\ \partial_t \phi_k^* = -\overline{\psi} P_{\mathrm{L}} \psi \partial_t \alpha_k. \tag{26}$$

$\alpha_k$ some function.

From the Yukawa interaction, we get from the last term in eq. 23 together with eq. 26

$$\int \frac{\delta\Gamma_k}{\delta\phi_k}\partial_t\phi_k + \int \frac{\delta\Gamma_k}{\delta\phi_k^*}\partial_t\phi_k^* = \int \left(m_k^2\phi_k^* + h\overline{\psi}P_{\mathrm{L}}\psi\right)\partial_t\phi_k + \left(m_k^2\phi_k - h\overline{\psi}P_{\mathrm{R}}\psi\right)\partial_t\phi_k^* \tag{27}$$

$$= \int \left(m_k^2\phi_k^* + h\overline{\psi}P_{\mathrm{L}}\psi\right)\overline{\psi}P_{\mathrm{R}}\psi\partial_t\alpha_k - \left(m_k^2\phi_k - h\overline{\psi}P_{\mathrm{R}}\psi\right)\overline{\psi}P_{\mathrm{L}}\psi\partial_t\alpha_k \tag{28}$$

$$= \int m_k^2\left(\overline{\psi}P_{\mathrm{R}}\phi_k^*\psi - \overline{\psi}P_{\mathrm{L}}\phi_k\psi\right)\partial_t\alpha_k + 2h\left(\overline{\psi}P_{\mathrm{L}}\psi\right)\left(\overline{\psi}P_{\mathrm{R}}\psi\right)\partial_t\alpha_k \tag{29}$$

$$= \int -m_k^2\left(\overline{\psi}P_{\mathrm{L}}\phi_k\psi - \overline{\psi}P_{\mathrm{R}}\phi_k^*\psi\right)\partial_t\alpha_k + \frac{1}{2}h\partial_t\alpha_k\left[\left(\overline{\psi}\psi\right)^2 - \left(\overline{\psi}\gamma_5\psi\right)^2\right] \tag{30}$$

Neglecting the second term of the RHS of Eq. 23 here and in the following as discussed above, we obtain the flow of the four-quark coupling at fixed $\phi_k$ (compare to the flow equation 12) as

$$\partial_t\lambda|_{\phi_k} = -c_\lambda \frac{1}{k^2} g^4 - h\partial_t\alpha_k \tag{31}$$

If we choose the transformation function $\alpha_k$ such that:

$$\partial_t\alpha_k = -c_\lambda \frac{1}{k^2}\frac{g^4}{h} \tag{32}$$

then

$$\boxed{\partial_t\lambda_k|_{\phi_k} = 0.} \tag{33}$$

The initial condition $\lambda_{k\to k}$, implies that $\lambda = 0$ holds for all scales $k$. The scale-dependent transformation Eq. 26 has removed the point-like four-quark interaction *in the scalar–pseudo-scalar sector* completely by **partial rebosonization**. The information about this interaction is transformed into the scalar sector

The scalar mass term of eq. 30, yields a contribution to the flow of the Yukawa coupling $h_k$. Together with the contribution from the standard flow term, we obtain

$$\partial_t h|_{\phi_k} = -\frac{1}{2}c_h g^2 h + m^2\partial_t\alpha \tag{34}$$

$$= -\frac{1}{2}c_h g^2 h - c_\lambda \frac{m^2}{k^2}\frac{g^4}{h} \tag{35}$$

where the coefficient $c_h$ is a result of the diagram shown in Fig. 4. For the Litim regulator and in the Landau gauge, the result is $c_h = \frac{1}{\pi^2}$ for $SU(3)$.

Only the standard flow-equation term contributes to the flow of the scalar mass term

$$\partial_t m^2 = c_m k^2 h^2, \tag{36}$$

where the coefficient $c_m$ yields for the Litim regulator $c_m = N_c/(8\pi^2)$, resulting from the diagram in Fig 5.

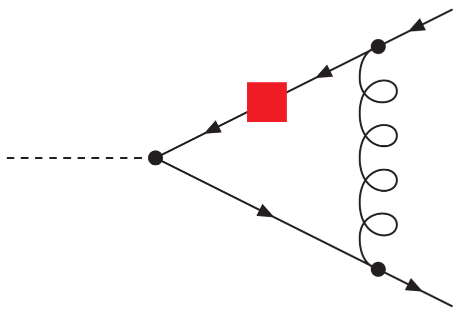

Figure 4: Diagrams contributing to the RG flow of the scalar sector flow of the Yukawa coupling, see Eq. 35. Only one diagram per topology is shown; further diagrams exhibit the regulator insertion (filled box) attached to other internal lines; (Figure taken from [1])

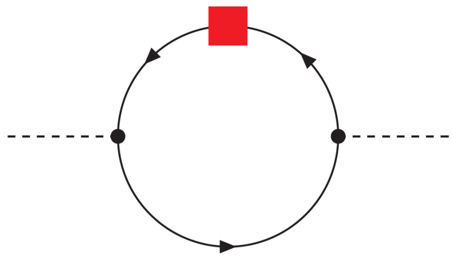

Figure 5: Diagrams contributing to the RG flow of the scalar sector flow of the scalar mass, see Eq. 36. Only one diagram per topology is shown; further diagrams exhibit the regulator insertion (filled box) attached to other internal lines. (Figure taken from [1])

The physical properties of the resulting boson field can best be illustrated with the convenient dimensionless composite coupling

$$\tilde{\epsilon} := \frac{m^2}{k^2 h^2}. \tag{37}$$

Its $\beta_\epsilon$ function is

$$\begin{aligned}
\partial_t \tilde{\epsilon} &= -2\frac{m^2}{k^2h^2} + \frac{\partial_t m^2}{k^2h^2} - 2\frac{m^2}{k^2h^3}\partial_t h && (38)\\
&= -2\tilde{\epsilon} + \frac{c_m k^2 h^2}{k^2h^2} - 2\frac{m^2}{k^2h^3}(-\frac{1}{2}c_h g^2 h - c_\lambda \frac{m^2}{k^2}\frac{g^4}{h}) && (39)\\
&= -2\tilde{\epsilon} + c_m + c_h g^2 \frac{m^2}{k^2h^2} + 2c_\lambda g^4 \frac{m^4}{k^4h^4} && (40)\\
&= c_m + (c_h g^2 - 2)\tilde{\epsilon} + 2c_\lambda g^4 \tilde{\epsilon}^2 && (41)\\
& && (42)
\end{aligned}$$

Notice that the last term comes directly from The last term comes directly from rebosonization Since all $c_i > 0$, the $\beta_\epsilon$ function looks like a parabola. see Fig. 6. Without rebosonization, this function would have corresponded to a straight line. The $\beta_\epsilon$ function exhibits two fixed points: $\tilde{\epsilon}_{*,1}$ is IR repulsive and $\tilde{\epsilon}_{*,2}$ is IR attractive.

$$\tilde{\epsilon}_{*,1} = \frac{1}{4c_\lambda g^4}\left(2 - c_h g^2 - \sqrt{4 - 4c_h g^2 + (c_h^2 - 8c_m c_\lambda)g^4}\right)$$
$$\tilde{\epsilon}_{*,2} = \frac{1}{4c_\lambda g^4}\left(2 - c_h g^2 + \sqrt{4 - 4c_h g^2 + (c_h^2 - 8c_m c_\lambda)g^4}\right)$$

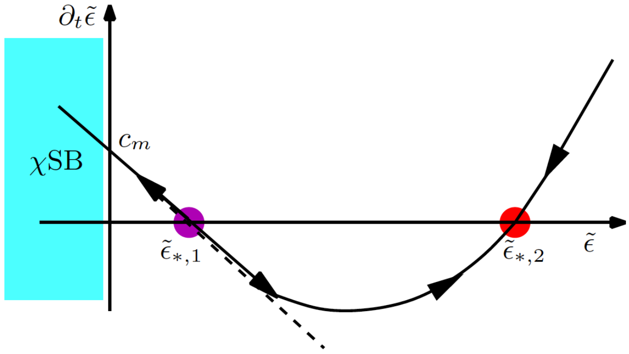


Figure 6: Schematic plot of the $\beta_\epsilon$ function for the composite coupling $\tilde{eps}$ with arrows pointing along the flow towards the IR. The fixed point $\tilde{\epsilon}_{*,1}$ is IR repulsive; in its vicinity, the scalar field behaves as a fundamental scalar (dashed line). If the flow is initiated with $\tilde{\epsilon}|_{k=\Lambda} < \tilde{\epsilon}_{*,1}$, $\tilde{\epsilon}$ drops quickly below zero and the system runs into the regime with chiral symmetry breaking ($\chi$SB). For $\tilde{\epsilon}|_{k=\Lambda} < \tilde{\epsilon}_{*,1}$, $\tilde{\epsilon}$, the system rapidly approaches the bound-state IR fixed point $\tilde{\epsilon}_{*,2}$, where the scalar exhibits bound-state behaviour. QCD initial conditions correspond to $\tilde{\epsilon}|_{k\to\Lambda} \to \infty$ (Figure taken from [1])

In a small-gauge-coupling expansion, we get

$$\tilde{\epsilon}_{*,1} \simeq \frac{c_m}{2} - \frac{c_h^2}{16c_\lambda} + \mathcal{O}(g^2)$$
$$\tilde{\epsilon}_{*,2} \simeq \frac{1}{c_\lambda g^4} - \frac{c_h}{2c_\lambda g^2} - \frac{c_m}{2} + \frac{c_h^2}{16c_\lambda} - \mathcal{O}(g^2)$$

Without rebosonization (the limit $g \to 0$) only $\tilde{\epsilon}_{*,1}$ is present, while $\tilde{\epsilon}_{*,2} \to \infty$.
If we start with initial conditions such that
There are two different classes of initial conditions:

- $\tilde{\epsilon}|_{k\to\Lambda} < \tilde{\epsilon}_{*,1}$, obtained only if either the scalar mass is small or the Yukawa coupling is large or both. Then $\tilde{\epsilon}$ quickly becomes negative, corresponding to the bosonic mass term dropping below zero, $m^2 < 0$. This indicates that the potential develops a nonzero minimum, giving rise to *chiral symmetry breaking* and *quark mass generation*.
- $\tilde{\epsilon}|_{k\to\Lambda} > \tilde{\epsilon}_{*,1}$, the system quickly approaches $\tilde{\epsilon}_{*,2}$ either from above or below rather independently of the initial values of $m, h|_{k\to\Lambda}$. Also for much smaller initial $\tilde{\epsilon}$, the system rapidly flows to $\tilde{\epsilon}_{*,2}$, and the memory of the precise initial values gets lost. There, the system is solely determined by the gauge coupling $g^2$ which governs the fixed-point position.

Flow behaviour near the fixed points

- **Near** $\tilde{\epsilon}_{*,1}$ , the slope of the $\beta_\epsilon$ function is -2, which is nothing but a typical quadratic renormalization of a bosonic mass term; the scalar behaves like an ordinary fundamental scalar here. In fact, there is a correspondence between $\tilde{\epsilon}_{*,1}$ and the critical coupling of NJL-like systems,

$$\tilde{\epsilon}_{*,1} \simeq \frac{N_C}{2k^2\lambda_{\rm cr}}, \tag{43}$$

  where the factor $N_C$ takes care of the additional color degree of freedom of the quarks which was not present in the NJL above. The initial condition $\tilde{\epsilon}|_{k\to\Lambda} < \tilde{\epsilon}_{*,1}$ agrees with $\lambda|_{k\to\Lambda} > \lambda_{\rm cr}$, and the system is in the broken phase of the NJL model.
- **Near** $\tilde{\epsilon}_{*,2}$, the boson is not really a fully developed degree of freedom. The flow does not at all remind us of the flow of a fundamental scalar, but points to the composite nature of the scalar. This justifies to call $\tilde{\epsilon}_{*,2}$ the *bound-statefixed point*. For instance, in weakly coupled systems such as QED, the boson at the bound-state fixed point describes a positronium-like bound state.

It is a particular strength of the RG approach with scale-dependent field transformations that one and the same field can describe bound state formation on the one hand and condensate formation as well as meson excitations on the other hand; whether the field behaves as a bound state or as a fundamental scalar is solely governed by the dynamics and the coupling strength of the system.

QCD initial conditions with large initial $m$ and small initial $h$ correspond to $\tilde{\epsilon}|_{k\to\Lambda} \to \infty$.

Then, the initial condition $\tilde{\epsilon}_{k\to\Lambda} < \tilde{\epsilon}_{*,1}$ is in conflict with our QCD initial conditions and the system is not in the QCD universality class.

the system is solely determined by the gauge coupling $g^2$ which governs the fixed-point position. This is precisely how it should be in QCD.

#### 2.1.1 Comments on Running of g

So far, our analysis of the system was essentially based on weak-coupling arguments, revealing that QCD at initial stages of the flow ($k \to \lambda \to \infty$) approaches the bound-state fixed point. But, we still have to answer a crucial question: **how does QCD leave the bound-state fixed point and ultimately approach the chiral-symmetry broken regime?**

The answer is again given by the gauge coupling, $g$, which controls the whole flow. For increasing gauge coupling, the parabola characterising the $\beta_\epsilon$ function is lifted, as depicted in figure 7.

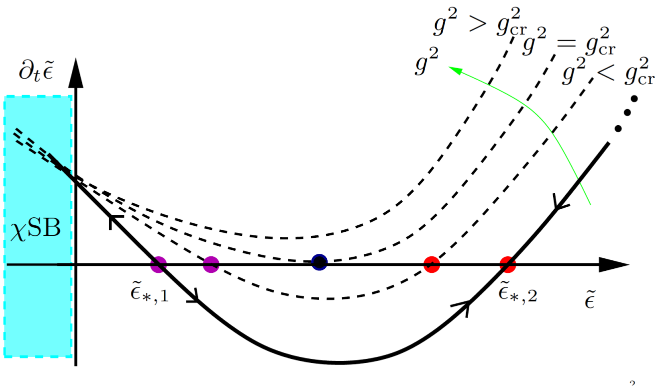


Figure 7: Schematic plot of the $\beta_\epsilon$ function for the composite coupling $e\tilde{p}s$ with arrows pointing along the flow towards the IR. At weak gauge coupling, QCD like systems first flow to the bound-state fixed point $\tilde{\epsilon}_{*,2}$ where they remain over a wide range of scales. For increasing gauge coupling $g^2$, the $\beta_\epsilon$ function is lifted (dashed lines). At the critical coupling $g^2 = g^2_{\rm cr}$, the fixed points are destabilised and the system rapidly runs into the chiral symmetry broken regime ($\chi$SB). (Figure taken from [1])

At a critical coupling value, $g^2 = g^2_{\rm cr}$, the fixed points $\tilde{\epsilon}_{*,1}$ and $\tilde{\epsilon}_{*,2}$ annihilate each other and the system runs towards the chiral-symmetry broken regime. This transition is unambiguously triggered by gluonic interactions. For instance in the present case of one-flavor QCD with gauge group $SU(3)$, the critical coupling is given by

$$\alpha_{\rm cr} \equiv \frac{g^2_{\rm cr}}{4\pi} = \frac{2\pi}{1+\sqrt{5/12N_C}} \simeq 0.74$$

for the Litim regulator. Since this coupling value is not a universal quantity, one should not overemphasise its meaning. However, it is interesting to observe that this coupling strength is in the non-perturbative domain, as expected, but not very deeply. In particular, since loop expansions go along with the expansion parameter $\frac{\alpha}{\pi}$, the critical coupling and thus the approach to chiral symmetry breaking appears still to be in reach of weak-coupling methods *(not to be confused with perturbation theory)*.

**Conclusion** We conclude that the continuous scale-dependent translation allows for a controllable transition between microscopic to macroscopic degrees of freedom and between different dynamical regimes of a system. From a quantitative viewpoint, no spurious dependence on a bosonization scale, i.e., a scale at which degrees of freedom are discretely changed, is introduced, because field transformations are continuously performed on all scales. This helps maintaining the predictive power of truncated RG flows. As a result, macroscopic parameters can quantitatively be related to microscopic input.

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