Three-Point Amplitudes, On-Shell Recursion Relations and Double Copy

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Abstract

We are going to start where we ended last time and see how we can systematically use the derived form of three-point amplitudes for particles of different helicities to reconstruct the structure of the theory from which they originated. We will then introduce the notion of on-shell recursion relations and use them to prove the Parke-Taylor formula for maximally helicity violating color-ordered partial amplitudes in Yang-Mills theory and finally conclude with a brief discussion of color-kinematics duality and the double copy procedure.

1 Where We Left Off

We saw in the last seminar: three-particle special kinematics

$$p_1 + p_2 + p_3 = 0\,, \tag{1.1}$$
$$\langle 12\rangle[12] = 0\,, \quad \langle 23\rangle[23] = 0\,, \quad \langle 31\rangle[31] = 0 \tag{1.2}$$

and the requirement of little group covariance of a scattering amplitude

$$\mathcal{M}(1^{h_1}, ..., n^{h_n}) \to \prod_{j=1}^{n} z_j^{-2h_j}\mathcal{M}(1^{h_1}, ..., n^{h_n}) \tag{1.3}$$

fix the general form for three-particle scattering amplitudes, which are non-vanishing for complex momenta,

$$\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) = \begin{cases} c\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_1-h_3}\,, & h \leq 0 \\ c'[12]^{-h_3+h_1+h_2}[23]^{-h_1+h_2+h_3}[31]^{-h_2+h_1+h_3}\,, & h > 0 \end{cases} \tag{1.4}$$

nonperturbatively. We considered some examples and by dimensional analysis, knowing that

$$[\mathcal{M}_n]_m = 4 - n\,, \quad [[ij]]_m = [\langle ij\rangle]_m = 1\,, \tag{1.5}$$

we pinned down $[c]_m$ and were able to trace the amplitude back to a renormalizable or to an effective theory.

Reminder: Why is $[\mathcal{M}_n]_m = 4 - n$? Consider the formula for the differential scattering cross-section of a $2 \to n-2$ process:

$$\mathrm{d}\sigma_{2\to n-2} = \frac{1}{4E_{\boldsymbol{p_1}}E_{\boldsymbol{p_2}}|\boldsymbol{v_1} - \boldsymbol{v_2}|}|\mathcal{M}_n|^2 \prod_{i=1}^{n-2} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_{\boldsymbol{p}_i}}(2\pi)^4\delta^{(4)}\left(\sum_{j\text{ out}} p_j - \sum_{k\text{ in}} p_k\right). \tag{1.6}$$

By dimensional analysis:

$$-2 = [\mathrm{d}\sigma]_m = -2 + 2[\mathcal{M}_n]_m + 2(n-2) - 4 \quad \Longrightarrow \quad [\mathcal{M}_n]_m = 4 - n \checkmark\,. \tag{1.7}$$

2 From Amplitudes to Lagrangians

This analysis can be made systematic and we can use massless three-point scattering amplitudes to reconstruct ("bootstrap") the structure of the Lagrangian, and hence of the theory, from which they arose. We are going to discuss the cases of massless scalar, vector and tensor particles as examples of this.

2.1 Scalars

If all particles have $h_i = 0$ and they are identical, we have just

$$\mathcal{M}(123) = i\lambda = \text{const.}\,, \tag{2.1}$$

which stems from the trilinear self-interaction

$$\mathcal{L}_{\text{int}} \supset \frac{\lambda}{3!}\phi^3\,. \tag{2.2}$$

If the theory contains different species of scalars then

$$\mathcal{M}(1_a 2_b 3_c) = i\lambda_{abc}\,, \tag{2.3}$$

with λ_{abc} totally symmetric, coming from

$$\mathcal{L}_{\text{int}} \supset \lambda_{abc}\phi_a\phi_b\phi_c\,. \tag{2.4}$$

We can also think of trilinear couplings containing derivatives, e.g.

$$\mathcal{L}_{\text{int}} \supset \phi\partial_\mu\phi\partial^\mu\phi\,, \tag{2.5}$$

however amplitudes coming from these interaction terms vanish by three-particle kinematics, which reflects the fact that these terms can be eliminated via a field redefinition using the equations of motion of ϕ.

2.2 Vectors

Vector particles have $h_i = \pm 1$, and as we saw last time we have four possible helicity configurations, $(+++)$, $(++-)$, $(--+)$ and $(---)$. These correspond to the amplitudes

$$\mathcal{M}(1^+2^+3^+) = ic_1[12][23][31]\,, \quad \mathcal{M}(1^+2^+3^-) = ic_2\frac{[12]^3}{[23][31]}\,, \tag{2.6}$$

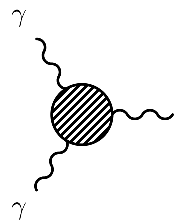
which, by crossing symmetry, are the same as

$$\mathcal{M}(1^-2^-3^-) = ic_1\langle 12\rangle\langle 23\rangle\langle 31\rangle\,, \quad \mathcal{M}(1^-2^-3^+) = ic_2\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,. \tag{2.7}$$

They contain odd powers of the brackets, so they are all totally antisymmetric under the exchange of any pair of particles (since $[ij] = -[ji]$, $\langle ij\rangle = -\langle ji\rangle$), but the total amplitude must be symmetric, since it is bosonic. This means that the coefficients must be totally antisymmetric under the exchange of any pair of particles as well, i.e. they must carry species indices,

$$c_i = c_i^{abc} \tag{2.8}$$

and the vector particles must come in different species as well. This means that there is no theory containing a single species of self-interacting vector particle which allows for a non-vanishing three-point amplitude, which is in agreement with the fact that



$= 0$

in QED due to the requirement of charge conjugation invariance. We therefore have

$$\mathcal{M}(1_a^+ 2_b^+ 3_c^+) = i c_1^{abc} [12][23][31]\,, \quad \mathcal{M}(1_a^+ 2_b^+ 3_c^-) = i c_2^{abc} \frac{[12]^3}{[23][31]} \tag{2.9}$$

and analogously for the $(--+)$ and $(---)$ amplitudes. By dimensional analysis, since $[\mathcal{M}]_m = 1$, we find

$$[c_1^{abc}]_m = -2\,, \quad [c_2^{abc}]_m = 0 \tag{2.10}$$

and since the mass dimension of the kinematic piece of $\mathcal{M}$ reflects the number of derivatives appearing in the corresponding trilinear interaction Lagrangian the $(+++)$ and $(---)$ amplitudes stem from an effective theory with a trilinear coupling containing three derivatives $(\partial A)^3$ and the $(++-)$ and $(--+)$ amplitudes stem from a renormalizable theory with a trilinear coupling containing one derivative $A^2 \partial A$:

$$(++-)\,,\ (--+): \quad \mathcal{L}_{\text{int}} \supset c_2^{abc} A_\mu^b A_\nu^c \partial^\mu A^{a\nu} \quad \text{contained in the Yang-Mills Lagrangian,} \tag{2.11}$$

$$(+++)\,,\ (---): \quad \mathcal{L}_{\text{int}} \supset c_1^{abc} F^{a\mu}_{\ \ \ \nu} F^{b\nu}_{\ \ \ \rho} F^{c\rho}_{\ \ \ \mu}\,, \quad \text{induced by loops of massive colored vector particles.} \tag{2.12}$$

Studying the three-point amplitudes therefore allowed us to uniquely fix the trilinear interaction term of the Yang-Mills Lagrangian.

2.3 Tensors

Tensor particles have $h_i = \pm 2$ and we have the same possible helicity configurations as in the vector case. From (1.4) we see that only even exponents of all the brackets will appear, so the amplitude is totally symmetric under exchange of any pair of particles. We have the amplitudes

$$\mathcal{M}(1^{++} 2^{++} 3^{++}) = i k_1 [12]^2 [23]^2 [31]^2\,, \quad \mathcal{M}(1^{++} 2^{++} 3^{--}) = i k_2 \frac{[12]^6}{[23]^2 [31]^2}\,, \tag{2.13}$$

which cross to the $(---)$ and $(--+)$ amplitudes. By dimensional analysis we find

$$[k_1]_m = -5\,, \quad [k_2]_m = -1\,, \tag{2.14}$$

so all three-point amplitudes stem from effective theories, one with a six-derivative trilinear interaction $(\partial^2 h)^3$, one with a two-derivative trilinear interaction $h(\partial h)^2$:

$$(++-)\,,\ (--+): \quad \mathcal{L}_{\text{int}} \supset k_2 h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} + \ldots \quad \text{contained in the Einstein-Hilbert Lagrangian}\,, \tag{2.15}$$

$$(+++)\,,\ (---): \quad \mathcal{L}_{\text{int}} \supset k_1 R_{\mu\nu}^{\ \ \ \alpha\beta} R_{\alpha\beta}^{\ \ \ \rho\sigma} R_{\rho\sigma}^{\ \ \ \mu\nu} \quad \text{induced at two loops by Einstein gravity.} \tag{2.16}$$

In fact, knowing the structure of three-point amplitudes is in some cases enough to fully reconstruct the entire tree-level S-matrix of a theory using on-shell recursion relations. We will see what they are in the next section.

3 On-Shell Recursion Relations

We consider a generic tree-level n-point scattering amplitude $\mathcal{M}_n(p_1, ..., p_n)$ of massless particles, so for the momenta in the set $P \equiv \{p_i\}_{i=1}^n$ the identities

$$p_i^2 = 0\,, \quad \sum_{i=1}^{n} p_i = 0 \tag{3.1}$$

hold and we are going to assume that $\mathcal{M}_n$ stems from a local, unitary theory. This means that the poles of the amplitude can be only simple poles: they arise from propagators of intermediate particles, which for special momentum configurations can go on-shell, and a given diagram contributing to $\mathcal{M}_n$ can only contain one power of a specific propagator. Any propagator derived from a unitary field theory with momentum p flowing through it diverges like $1/p^2$ in the on-shell limit $p^2 \to 0$, so we have:

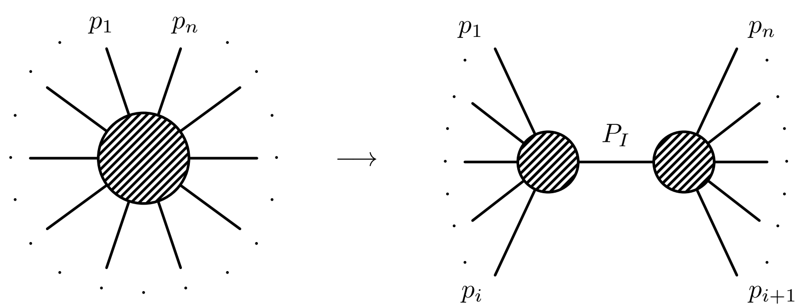
$$\text{Locality} + \text{unitarity} \quad \Longrightarrow \quad \mathcal{M}_n \text{ contains only simple poles of the form } \frac{1}{P_I^2}\,,$$

with

$$P_I = \sum_{p_i \in I \subset P} p_i\,. \tag{3.2}$$

Further, in the kinematic limit where $\mathcal{M}_n$ does become singular, it factorizes as follows:

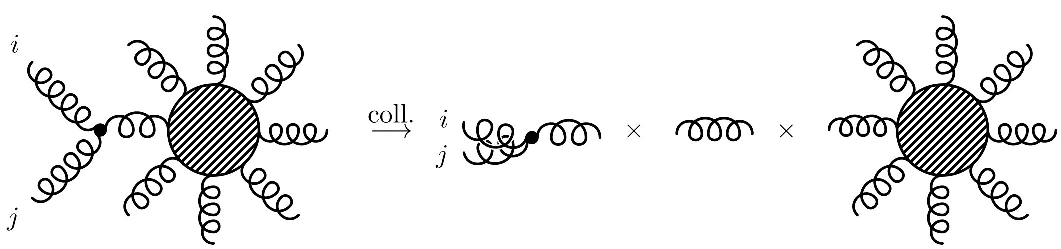
$$\mathcal{M}_n(p_1, ..., p_n) \to \mathcal{M}^L_{i+1}(p_1, ..., p_i, P_I)\frac{1}{P_I^2}\mathcal{M}^R_{n-i+1}(-P_I, p_{i+1}, ..., p_n)\,, \tag{3.3}$$



$2 \leq i \leq n-2$, where now

$$P_I = p_1 + ... + p_i\,, \tag{3.4}$$

so in this limit we can actually build the higher-point amplitude $\mathcal{M}_n$ from a lower-point left-amplitude $\mathcal{M}^L_{i+1}$, a right-amplitude $\mathcal{M}^R_{n-i+1}$ and a scalar propagator $1/P_I^2$. One well-known example of this behavior is the collinear limit of gluon amplitudes (here we show a tree-level example):



The main idea behind the on-shell recursion relations is to use this fact and the knowledge of the residue theorem in complex analysis to recursively construct higher-point amplitudes from already known lower-point ones.

To this end we introduce n complex-valued vectors r_i^μ, some of which can be chosen to vanish, which satisfy:

1) $\sum_{i=1}^{n} r_i = 0,$

2) $r_i \cdot r_j = 0$ for all $i, j = 1, 2, ..., n$, which also implies $r_i^2 = 0$,

3) $p_i \cdot r_i = 0$ separately for each i

and we use them to define the shifted complex momenta, with the complex parameter $z \in \mathbb{C}$:

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu \,. \tag{3.5}$$

Our definition of the r_i still ensures that

$$\hat{p}_i^2 = 0 \,, \quad \sum_{i=1}^{n} \hat{p}_i = 0 \tag{3.6}$$

and the shifted $\hat{P}_I$, with $I \subset P$ a nontrivial subset (i.e. $I \neq \varnothing$, $I \neq \hat{P}$), now satisfy

$$R_I \equiv \sum_{i \in I} r_i \,, \quad \hat{P}_I^2 = \Bigg(\sum_{\hat{p}_i \in I \subset \hat{P}} \hat{p}_i \Bigg)^2 = P_I^2 + 2z P_I \cdot R_I + \underset{\substack{\uparrow \\ =0 \text{ by 2)}}}{z^2 R_I^2} \,, \tag{3.7}$$

with the unshifted P_I, so we can write

$$\hat{P}_I^2 = -P_I^2 \frac{z - z_I}{z_I} \,, \quad z_I \equiv -\frac{P_I^2}{2 P_I \cdot R_I} \tag{3.8}$$

and the shifted amplitude $\hat{\mathcal{M}}_n(\hat{p}_1, ..., \hat{p}_n)$ has simple poles at $z = z_I$. Since we are interested in the analytic structure of the amplitude, which is defined by its poles, in the following it will be useful to consider it as a function of z, $\hat{\mathcal{M}}_n(z)$.

The original, unshifted amplitude is obtained by setting $z = 0$, $\mathcal{M}_n = \hat{\mathcal{M}}_n(z = 0)$; since the P_I are defined via non-trivial subsets $I \subset P$ we know that the shifted amplitude is regular there, so if we consider the quantity

$$\frac{\hat{\mathcal{M}}_n(z)}{z} \tag{3.9}$$

we know that it will have all the simple poles at z_I that $\hat{\mathcal{M}}_n(z)$ has, plus additionally the simple pole at $z = 0$, which we added by hand, and which has as a residue the unshifted amplitude $\mathcal{M}_n$, which is what we are interested in. By picking a closed integration contour C which encompasses all the poles of $\hat{\mathcal{M}}_n(z)/z$ by the residue theorem we get

$$\frac{1}{2\pi i} \oint_C \mathrm{d}z \, \frac{\hat{\mathcal{M}}_n(z)}{z} = \operatorname*{Res}_{z=0} \frac{\hat{\mathcal{M}}_n(z)}{z} + \sum_I \operatorname*{Res}_{z=z_I} \frac{\hat{\mathcal{M}}_n(z)}{z} = \mathcal{M}_n + \sum_I \operatorname*{Res}_{z=z_I} \frac{\hat{\mathcal{M}}_n(z)}{z} \tag{3.10}$$

and if we push the contour C to infinity the integral on the left-hand side of the equation gives a possibly non-vanishing contribution B_∞ and we can solve for $\mathcal{M}_n$:

$$\mathcal{M}_n = -\sum_I \operatorname*{Res}_{z=z_I} \frac{\hat{\mathcal{M}}_n(z)}{z} + B_\infty \,. \tag{3.11}$$

The reason why this is useful is that we know what the residues of the shifted amplitudes at z_I will be: the shifted momentum $\hat{P}_I$ goes on shell at $z = z_I$ and $\hat{\mathcal{M}}_n(z)$ factorizes, so:

$$\lim_{z \to z_I} \hat{\mathcal{M}}_n(z) = \lim_{z \to z_I} \hat{\mathcal{M}}_L(z) \frac{1}{\hat{P}_I^2} \hat{\mathcal{M}}_R(z) = -\lim_{z \to z_I} \hat{\mathcal{M}}_L(z) \frac{z_I}{(z - z_I) P_I^2} \hat{\mathcal{M}}_R(z) \tag{3.12}$$

and therefore

$$\operatorname*{Res}_{z=z_I} \frac{\hat{\mathcal{M}}_n(z)}{z} = -\hat{\mathcal{M}}_L(z_I)\frac{1}{P_I^2}\hat{\mathcal{M}}_R(z_I)\,, \tag{3.13}$$

so we obtain a recursive result for $\mathcal{M}_n$, which reads

$$\mathcal{M}_n = \sum_I \hat{\mathcal{M}}_L(z_I)\frac{1}{P_I^2}\hat{\mathcal{M}}_R(z_I) + B_\infty\,. \tag{3.14}$$

There is up to now no general constructive method to obtain B_∞, however in some theories and for a particular choice of shift vectors r_i^μ it can be shown to vanish by showing that

$$\lim_{z\to\infty} \hat{\mathcal{M}}_n(z) = 0\,. \tag{3.15}$$

If this holds, the shift is called a *good shift* and the tree-level amplitude $\mathcal{M}_n$ is called *on-shell constructible* from lower-point amplitudes by the recursion formula

$$\boxed{\mathcal{M}_n = \sum_I \hat{\mathcal{M}}_L(z_I)\frac{1}{P_I^2}\hat{\mathcal{M}}_R(z_I)\,,} \tag{3.16}$$

where the sum runs over all possible factorization channels and helicity configurations of the particle associated with the intermediate propagator $1/P_I^2$.

The simplest known shift is the BCFW (Britto-Cachazo-Feng-Witten) shift, which defines the BCFW recursion relations: we leave all but two lines, i and j, unchanged and the shift of these two is implemented on their angle and square brackets (called $[i,j\rangle$-shift):

$$\hat{i}] = i] + zj]\,, \quad \hat{j}] = j]\,, \quad \hat{i}\rangle = i\rangle\,, \quad \hat{j}\rangle = j\rangle - zi\rangle\,, \tag{3.17}$$

which correspond to the shift vectors

$$(r_j)_{\alpha\dot{\alpha}} = -(r_i)_{\alpha\dot{\alpha}} = i_\alpha \tilde{j}_{\dot{\alpha}}\,. \tag{3.18}$$

As an example and application of the BCFW recursion relations we are going to derive the Parke-Taylor formula for color-ordered gluon maximally helicity violating amplitudes.

3.1 Application: Inductive Proof of the Parke-Taylor Formula

Any tree-level n-point Yang-Mills amplitude can be written as a sum over products of traces of multiple generators and partial amplitudes:

$$\mathcal{M}_{n,\mathrm{YM}} = g_s^{n-2} \sum_{\sigma\in S_{n-1}} \mathrm{tr}(T^{a_1}T^{\sigma(a_2)}...T^{\sigma(a_n)})A_n[1,\sigma(2),...,\sigma(n)]\,. \tag{3.19}$$

The $A_n[1,...,n]$ are called *color-ordered partial amplitudes.* They are gauge invariant and further satisfy:

1) Cyclicity: $A_n[12...n] = A_n[2...n1] = A_n[...n12]$ etc.,

2) Reflection (anti-)symmetry: $A_n[12...n] = (-1)^n A_n[n...21]$,

3) U(1) decoupling identity: $A_n[123...n] + A_n[213...n] + A_n[231...n] + ... + A_n[23...n1] = 0$.

They are calculated in terms of planar (no lines crossing) Feynman diagrams with a fixed ordering of the n external gluons.

For tree-level maximally helicity violating (MHV) gluon amplitudes $\mathcal{M}_{n,\mathrm{MHV}}$, i.e. scattering amplitudes of two negative-helicity gluons and $(n-2)$ positive-helicity gluons

$$\mathcal{M}_{n,\mathrm{MHV}}(1^+,...,i^-,...,j^-,...,n^+)\,, \tag{3.20}$$

the Parke-Taylor formula gives us a very simple expression for the color-ordered partial amplitudes of $\mathcal{M}_{n,\text{MHV}}$:

$$\boxed{A_n[1^+, ..., i^-, ..., j^-, ..., n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle (n-1)n\rangle\langle n1\rangle}\,,} \tag{3.21}$$

and, analogously for anti-MHV:

$$\boxed{A_n[1^-, ..., i^+, ..., j^+, ..., n^-] = \frac{[ij]^4}{[12][23]...[(n-1)n][n1]}\,.} \tag{3.22}$$

Since in Yang-Mills theory at tree level we have

$$\mathcal{M}_{n\,,\text{tree}}(1^\pm 2^\pm ... n^\pm) = 0\,, \quad \mathcal{M}_{n\geq 4,\text{tree}}(1^\mp 2^\pm ... n^\pm) = 0 \tag{3.23}$$

for complex momenta, the MHV amplitude is the simplest possible non-vanishing amplitude (see appendix A for a proof of this statement).

The Parke-Taylor formula can be proven inductively using the BCFW recursion relations. The first step is to make sure that B_∞ vanishes for the BCFW shifts and for color-ordered gluon tree-level amplitudes it can be shown that if i and j are adjacent lines the partial amplitudes in the large z case behave as

$$\begin{array}{ccccc} [i,j\rangle & [-,-\rangle & [-,+\rangle & [+,+\rangle & [+,-\rangle \\ A_n(z)\sim & 1/z & 1/z & 1/z & z^3 \end{array}\,, \tag{3.24}$$

where we have denoted the particles i and j by their helicities $h_{i,j} = \pm$, and if they are not adjacent then

$$A_{n,\text{not adj.}}(z) \sim \frac{1}{z} A_{n,\text{adj.}}(z)\,, \tag{3.25}$$

therefore the shifts $[-,-\rangle$, $[-,+\rangle$ and $[+,+\rangle$ are good shifts in both cases.

Remark: This implies that the entire tree-level S-matrix of Yang-Mills theory can be recursively constructed using the three-point amplitudes as the fundamental building blocks. This is the case for Einstein gravity as well.

We are only going to prove the Parke-Taylor formula for adjacent negative-helicity gluons here, which we can write without loss of generality as $A_n[1^-, 2^-, 3^+, ..., n^+]$; the case for non-adjacent negative-helicity gluons can be proven in a similar way.

3.1.1 Base Case

We already saw that the 3-gluon MHV amplitude is

$$\mathcal{M}_3(1_a^-, 2_b^-, 3_c^+) = g_s \tilde{f}^{abc} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} = g_s \Big[\text{tr}(T^a T^b T^c) - \text{tr}(T^a T^c T^b)\Big] \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \tag{3.26}$$

where here we use the convention

$$\text{tr}(T^a T^b) = \delta^{ab}\,, \quad [T^a, T^b] = \tilde{f}^{abc} T^c\,, \quad \tilde{f}^{abc} \equiv i\sqrt{2} f^{abc}\,, \tag{3.27}$$

so we find

$$A_3[1^-, 2^-, 3^+] = -A_3[1^-, 3^+, 2^-] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \quad \checkmark\,. \tag{3.28}$$

3.1.2 Induction Step

We now assume that (3.21) is true up to $(n-1)$ gluons. We use the BCFW recursion with the particular shift $[1,2\rangle$:

$$\hat{1}] = 1] + z2]\,, \quad \hat{2}] = 2]\,, \quad \hat{1}\rangle = 1\rangle\,, \quad \hat{2}\rangle = 2\rangle - z1\rangle\,, \tag{3.29}$$

which is a $[-,-\rangle$, and hence good, shift for the considered helicity configuration, and it gives:

$$A_n[1^-,2^-,3^+,...,n^+] = \sum_{k=4}^{n}\sum_{h_I=\pm} \hat{A}_{n-k+3}[\hat{1}^-,\hat{P}_I^{h_I},k^+,...,n^+]\frac{1}{P_I^2}\hat{A}_{k-1}[-\hat{P}_I^{-h_I},\hat{2}^-,3^+,...,(k-1)^+] \tag{3.30}$$

with

$$P_I = p_2+p_3+...+p_{k-1}\,, \quad \hat{P}_I = \hat{p}_2+p_3+...+p_{k-1}\,. \tag{3.31}$$

The reason why the two negative-helicity gluons always appear in different lower-point amplitudes is that if they appeared in the same amplitude their respective shifts would cancel out and the internal momentum flowing out of the subamplitude would not be shifted, thus having no pole and consequently no contribution to the recursion formula.

$A_n[-+...+]$ vanishes except for the case $n=3$, so the only non-vanishing summands are

$$\begin{aligned} A_n[1^-,2^-,3^+,...,n^+] =& \hat{A}_3[\hat{1}^-,-\hat{P}_{1n}^+,n^+]\frac{1}{P_{1n}^2}\hat{A}_{n-1}[\hat{P}_{1n}^-,\hat{2}^-,3^+,...,(n-1)^+] \\ &+ \hat{A}_{n-1}[\hat{1}^-,\hat{P}_{23}^-,4^+,...,n^+]\frac{1}{P_{23}^2}\hat{A}_3[-\hat{P}_{23}^+,\hat{2}^-,3^+]\,, \end{aligned} \tag{3.32}$$

with

$$\hat{P}_{1n} \equiv \hat{p}_1+p_n\,, \quad \hat{P}_{23} \equiv \hat{p}_2+p_3 \tag{3.33}$$

and analogously for the unhatted momenta.

$\hat{A}_3[\hat{1}^-,-\hat{P}_{1n}^+,n^+]$ is completely determined by little group scaling, as we already saw:

$$\hat{A}_3[\hat{1}^-,-\hat{P}_{1n}^+,n^+] = \frac{[\hat{P}_{1n}n]^3}{[n\hat{1}][\hat{1}\hat{P}_{1n}]}\,, \tag{3.34}$$

where we have fixed the convention

$$(-p)\rangle = -p\rangle\,, \quad (-p)] = p]\,. \tag{3.35}$$

Since the lower-point amplitudes in the recursion are evaluated at the poles z_I where the shifted momenta $\hat{P}_I$ go on-shell, we have in particular $\hat{P}_{1n}^2 = 0$, which implies

$$0 = (\hat{p}_1+p_n)^2 = 2\hat{p}_1\cdot p_n = \langle\hat{1}n\rangle[\hat{1}n] \underset{\langle\hat{1}=\langle 1}{\overset{\uparrow}{=}} \langle 1n\rangle[\hat{1}n] \tag{3.36}$$

and since for generic momenta $\langle 1n\rangle \neq 0$ this implies $[\hat{1}n]=0$, so it looks like (3.34) diverges because the denominator is vanishing. However all is not lost, since this implies that every bracket appearing in (3.34) is vanishing:

$$\hat{P}_{1n}\rangle[\hat{P}_{1n}n] = \sigma^\mu(\hat{p}_1+p_n)_\mu n] = \hat{1}\rangle[\hat{1}n] = 0\,, \tag{3.37}$$

$$\hat{P}_{1n}\rangle[\hat{P}_{1n}\hat{1}] = \sigma^\mu(\hat{p}_1+p_n)_\mu \hat{1}] = n\rangle[n\hat{1}] = 0 \tag{3.38}$$

and since we have three powers of $[\hat{1}n]$ in the numerator and only two in the denominator we conclude:

$$\hat{A}_3[\hat{1}^-,-\hat{P}_{1n}^+,n^+] \sim [\hat{1}n] = 0\,, \tag{3.39}$$

which means that all we are left with is

$$A_n[1^-,2^-,3^+,...,n^+] = \hat{A}_{n-1}[\hat{1}^-,\hat{P}_{23}^-,4^+,...,n^+]\frac{1}{P_{23}^2}\hat{A}_3[-\hat{P}_{23}^+,\hat{2}^-,3^+] \tag{3.40}$$

$$= \frac{\langle\hat{1}\hat{P}_{23}\rangle^4}{\langle\hat{1}\hat{P}_{23}\rangle\langle\hat{P}_{23}4\rangle\langle 45\rangle...\langle n\hat{1}\rangle}\cdot\frac{1}{\langle 23\rangle[23]}\cdot\frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]}\,. \tag{3.41}$$

This can be shown to be independent of the shifted momentum $\hat{P}_{23}$ and we can do so in two steps: first by combining the brackets in the numerator as

$$\langle\hat{1}\hat{P}_{23}\rangle[3\hat{P}_{23}] = -\langle\hat{1}\hat{P}_{23}\rangle[\hat{P}_{23}3] = -\langle\hat{1}\sigma^\mu\hat{P}_{23\mu}3] = -\langle\hat{1}\sigma^\mu(\hat{p}_2+p_3)_\mu 3] = -\langle\hat{1}\hat{2}\rangle[\hat{2}3] \underset{\substack{\uparrow\\ \langle\hat{2}=\langle 2-z\langle 1\,,\ [\hat{2}=[2}}}{=} -\langle 1(2\rangle - z1\rangle)[23] = -\langle 12\rangle[23]\,, \tag{3.42}$$

and in the denominator as

$$\langle\hat{P}_{23}4\rangle[\hat{P}_{23}\hat{2}] = -\langle 4\sigma^\mu\hat{P}_{23\mu}\hat{2}] = -\langle 4\sigma^\mu p_{3\mu}2] = -\langle 43\rangle[32] = -\langle 34\rangle[23]\,, \tag{3.43}$$

which gives us, at last:

$$A_n[1^-,2^-,3^+,...,n^+] = \frac{-\langle 12\rangle^3[23]^3}{(-\langle 34\rangle[23])\langle 23\rangle[23]^2\langle 45\rangle...\langle n1\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle...\langle n1\rangle}\,, \tag{3.44}$$

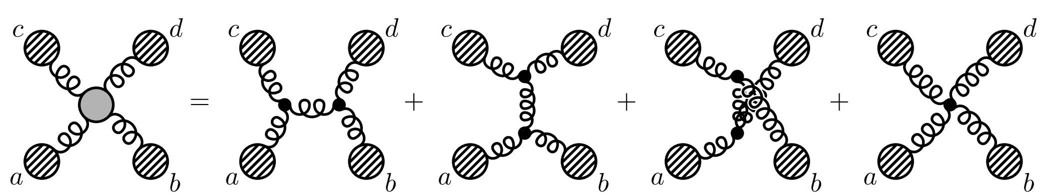
thus completing the induction step and proving the Parke-Taylor formula for the case of adjacent negative-helicty gluons.

4 Color-Kinematics Duality and Double Copy

The notion of color-kinematics duality first arose from the study of pure tree-level Yang-Mills scattering amplitudes, which we can write as

$$\mathcal{M}_{n,\mathrm{YM}} = g_s^{n-2}\sum_{i\in\Gamma}\frac{c_i n_i}{s_i}\,, \tag{4.1}$$

with Γ the set of Feynman diagrams contributing to the amplitude, c_i and n_i the color and kinematic numerators of the i-th diagram and in the denominator we have the contribution from the propagators appearing in the i-th diagram s_i. Here we assume that the local contributions from four-point vertices at any junction of the diagram have been distributed according to their color factors onto the non-local contributions containing three-point vertices, which can always be done, and this defines the kinematic numerators:



$$\mathcal{M}_i = M_{\text{blobs}}^{abcd}\left[\tilde{f}^{abe}\tilde{f}^{ecd}\frac{\hat{n}_s + s\hat{n}_{4,s}}{s} + \tilde{f}^{cae}\tilde{f}^{ebd}\frac{\hat{n}_t + t\hat{n}_{4,t}}{t} + \tilde{f}^{ade}\tilde{f}^{ebc}\frac{\hat{n}_u + u\hat{n}_{4,u}}{u}\right] \tag{4.2}$$

$$\equiv M_{\text{blobs}}^{abcd}\left[\tilde{f}^{abe}\tilde{f}^{ecd}\frac{n_s}{s} + \tilde{f}^{cae}\tilde{f}^{ebd}\frac{n_t}{t} + \tilde{f}^{ade}\tilde{f}^{ebc}\frac{n_u}{u}\right]. \tag{4.3}$$

What we understand under color-kinematics duality is:

1) The observation that the kinematic numerators have the same symmetry properties as the color factors under exchange of pairs of legs in a diagram, since these are induced by the c_i's themselves,

$$c_i \to -c_i \quad \Leftrightarrow \quad n_i \to -n_i \,, \tag{4.4}$$

2) The *conjecture* (called Bern-Carrasco-Johansson (BCJ) conjecture) that if a triple of color factors in an amplitude is found which satisfies a Jacobi-type relation,

$$c_i + c_j + c_k = 0 \,, \tag{4.5}$$

then a form of the corresponding kinematic numerators can always be found such that they do so as well:

$$n_i + n_j + n_k = 0 \,. \tag{4.6}$$

This can be achieved by means of a generalized gauge transformation

$$n_i \to n_i + \Delta_i \,, \quad \sum_{i\in\Gamma} \frac{c_i \Delta_i}{s_i} \equiv 0 \,, \tag{4.7}$$

which contains the usual gauge transformations as a subset:

$$\varepsilon^a_\mu(k) \to \varepsilon^a_\mu(k) + w^a k_\mu + \mathcal{O}(g_s) \,, \tag{4.8}$$

$$n_i \to n_i + \delta_i \,, \quad \sum_{i\in\Gamma} \frac{c_i \delta_i}{s_i} = 0 \,. \tag{4.9}$$

An example of the latter property is the four-gluon tree-level amplitude: we have

$$c_s = \tilde{f}^{abe}\tilde{f}^{ecd} \,, \quad c_t = \tilde{f}^{cae}\tilde{f}^{ebd} \,, \quad c_u = \tilde{f}^{ade}\tilde{f}^{ebc} \,, \tag{4.10}$$

$$c_s + c_t + c_u = \tilde{f}^{abe}\tilde{f}^{ecd} + \tilde{f}^{cae}\tilde{f}^{ebd} + \tilde{f}^{ade}\tilde{f}^{ebc} = 0 \tag{4.11}$$

and using the textbook three-gluon and four-gluon vertex Feynman rules one immediately finds

$$n_s + n_t + n_u = 0 \,. \tag{4.12}$$

Therefore, since the color and kinematic pieces of the amplitude exhibit the same algebraic properties, we say that they are *dual*.

The double copy procedure makes use of this to obtain a gravitational amplitude from a gauge theory one. Once we have found a suitable form of the kinematic numerators, such that they satisfy color-kinematics duality, we can replace the c_i with a second copy of the n_i, which we denote as $\tilde{n}_i$, and the couplings $g_s \to \kappa/2$, to get

$$\boxed{\mathcal{M}_{n,\mathrm{YM}} \to \mathcal{M}_{n,\mathrm{grav}} = \left(\frac{\kappa}{2}\right)^{n-2} \sum_{i\in\Gamma} \frac{n_i \tilde{n}_i}{s_i}} \tag{4.13}$$

and we identify the direct product of the two copies of the color-stripped polarization vectors as a polarization tensor:

$$\varepsilon_\mu \tilde{\varepsilon}_\nu \equiv E_{\mu\nu} \,. \tag{4.14}$$

We claim this to be a gravitational amplitude, which should satisfy the minimal requirement of being gauge invariant under linearized diffeomorphisms; it is in fact invariant under an enhanced version of linearized diffeomorphisms, which is induced by the transformation behavior of the two copies of the kinematic numerators and by color-kinematics duality. We know that the color-stripped polarization vectors and the n_i's transform as

$$\varepsilon_\mu(k) \to \varepsilon_\mu(k) + k_\mu \,, \quad \tilde{\varepsilon}_\mu(k) \to \tilde{\varepsilon}_\mu(k) + k_\mu \,, \tag{4.15}$$

$$n_i \to n_i + \delta_i \,, \quad \tilde{n}_i \to \tilde{n}_i + \tilde{\delta}_i \,, \tag{4.16}$$

with both δ_i and $\tilde{\delta}_i$ belonging to the *same* set of gauge transformations, so

$$\mathcal{M}_{n,\text{grav}} \to \mathcal{M}_{n,\text{grav}} + \sum_{i\in\Gamma} \frac{\delta_i \tilde{n}_i}{s_i} + \sum_{i\in\Gamma} \frac{n_i \tilde{\delta}_i}{s_i} + \mathcal{O}(\delta_i^2)\,. \tag{4.17}$$

We expressed the gauge invariance of the Yang-Mills amplitude through (4.9), and the vanishing of the sum must be purely due to the algebraic properties of the color factors, which are shared by the kinematic numerators in their dual form, which means that

$$\sum_{i\in\Gamma} \frac{\delta_i \tilde{n}_i}{s_i} = \sum_{i\in\Gamma} \frac{n_i \tilde{\delta}_i}{s_i} = 0 \tag{4.18}$$

and the amplitude is indeed gauge invariant. Assuming the validity of color-kinematics duality the double copy formula can be proven using on-shell recursion relations.

Remark: The reason why this is an enhanced version of the usual linearized diffeomorphisms is that $E_{\mu\nu}$ transforms as

$$E_{\mu\nu}(k) \to E_{\mu\nu}(k) + \varepsilon_\mu(k) k_\nu + k_\mu \tilde{\varepsilon}_\nu(k) + \mathcal{O}(k^2)\,, \tag{4.19}$$

so we have two different gauge parameters, ε_μ, $\tilde{\varepsilon}_\mu$, instead of the usual transformation behavior of the graviton polarization vector

$$\varepsilon_{\mu\nu}(k) \to \varepsilon_{\mu\nu}(k) + \xi_\mu(k) k_\nu + \xi_\nu(k) k_\mu \tag{4.20}$$

with only one ξ_μ.

Since the two copies of the polarization vectors are independent, the tensor field $H_{\mu\nu}$ associated to $E_{\mu\nu}$ has no particular symmetry in its indices, so we can split it into a symmetric-traceless ($h_{\mu\nu}$), antisymmetric ($B_{\mu\nu}$) and trace (ϕ) part:

$$H_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu}\phi\,, \tag{4.21}$$

with $C_{\mu\nu}$ a symmetric coefficient. We identify $h_{\mu\nu}$ with the graviton field, $B_{\mu\nu}$ is called the Kalb-Ramond field and ϕ is identified with the dilaton. Hence the double copy procedure gives us an "$\mathcal{N}=0$"-supergravity amplitude from a Yang-Mills amplitude. If we only want to keep the pure Einstein gravity piece of the amplitude, we just replace the polarization tensor with its symmetric-traceless part:

$$E_{\mu\nu} \to \varepsilon_{\mu\nu}\,, \tag{4.22}$$

which is equivalent to assuming that all external states are graviton states, and this is enough to project out the pure Einstein gravity piece.

Example 1: We saw that the only two possible three-point amplitudes in Yang-Mills theory read

$$\mathcal{M}(1_a^-, 2_b^-, 3_c^+) = g_s \tilde{f}^{abc} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \quad \mathcal{M}(1_a^+, 2_b^+, 3_c^-) = g_s \tilde{f}^{abc} \frac{[12]^3}{[23][31]} \tag{4.23}$$

and the analogous gravity amplitudes are

$$\mathcal{M}(1^{--}, 2^{--}, 3^{++}) = \frac{\kappa}{2} \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,, \quad \mathcal{M}(1^{++}, 2^{++}, 3^{--}) = \frac{\kappa}{2} \frac{[12]^6}{[23]^2[31]^2}\,, \tag{4.24}$$

so we just need to replace

$$\tilde{f}^{abc} \to \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \text{ or } \frac{[12]^3}{[23][31]}\,, \quad g_s \to \frac{\kappa}{2}\,. \tag{4.25}$$

This is the basis for the recursive proof of the double copy formula.

Example 2: Since the kinematic numerators of the four-gluon amplitude

$$\mathcal{M}_{4,\mathrm{YM}} = ig_s^2\left[\tilde{f}^{abe}\tilde{f}^{ecd}\frac{n_s}{s} + \tilde{f}^{cae}\tilde{f}^{ebd}\frac{n_t}{t} + \tilde{f}^{ade}\tilde{f}^{ebc}\frac{n_u}{u}\right] \tag{4.26}$$

are already in their dual form, we just use them to replace the color factors and get

$$\mathcal{M}_{4,\mathrm{grav}} = i\left(\frac{\kappa}{2}\right)^2\left[\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right]. \tag{4.27}$$

A Vanishing of All-Plus and Single-Minus Tree-Level Amplitudes in Yang-Mills Theory

We can write any tree-level n-point Yang-Mills amplitude schematically as

$$\mathcal{M}_n(1,...,n) \sim \sum_{\text{diagrams}} \frac{\sum \left(\prod \varepsilon_i \cdot \varepsilon_j\right)\left(\prod \varepsilon_i \cdot p_j\right)\left(\prod p_i \cdot p_j\right)}{\prod P_I^2} \tag{A.1}$$

and we have the following mass dimensions in the numerator and denominators:

$$[\mathcal{M}_n]_m = 4 - n = [\text{numerator}]_m - [\text{denominator}]_m = (n-2) - 2(n-3)\,, \tag{A.2}$$

therefore we see that the highest allowed number of gluon momenta appearing in the numerator is $(n-2)$. Using

$$\varepsilon_-^{*\,\mu}(k,r) = -\frac{1}{\sqrt{2}}\frac{[r\gamma^\mu k\rangle}{[rk]}\,, \quad \varepsilon_+^{*\,\mu}(k,r) = -\frac{1}{\sqrt{2}}\frac{[k\gamma^\mu r\rangle}{\langle rk\rangle} \tag{A.3}$$

we find:

$$\varepsilon_{i+} \cdot \varepsilon_{j+} \sim \langle r_i r_j\rangle\,, \quad \varepsilon_{i-} \cdot \varepsilon_{j-} \sim [r_i r_j]\,, \quad \varepsilon_{i-} \cdot \varepsilon_{j+} \sim \langle i r_j\rangle [j r_j]\,. \tag{A.4}$$

A.1 All-Plus Amplitude

We choose all the reference vectors to be the same: $r_1 = r_2 = ... = r_n = r$, which achieves $\varepsilon_{i+} \cdot \varepsilon_{j+} = 0$ for all i, j. All open polarization vector indices thus need to be contracted using gluon momenta, $\varepsilon_i \cdot k_j$, which means that we need n of them. But this exceeds the maximal allowed number, which is $(n-2)$, so we conclude that:

$$\mathcal{M}_{n,\text{tree}}(1^+, ..., n^+) = 0\,. \tag{A.5}$$

A.2 Single-Minus Amplitude

For $n \geq 4$ we can choose $r_2 = r_3 = ... = r_n = p_1$, which gives us $\varepsilon_{i+} \cdot \varepsilon_{j+} = 0$ and $\varepsilon_{1-} \cdot \varepsilon_{j+} = 0$ for all i, j. Therefore we would again need to contract all polarization vector indices with momenta, so we would need n of them, which is more than allowed, so we again conclude

$$\mathcal{M}_{n\geq 4,\text{tree}}(1^-, 2^+, ..., n^+) = 0\,. \tag{A.6}$$

Notice that the choice $r_2 = r_3 = p_1$ is not allowed for the three-particle case, since three-particle kinematics for complex momenta enforces

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0 \quad \text{or} \quad [12] = [23] = [31] = 0\,, \tag{A.7}$$

therefore, depending on whether we choose the holomorphic or anti-holomorphic configuration, the polarization vectors of positive or negative helicity of the gluons 2 and 3 are ill-defined because their denominator would vanish, confront (A.3). Therefore this trick does not work in the three-particle case and the amplitudes are non-vanishing for complex momenta, which is indeed what we found in the previous seminar.

A.3 Maximally Helicity Violating Amplitude

For the MHV helicity configuration we can achieve the largest number of vanishing polarization vector inner products by choosing $r_1 = r_2 = p_n$ and $r_3 = r_4 = ... = r_n = p_1$, which lets all $\varepsilon_i \cdot \varepsilon_j = 0$ except for $\varepsilon_{2-} \cdot \varepsilon_{i+}$ for $i = 3, ..., n-1$. So we can contract the open index of ε_{2-} with another polarization vector and the rest of the polarization vectors need to be contracted using the gluon momenta, so

$$\mathcal{M}_n(1^-, 2^-, 3^+, .., n^+) \sim \sum_{\text{diagrams}} \frac{\sum (\varepsilon_{2-} \cdot \varepsilon_{i+})(\varepsilon_j \cdot k_l)^{n-2}}{\prod P_I^2} \tag{A.8}$$

and we exactly saturate the number of allowed momenta appearing in the numerator, so the MHV amplitude is the first non-vanishing amplitude and it has the simplest possible helicity configuration.

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