Amplitudes and the Spinor-Helicity Formalism

Thomas Bader

Overview

In this seminar, we will introduce the spinor-helicity formalism and work out a few important implications. We will do so as follows:

- We will start by discussing the motivation for a new formalism.
- Next, we will introduce the spinor-helicity formalism and show several identities.
- Third, we will discuss vector boson polarizations, how to represent them in the spinor helicity formalism and how we can directly see their appearance connected to the Ward identity.
- Furthermore, we will discuss little group covariance, and what the implications are for scattering amplitudes.
- Last, we will learn how to construct three-point-amplitudes in the spinor-helicity formalism simply by considering the little group scaling of the external particles and a tiny bit of dimensional analysis. In that part, we will also show a few examples.

In the next lectures, we will see important applications. For example, we can use this formalism to easily prove the uniqueness of Yang-Mills theory, calculate n-point functions by only considering three-point amplitudes and using recursion relations and also see its power in quantum gravity and determining whether theories can be consistently defined.

1 Motivation

Consider the general process of obtaining Amplitudes in scattering processes in quantum field theory:

simple action	$\longrightarrow$	**complicated** and **redundant** Feynman rules	$\longrightarrow$	**simple** amplitude squares and cross sections

$$S = -\frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^{a\mu\nu} \longrightarrow \mathcal{M} = -ig_s^2\left\{\text{horrible mess }^1\right\} \longrightarrow \overline{|\mathcal{M}|}^2 = \frac{9g_s^4}{2}\left(3 - \frac{su}{t^2} - \frac{ut}{s^2} - \frac{st}{u^2}\right)$$

The redundancy is a result of the gauge redundancy. Note that for a gauge field A or the graviton field h, the following transformations have to leave the theory invariant:

$$A_\mu \longrightarrow UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \qquad \text{(gauge invariance)},$$
$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu\theta_\nu + \partial_\nu\theta_\mu \qquad \text{(diffeomorphism)}.$$

So can we somehow eliminate the complicated step in between by eliminating gauge redundancy? The solution to this problem turns out to be the use of an on-shell formalism. By only considering the outgoing particles, we eliminate the connection to quantised fields and therefore gauge redundancies. Furthermore, we simultaneously reduce the degrees of freedom in the same step. I.e. for the above examples, a priori the gauge field A_μ has four degrees of freedom and the graviton field even has ten, which both reduce down to two in the case of on-shell particles.

We can even dig down deeper and state that we can have different actions resulting in the same S-matrix and thus the same physics, however we will not discuss this further.

During this seminar series, we will introduce the spinor-helicity formalism and show its power in calculating amplitudes simply from symmetry and dimensional analysis massively simplifying complicated amplitude computations. For example, we will see in the next lecture that the n

spin-1 particle amplitude with only two "−" helicities will have the simple form of the Parke-Taylor formula:

$$\mathcal{M}(1^+,...,i^-,...,j^-,...,n^+) = \frac{\langle i\,j\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle n1\rangle}.$$

2 Introduction

Let us first think about the way momenta transform under Lorentz transformation. Their representation as four-vectors p^μ is $\left(\frac{1}{2},\frac{1}{2}\right)$, however, we can equivalently represent them by bispinors $\not{p}$ (more precisely $p_{\alpha\dot{\alpha}}$) corresponding to the representation $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$. For all real momenta this means that we can write the contraction of the momentum with the Pauli vector $\sigma^\mu = (\mathbb{1},\sigma^i)$ and $\overline{\sigma}^\mu = (\mathbb{1},-\sigma^i)$.

$$\sigma^\mu p_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}$$

as a product of two Weyl spinors

$$\sigma^\mu p_\mu = u(p)u^\dagger(p)^2, \tag{1}$$

and by means of $\overline{\sigma}^\mu = \sigma^2\sigma^{\mu*}\sigma^{23}$ and $\left(\sigma^2\right)^\dagger = \sigma^2$ equivalently

$$\overline{\sigma}^\mu p_\mu = \sigma^2\sigma^{\mu*}\sigma^2 p_\mu = i\sigma^2\left(u(p)u^\dagger(p)\right)^*\left(i\sigma^2\right)^\dagger = i\sigma^2 u^*(p)\left(i\sigma^2 u^*(p)\right)^\dagger. \tag{2}$$

That these representations are indeed equivalent is due to the fact, that the four Pauli matrices form a basis for the vector space of 2×2 Hermitian matrices witch is isomorphic to $\mathbb{R}^4$.

Now note that for real momenta, external spin-$\frac{1}{2}$ states are described by Dirac spinors

$$U(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix},$$

where for real momenta

$$u_L(p) = i\sigma^2 u_R^*(p) \tag{3}$$

holds. Combining Eqs. (1), (2) and (3), we find that we can write the product of the momentum with the gamma matrices (in the Weyl representation) as

$$\not{p} = p_\mu\gamma^\mu = p_\mu\begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_L(p)u_L^\dagger(p) \\ u_R(p)u_R^\dagger(p) & 0 \end{pmatrix}$$

[1] The given example considers the scattering process $g_{1,a}g_{2,b} \to g_{3,c}g_{4,d}$:

$$\begin{aligned}
\text{horrible mess} =& \frac{f^{abe}f^{cde}}{s_{12}}\left[(\epsilon_1\cdot\epsilon_2)(p_1-p_2)^\mu + 2\epsilon_2^\mu(p_2\cdot\epsilon_1) - 2\epsilon_1^\mu(p_1\cdot\epsilon_2)\right] \\
&\times\left[(\epsilon_3\cdot\epsilon_4)(p_3-p_4)_\mu + 2\epsilon_{4\mu}(p_4\cdot\epsilon_3) - 2\epsilon_{3\mu}(p_3\cdot\epsilon_4)\right] \\
&+ \frac{f^{ade}f^{cbe}}{s_{14}}\left[(\epsilon_1\cdot\epsilon_4)(p_1-p_4)^\mu + 2\epsilon_4^\mu(p_4\cdot\epsilon_1) - 2\epsilon_1^\mu(p_1\cdot\epsilon_4)\right] \\
&\times\left[(\epsilon_3\cdot\epsilon_2)(p_3-p_2)_\mu + 2\epsilon_{2\mu}(p_2\cdot\epsilon_3) - 2\epsilon_{3\mu}(p_3\cdot\epsilon_2)\right] \\
&+ \frac{f^{ace}f^{bde}}{s_{13}}\left[(\epsilon_1\cdot\epsilon_3)(p_1-p_3)^\mu + 2\epsilon_3^\mu(p_3\cdot\epsilon_1) - 2\epsilon_1^\mu(p_1\cdot\epsilon_3)\right] \\
&\times\left[(\epsilon_2\cdot\epsilon_4)(p_2-p_4)_\mu + 2\epsilon_{4\mu}(p_4\cdot\epsilon_2) - 2\epsilon_{2\mu}(p_2\cdot\epsilon_4)\right] \\
&+\left[f^{abe}f^{cde}(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4) - (\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\right] \\
&+\left[f^{ace}f^{bde}(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4) - (\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\right] \\
&+\left[f^{ade}f^{bce}(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4) - (\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)\right]
\end{aligned}$$

[2] An explicit decomposition for a massless four-vector is

$$u(p) = \frac{z}{\sqrt{p^0-p^3}}\begin{pmatrix} p^0-p^3 \\ -p^1-ip^2 \end{pmatrix}; \quad u^\dagger(p) = \frac{z^{-1}}{\sqrt{p^0-p^3}}\left(p^0-p^3\ ,\ -p^1+ip^2\right).$$

[3] Reminder:

$$i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

What we see here is essentially just the completeness relation for massless external spin-$\frac{1}{2}$ fermions

$$\not{p} = U_L(p)\overline{U}_L(p) + U_R(p)\overline{U}_R(p) = \begin{pmatrix} 0 & u_L(p)u_L^\dagger(p) \\ u_R(p)u_R^\dagger(p) & 0 \end{pmatrix},$$

where

$$U_{L/R}(p) = P_{L/R}U(p) = \left(\frac{1 \mp \gamma_5}{2}\right) U(p).$$

Now, we will simplify the notation by introducing angle and square brackets:

$$p\rangle \equiv \begin{pmatrix} 0 \\ u_R(p) \end{pmatrix} \; ; \quad p] \equiv \begin{pmatrix} u_L(p) \\ 0 \end{pmatrix} \; ; \quad [p \equiv \left(u_R^\dagger(p) \; , \; 0\right) \; ; \quad \langle p \equiv \left(0 \; , \; u_L^\dagger(p)\right) \tag{4}$$

Note that we can rewrite $\not{p}$ by

$$\not{p} = U_L(p)\overline{U}_L(p) + U_R(p)\overline{U}_R(p) = p\rangle[p + p]\langle p.$$

We also see the mass dimension of the angle and square brackets given by $[p\rangle]_{\mathrm{M}} = [\langle p]_{\mathrm{M}} = [p]]_{\mathrm{M}} = [[p]_{\mathrm{M}} = \frac{1}{2}$[4]. We will proceed by showing a few identities.

1. Using the definition in Eq. (4), we immediately see

$$0 = \langle pq] = [pq\rangle = \langle p\gamma^\mu q\rangle = [p\gamma^\mu q] = 0 \;\; \forall p, q$$

2. For real momenta, the brackets are conjugate to each other:

$$\begin{aligned} \langle pq\rangle &= \overline{U}_L(p)U_R(q) \\ &= u_{R\alpha}(q)u_{L\alpha}^*(p) \\ &= \left(u_R^\dagger(q)u_L(p)\right)^* \\ &= \left(\overline{U}_R(q)U_L(p)\right)^* \\ &= [qp]^*. \end{aligned} \tag{5}$$

Note that the spinors $u_{L/R}$ have to be viewed as two-component vectors of real or complex numbers and not as Grassmann numbers, hence we can exchange their position without an additional minus sign.

3. For real momenta, the brackets are "square roots" of the momentum product:

$$\langle pq\rangle[qp] = \mathrm{tr}\{U_L(p)\overline{U}_L(p)U_R(q)\overline{U}_R(q)\} = \mathrm{tr}\left\{\not{p}\left(\frac{1+\gamma_5}{2}\right)\not{q}\left(\frac{1-\gamma_5}{2}\right)\right\} = 2p\cdot q, \tag{6}$$

where in the last step, we have used $\mathrm{tr}\{\gamma^\mu\gamma^\nu\} = 4\eta^{\mu\nu}$ $\mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma_5\} = 0$. By now combining Eqs. (5) and (6), we can conclude that for all real momenta

$$\langle pq\rangle = \sqrt{p\cdot q}e^{i\phi} \; , \qquad [qp] = \sqrt{p\cdot q}e^{-i\phi}$$

holds. Note that this again verifies $[p\rangle]_{\mathrm{M}} = [\langle p]_{\mathrm{M}} = [p]]_{\mathrm{M}} = [[p]_{\mathrm{M}} = \frac{1}{2}$.

4. The brackets are antisymmetric:

$$\langle pq\rangle = u_L^\dagger(p)u_R(q) = u_{L\alpha}^*(p)\left(i\sigma^2\right)_{\alpha\beta}u_{L\beta}^*(q) = -u_{L\beta}^*(q)\left(i\sigma^2\right)_{\beta\alpha}u_{L\alpha}^*(p) = -\langle qp\rangle \tag{7}$$

$$[pq] = u_R^\dagger(p)u_L(q) = u_{R\alpha}^*(p)\left(i\sigma^2\right)_{\alpha\beta}u_{R\beta}^*(q) = -u_{R\beta}^*(q)\left(i\sigma^2\right)_{\beta\alpha}u_{R\alpha}^*(p) = -[qp]. \tag{8}$$

One important result of antisymmetry is that

$$\langle pp\rangle = [pp] = 0. \tag{9}$$

[4] As the usual notation for the mass dimension [...] and the product of two angle brackets look the same, we denote the mass dimension with an index M.

5. For vector currents, we find

$$\begin{aligned}\langle p\gamma^\mu q] =& u_L^\dagger(p)\overline{\sigma}^\mu u_L(q) = u_L^\dagger(p)\left(i\sigma^2\right)^\dagger \sigma^{\mu *} i\sigma^2 u_L(q)\\ =& u_R^T(p)\sigma^{\mu *} u_R^*(q) = u_R^\dagger(q)\sigma^\mu u_R(p) = [q\gamma^\mu p\rangle,\end{aligned}$$

where we again used $\overline{\sigma}^\mu = \left(i\sigma^2\right)^\dagger \sigma^{\mu *} i\sigma^2$, as well as $\sigma^{\mu *} = \sigma^{\mu T}$. It also follows that

$$\langle p\gamma^\mu q]^* = \left(u_R^T(p)\sigma^{\mu *}u_R^*(q)\right)^* = u_R^\dagger(p)\sigma^\mu u_R(q) = [p\gamma^\mu q\rangle \tag{10}$$

6. Furthermore we find the Dirac algebra becoming particularly easy

$$\begin{aligned}\langle p\gamma^\mu q]\langle r\gamma_\mu s] =& u_L^\dagger(p)\overline{\sigma}^\mu u_L(q) u_L^\dagger(r)\overline{\sigma}_\mu u_L(s)\\ =& 2u_L^\dagger(p) i\sigma^2 u_L^*(r) u_L^T(q) i\sigma^2 u_L(s)\\ =& -2u_L^\dagger(p)u_R(r)u_R^\dagger(q)u_L(s)\\ =& -2\langle pr\rangle[qs]\\ =& 2\langle pr\rangle[sq],\end{aligned} \tag{11}$$

where we used $\left(\overline{\sigma}^\mu\right)_{\alpha\beta}\left(\overline{\sigma}_\mu\right)_{\gamma\delta} = 2\left(i\sigma^2\right)_{\alpha\gamma}\left(i\sigma^2\right)_{\beta\delta}$.

7. It is useful to note

$$\langle pk\rangle[kq] = \langle p\not{k}q].$$

8. Considering momentum conservation $\sum_i p_i = 0$, hence $\sum_i \not{p}_i = 0$, we find the following equation has to be fulfilled for a scattering process:

$$\sum_j p_j\rangle[p_j = \sum_j p_j]\langle p_j = 0.$$

9. Since spinors are two-dimensional, we can express every spinor in terms of any two others that are not proportional to each other:

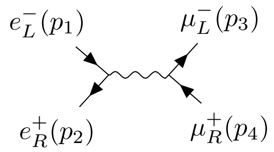
$$1\rangle = \frac{\langle 13\rangle}{\langle 23\rangle}2\rangle - \frac{\langle 12\rangle}{\langle 23\rangle}3\rangle.$$

Contracting with an arbitrary $\langle 4$ gives the Schouten identity

$$\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle = 0.$$

Let us have a look at a simple example now: $e_R^+ + e_L^- \longrightarrow \mu_R^+ + \mu_L^-$ in massless QED, for simplicity we will write $p_i\rangle \equiv i\rangle$:

$i\mathcal{M} =$



$$= \langle 2(-ie\gamma^\mu)1]\frac{-i\eta_{\mu\nu}}{s}\langle 3(-ie\gamma^\nu)4] = \frac{ie^2}{s}\langle 2\gamma^\mu 1]\langle 3\gamma_\mu 4] = \frac{2ie^2}{s}[41]\langle 23\rangle.$$

The amplitude square now is given by

$$|\mathcal{M}|^2 = 4e^4\frac{[41]\langle 14\rangle\langle 23\rangle[32]}{s^2} = 4e^4\frac{u^2}{s^2}.$$

Compare this to the Feynman method, i.e. writing out the angle and square brackets as spinors $U(p)$, squaring the expression, expressing it as a trace and slowly work out all the factors, we can already see the effectiveness of this method.

3 Vector Boson Polarizations

The spinor-helicity formalism is especially powerful when discussing massless vector boson polarizations. We will postulate the following form for the polarization vectors:

$$\epsilon_-^{\mu *}(k,r) = -\frac{1}{\sqrt{2}}\frac{[r\gamma^\mu k\rangle}{[rk]}\,; \qquad \epsilon_+^{\mu *}(k,r) = \frac{1}{\sqrt{2}}\frac{[k\gamma^\mu r\rangle}{\langle rk\rangle} \tag{12}$$

Here, r is a so-called reference momentum, which is arbitrary except for the restrictions $r^2 = 0$ and $k \cdot r \neq 0$. We will later see that the arbitrariness of this vector is essentially a manifestation of gauge invariance. We will start by proving that we can indeed write polarization vectors as given in Eq. (12):

1. First, we require $[\epsilon_+^*(k,r)]^* = \epsilon_-^*(k,r)$:
$$[\epsilon_+^*(k,r)]^* = \frac{1}{\sqrt{2}} \frac{[k\gamma^\mu r\rangle^*}{\langle rk\rangle^*} = \frac{1}{\sqrt{2}} \frac{[r\gamma^\mu k\rangle}{[kr]} = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu k\rangle}{[rk]}, \tag{13}$$
where we have used Eqs. (5), (7), (8) and (10).

2. We require orthogonality: $k_\mu \epsilon_\pm^{*\mu}(k,r) = 0$. As $\not{k}k\rangle = [k\not{k} = 0$, we immediately see
$$k_\mu \epsilon_-^{*\mu}(k,r) = -\frac{1}{\sqrt{2}} \frac{[r\not{k}k\rangle}{[rk]} = 0,$$
$$k_\mu \epsilon_+^{*\mu}(k,r) = \frac{1}{\sqrt{2}} \frac{[k\not{k}r\rangle}{\langle rk\rangle} = 0.$$

3. The third aspect of polarization vectors is $\epsilon_\pm^*(k,r) \cdot \left(\epsilon_\mp^*(k,r)\right)^* = \epsilon_\pm^*(k,r) \cdot \epsilon_\pm^*(k,r) = 0$:
$$\epsilon_+^*(k,r) \cdot \epsilon_+^*(k,r) \propto [k\gamma^\mu r\rangle[k\gamma_\mu r\rangle = 2\langle rr\rangle[kk] = 0.$$
Where in the last step, we used Eq. (9).

4. The polarization vectors have to be space-like and normalized to unit length $\left|\epsilon_\pm^*(k,r)\right|^2 = \epsilon_\pm^*(k,r) \cdot \epsilon_\pm(k,r) = -1$. By means of Eq. (11), we find
$$\epsilon_+^*(k,r) \cdot \epsilon_+(k,r) = \frac{1}{2} \frac{\langle r\gamma^\mu k]\langle k\gamma_\mu r]}{\langle rk\rangle[kr]} = \frac{\langle rk\rangle[rk]}{\langle rk\rangle[kr]} = -1.$$

5. The last requirement is the freedom of choice of the reference momenta. For this, we may observe the difference between two polarization vectors with different reference momenta:
$$\begin{aligned}\sqrt{2}\left(\epsilon_+^{*\mu}(k,r) - \epsilon_+^{*\mu}(k,s)\right) =& \frac{[k\gamma^\mu r\rangle}{\langle rk\rangle} - \frac{[k\gamma^\mu s\rangle}{\langle sk\rangle} \\ =& \frac{1}{\langle rk\rangle\langle sk\rangle}\left(\langle sk\rangle[k\gamma^\mu r\rangle - \langle rk\rangle[k\gamma^\mu s\rangle\right) \\ =& \frac{1}{\langle rk\rangle\langle sk\rangle}\langle s\left(\not{k}\gamma^\mu + \gamma^\mu\not{k}r\rangle\right) \\ =& \frac{2k^\mu\langle sr\rangle}{\langle rk\rangle\langle sk\rangle}.\end{aligned}$$
Thus any change in reference momentum only leads to a change of the polarization vector proportional to the momentum. By means of gauge invariance, this does not lead to a change of physics, as this is essentially provided by the Ward identity $k_\mu \mathcal{M}^\mu = 0$.

4 Little Group Covariance

As we saw, representing momenta as four-vectors p^μ and as bispinors $p_{\alpha\dot{\alpha}}$ is equivalent. When discussing amplitudes, we also know there is an $E(2)$ (sometimes denoted $ISO(2)$) group of Lorentz transformations, called the **little group**, which leaves the momentum of one outgoing particle fixed. In bispinor representation, the little group transformations appear as

$$p\rangle \to zp\rangle\,; \qquad p] \to z^{-1}p] \qquad \Rightarrow \qquad p\rangle[p\,, p]\langle p \text{ invariant.} \tag{14}$$

For real momenta, we know that Eq. (3) has to be satisfied, hence z can only be a phase ($|z| = 1$) in that case. For complex momenta, there is no relation between u_L and u_R, hence z can be an arbitrary number.

This is already all we need in order to construct scattering amplitudes. First, note that little group transformations do not affect internal lines or vertices, as they only depend on constants or momenta that are left unchanged. Thus, the only effect of little group transformations is a rescaling of the amplitude depending on external lines. Second, recall that there is a (different) little group transformation for each external line and therefore the scaling of the amplitude has to be fulfilled for every external line separately.

To now construct amplitudes, let us take a look at the polarization vectors again. A little group transformation acts on them in the following way:

$$\epsilon_-^*(k,r) = -\frac{1}{\sqrt{2}}\frac{[r\gamma^\mu k\rangle}{[rk]} \to z^2\epsilon_-^*(k,r)\,; \qquad \epsilon_+^{\mu *}(k,r) = \frac{1}{\sqrt{2}}\frac{[k\gamma^\mu r\rangle}{\langle rk\rangle} \to z^{-2}\epsilon_+^*(k,r)$$

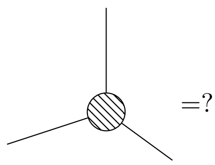
We see that the rescaling goes as $z_j^{-2h_j}$ with the helicity h_j for the external particle j. As it turns out, this is the general rule, hence we require the amplitude to fulfill the relation

$$\mathcal{M}(1^{h_1},...,n^{h_n}) \to \prod_{j=1}^{n} z_j^{-2h_j}\mathcal{M}(1^{h_1},...,n^{h_n})$$[5].

We can also explicitly write down amplitudes by considering the following rule: The power of $j\rangle$ plus the power of $\langle j$ minus the power of $j]$ and the power of $[j$ has to equal $-2h_j$. We will use this rule next to determine the general formula for a three-point amplitude.

5 Three-point Amplitude

In the following, we will learn how to construct three-point amplitudes just by considering the in- and outgoing helicites instead of deriving the corresponding Feynman rules:



For the following discussion, we will use the convention that all momenta are outgoing.

5.1 Three-Point Amplitude for Arbitrary Particles

First, let us have a look at momentum conservation:

$$1\rangle[1 + 2\rangle[2 + 3\rangle[3 = 0.$$

Multiplying this from the left with $\langle 1$ and $\langle 2$, we find the following set of equations

$$\langle 12\rangle[2 = -\langle 13\rangle[3\,; \qquad \langle 21\rangle[1 = -\langle 23\rangle[3.$$

There are two possible solutions to these equations. The first one is that all the $[j$ are proportional to each other and hence $[12] = [23] = [31] = 0$, the second is that $\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0$. Either way, the three particle amplitude can only depend on $\langle ij\rangle$ **or** $[ij]$. Additionally, we can immediately see that any three particle amplitude has to vanish in the limit of real momenta as in that case Eq. (5) holds. Nevertheless, we will continue our discussion, as the three-point amplitude appears in many other n-point amplitudes and we can even use it to determine the unique local interaction with real fields (in the next seminar). For example, we can put one of the momenta slightly off-shell into the complex plane, taking the limit towards real momenta at the end of the calculation.

We call the configurations:

$$\text{holomorphic: } [12] = [23] = [31] = 0,$$
$$\text{anti-holomorphic: } \langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0$$[6].

[5] We have seen that this is fulfilled for vector bosons. For scalars, the little group does not change anything about the scattering amplitude and for spin-1/2 fermions, we can easily see this fulfilled by means of Eq. (14). It is also possible to show this for spin-2 particles, but the derivation would go beyond the scope of this lecture.

[6] The reason to call the configurations holomorphic and anti-holomorphic is the way that Weyl-spinors transform under Lorentz transformations. There we have $\psi_{L/R} \to \Lambda_{L/R}\psi_{L/R} = e^{i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}\mp i\vec{\phi})}\psi_{L/R}$, where Λ_L (Λ_R) is (anti-) holomorphic with respect to $\vec{z} = \vec{\theta} + i\vec{\phi}$.

As any physical amplitude should only contain coupling constants and momenta for massless particles, we know that we can write every amplitude as

$$\mathcal{M} = c\langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2}\,; \qquad \text{or} \qquad \mathcal{M} = c[12]^{n_3'}[23]^{n_1'}[31]^{n_2'}. \tag{15}$$

In the following, we will only take the first form of the amplitude in Eq. (15), as we will see that these results are related by CP transformation.

Next, we require that the amplitude has to scale appropriately under the little group transformation for each external momentum. From this we find the helicities of the particles being related to the exponents by

$$\begin{aligned} h_1 &= -\frac{n_3+n_2}{2}, \\ h_2 &= -\frac{n_3+n_1}{2}, \\ h_3 &= -\frac{n_1+n_2}{2}. \end{aligned}$$

This is easy to see, when going back to the rule that the power of $j\rangle$ plus the power of $\langle j$ minus the power of $j]$ and the power of $[j$ has to equal $-2h_j$. Rewriting this expression, we find the appropriate exponents

$$\begin{aligned} n_1 =& h_1 - h_2 - h_3, \\ n_2 =& h_2 - h_3 - h_1, \\ n_3 =& h_3 - h_1 - h_2. \end{aligned}$$

Here, we can see that $n_j' = -n_j$ by CP. Furthermore, we can require $n_1+n_2+n_3 \geq 0$ and $n_1'+n_2'+n_3' \geq 0$, as otherwise the amplitude would diverge in the limit of real momenta according to Eq. (5). Hence we find the first possibility in Eq. (15) (angle brackets) is applied whenever the total helicity $h = h_1+h_2+h_3 \leq 0$, the second case (square brackets) whenever $h \geq 0$ and under CP we simply exchange square and angle brackets[7]. Put together, we find

$$\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) = \begin{cases} c\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} & h \leq 0, \\ c[12]^{-h_3+h_1+h_2}[23]^{-h_1+h_2+h_3}[31]^{-h_2+h_3+h_1} & h \geq 0. \end{cases} \tag{16}$$

The important point to this amplitude is that it is derived nearly completely from symmetry. Hence it does not only hold to any order in perturbation theory, but it holds **non-perturbatively**. We will put this to work in a few examples now.

5.2 Examples

Three Gluon Amplitude

Let us precisely discuss the three gluon amplitude, as we will look at color ordering in the next section. In general, we have four possible helicity configurations: $+++$, $++-$, $--+$ and $---$. As the latter two are simply the CP transformation of the first two, we will not go through them explicitly. We will therefore start with the helicities $+++$.

We know that the total helicity is greater than zero, and plugging in the helicities into Eq. (16), we can instantly write the amplitude as

$$\mathcal{M}(1^{a+}2^{b+}3^{c+}) = c^{abc}[12][23][31]^{8}, \tag{17}$$

where c^{abc} is some color structure. However, we need to have $[\mathcal{M}]_{\mathrm{M}} = 1$ for a three point function. Thus by $[[12][23][31]]_{\mathrm{M}} = 3$, we have $\left[c^{abc}\right]_{\mathrm{M}} = -2$. If we only consider renormalizable theories, the only possible solution is $c^{abc} = 0$.

In the case $++-$, we still have a positive total helicity h, but now the amplitude is given by

$$\mathcal{M}(1^{a+}2^{b+}3^{c-}) = c^{abc}\frac{[12]^3}{[23][31]}, \tag{18}$$

[7] In an arbitrary theory, the coupling constants can be different under CP. However, that merely changes the magnitude of the three-point amplitude, not its behavior.

[8] The corresponding operator is

$$f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}$$

where c^{abc} is not the same as in Eq. (17). By dimensional analysis, we can now conclude that $\left[c^{abc}\right]_{\mathrm{M}} = 0$ in this case. Furthermore, as the amplitude must be symmetric under the exchange of two particles, and $\frac{[12]^3}{[23][31]}$ is antisymmetric under the exchange of any two, also c^{abc} has to be antisymmetric.

Two Fermion, One Vector Boson Amplitude

Let us consider the process $f(1)\overline{f}(2) \to \gamma(3)$. With the helicity configuration $+++$, we have

$$\mathcal{M}(1^+2^+3^+) = c[23][31].$$

Again, this can only be achieved in an effective theory, as $[c]_{\mathrm{M}} = -1$ due to $[\mathcal{M}]_{\mathrm{M}} = 1$ for this process [9]. In the configuration $+-+$, we have

$$\mathcal{M}(1^+2^-3^+) = c\frac{[31]^2}{[12]}.$$

This result has the proper mass dimension, and this is actually the same result we would obtain from QED with $c = \sqrt{2}e$.

Bonus Box:

$$\begin{aligned} i\mathcal{M}(1^+2^-3^+) =& (-ie)\langle 2\gamma^\mu 1] \frac{1}{\sqrt{2}} \frac{\langle r\gamma_\mu 3]}{\langle r3\rangle} \\ =& -i\sqrt{2}e\frac{[13]\langle r2\rangle}{\langle r3\rangle} \\ =& -i\sqrt{2}e\frac{[13]\langle r2\rangle[21]}{\langle r3\rangle[21]} \\ =& -i\sqrt{2}e\frac{[13]\langle r\not{2}1]}{\langle r3\rangle[21]} \\ =& i\sqrt{2}e\frac{[13]\langle r(\not{1}+\not{3})1]}{\langle r3\rangle[21]} \\ =& i\sqrt{2}e\frac{[13]\langle r3\rangle[31]}{\langle r3\rangle[21]} \\ =& i\sqrt{2}e\frac{[31]^2}{[12]}. \end{aligned}$$

Three Graviton Amplitude

We can even extend the formalism onto the case of massless spin-2 particles (gravitons). As these are massless, we again only have two possible helicities for each particle. For the different helicity configurations we find:

$$\begin{aligned} \mathcal{M}(1^+2^+3^+) &= c[12]^2[23]^2[31]^2, \\ \mathcal{M}(1^+2^+3^-) &= c\frac{[12]^6}{[23]^2[31]^2}. \end{aligned} \tag{19}$$

We already find that any theory containing massless spin-2 particles can only be an effective theory, as necessarily $[c]_{\mathrm{M}} \leq -1$. It turns out, Eq. (19) is the same as the one derived from the Einstein-Hilbert action given through $\mathcal{L}_{\mathrm{EH}} = \frac{2}{\kappa^2}\sqrt{-g}R$ with $\kappa = \sqrt{32\pi G}$. An interesting and yet not fully understood result we also see is that we basically have a relation Gravitation = (Gauge)^2, when we compare the three-graviton amplitude in Eq. (19) to the three-gluon amplitude in Eq. (18).

References

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[9] The operator resulting in the given amplitude is known as the magnetic operator:

$$\mathcal{O}_m = -\frac{c_m m e}{4}\overline{\psi}F_{\mu\nu}\sigma^{\mu\nu}\psi.$$

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