Technische Universität München | **Theoretische Physik T70**
Physik Department | **B. Garbrecht**
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Quantum Mechanics II

Winter Term 2015/16

Hand in until Thursday, 3.12.15, 12:00 next to PH 3218.

Exercise Sheet No. 07 — To be discussed from 7.12. - 11.12.15.

Problem 1: Solar Neutrino Oscillations — 5 Points

Neutrinos are elementary particles of the standard model, they are almost massless and have only weak interactions with the rest of the standard model.
In the sun they are mainly produced via the proton-proton interaction

$$p + p \to d + e^+ + \nu_e$$

However, experimentally it was observed that roughly a third of the electron neutrinos produced in the sun reached earth. This was known as the solar neutrino problem. It turns out that neutrinos come in three flavors, ν_e, ν_μ and ν_τ, corresponding to the leptons that they interact with. The problem was solved when it was discovered that the neutrino's mass (energy) eigenstates $|\nu_i\rangle, i \in \{1, 2, 3\}$ are a superposition of their flavor states $|\nu_i\rangle = \sum_{a=e,\mu,\tau} c_{i,a} |\nu_a\rangle$.

(a) Consider only the ν_e and ν_μ neutrinos, if the effective Hamiltonian of their interaction is given by:

$$H = \frac{\delta m^2 c^4}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \tag{1}$$

where $\delta m^2 = m_2^2 - m_1^2$ is the mass (energy) difference between the eigenstates, E is the average energy at which they are produced. Determine the eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$.

(b) Show that the effective Hamiltonian is the first order expansion of the relativistic energy for $|\mathbf{p}| \gg m$

$$\tilde{H} = \begin{pmatrix} \sqrt{\mathbf{p}^2 c^2 + m_1^2 c^4} & 0 \\ 0 & \sqrt{\mathbf{p}^2 c^2 + m_2^2 c^4} \end{pmatrix} \tag{2}$$

In matter, the electron neutrino ν_e can interact with electrons. This introduces an effective interaction term into the Hamiltonian:

$$V_I = \begin{pmatrix} \sqrt{2} G_F \rho(x) & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} \sqrt{2} G_F \rho(x)/2 & 0 \\ 0 & -\sqrt{2} G_F \rho(x)/2 \end{pmatrix} \tag{3}$$

which depends on the local density of electrons $\rho(x)$. In the second step we subtracted the term proportional to the unit matrix, which leaves the dynamics of the problem unchanged. The neutrinos are produced in the core of the sun, where the electron density can be very high. Then they move outwards, to the surface of the sun, where the electron density vanishes.

(c) Assuming that the neutrinos move at the speed of light, replace $t \to x/c$, where x is the position of the neutrino. Write down the complete Hamiltonian for a neutrino going through matter.

(d) If the matter density changes slowly, we can assume that the neutrinos evolve *adiabatically*, which means that they follow the eigenstate in which they started.
Find the local heavy and light eigenstates:

$$|\nu_L(x)\rangle = \cos\theta(x)|\nu_e\rangle - \sin\theta(x)|\nu_\mu\rangle \tag{4}$$
$$|\nu_H(x)\rangle = \sin\theta(x)|\nu_e\rangle + \cos\theta(x)|\nu_\mu\rangle\,, \tag{5}$$

by determining the local angle $\theta(x)$.

(e) Determine the average probability of an electron neutrino produced in the core to be observed in the electron state once it exits the sun $\langle P(\nu_e \to \nu_e)\rangle = P(\nu_e \to \nu_H \to \nu_e) + P(\nu_e \to \nu_L \to \nu_e)$.

(f) It is possible that the values on the diagonals of the full Hamiltonian cross at some critical density ρ_c at position x_c. Around that position we may expand the density $\rho(x-x_c) = \rho_c + \mathrm{d}\rho/\mathrm{d}x(x-xc)$ Show that by substituting $x \to x - x_c$, and subtracting the part proportional to unity, the Hamiltonian assumes the form:

$$H(x) = \begin{pmatrix} -\alpha x & -f \\ -f & \alpha x \end{pmatrix}\,, \tag{6}$$

from the last sheet.

(g) If the electron density at the core of the sun is $\rho(0) = 100 N_A/\mathrm{cm}^3$, how big should the neutrino energy be for the values on the diagonals of the Hamiltonian never to cross?

(h) Use the Landau-Zener formula $P_{LZ} = \mathrm{e}^{-2\pi\gamma}$, with $\gamma = \frac{|f|^2}{2|\alpha|}$, for the probability of hopping between eigenstates to determine the probability of jumping from $|\nu_L(x)\rangle$ to $|\nu_H(x)\rangle$, and vice versa. Express γ in terms of the derivative of the electron density $\mathrm{d}\rho/\mathrm{d}x$, the neutrino energy E, the mass difference δm^2, and the vacuum mixing angle θ.

(i) How does $\langle P(\nu_e \to \nu_e)\rangle$ change if you include the Landau-Zener hopping probability?

Problem 2:
One-Photon Decay of the $2s$ State in the Hydrogen Atom — 5 Points

Calculate the lifetime of the $2s \to 1s$ transition via the emission of *one* photon. Note that the electron spin needs to flip to conserve angular momentum.

Hints:
Factor out the spin-dependent part of the matrix element.
Expand $e^{\mathbf{ik}\cdot\mathbf{r}}$ to the first term that yields a non-vanishing matrix element.