Technische Universität München **Theoretische Physik T70**
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Quantum Mechanics II

Winter Term 2015/16

Hand in until Thursday, 19.11.15, 12:00 next to PH 3218.
Exercise Sheet No. 05 To be discussed from 23.11. - 27.11.15.

Problem 1: Schrödinger Equation in Field Theory **7 Points**

There is another way to explore the connection between free quantum fields and the harmonic oscillator. In quantum mechanics the Schrödinger equation for a free particle depends on its location x and momentum $p \sim -i\hbar\partial_x$. If we interpret the quantum field as a set of infinitely many harmonic oscillators, then there are infinitely many coordinates $\phi_{\mathbf{p}}$ and conjugate momenta $\dot{\phi}_{\mathbf{p}}$. The Schrödinger equation for this system should depend on all of them. If we interpret $\phi_{\mathbf{p}}$ as a function of $\mathbf{p}$ (rather than just viewing $\mathbf{p}$ as a label for infinitely many numbers), then the Schrödinger equation can be viewed as a *functional equation* with only two arguments in function space: the functions $\phi_{\mathbf{p}}$ and $\dot{\phi}_{\mathbf{p}}$ (rather than a usual equation of the infinitely many coordinates labeled by $\mathbf{p}$ in configuration space).

In the lecture it was shown that the field theoretical equivalent to the usual Schrödinger wave function is the wave functional $\Psi[\phi, t]$. Its amplitude square gives the probability density to find a particular field configuration. The evolution of Ψ is governed by the equation

$$\mathrm{i}\hbar\frac{d}{dt}\Psi[\phi,t] = \int d^3\mathbf{x}\left(-\frac{\hbar^2c^2}{2}\frac{\delta^2}{\delta\phi(\mathbf{x},t)^2} + \frac{1}{2}(\nabla\phi(\mathbf{x},t))^2 + \frac{1}{2\hbar^2}m^2c^2\phi(\mathbf{x},t)^2\right)\Psi[\phi,t]\,, \tag{1}$$

where we have made all arguments explicit to avoid confusion.

Assuming the field ϕ to be inside a finite volume $V = \ell^3$, one can decompose $\phi(\mathbf{x}, t)$ into Fourier modes $\phi_{\mathbf{p}}(t)$:

$$\phi(\mathbf{x},t) = \sum_{\mathbf{p}}\left[\phi^+_{\mathbf{p}}(t)\cos\left(\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar}\right) + \phi^-_{\mathbf{p}}(t)\sin\left(\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar}\right)\right]\,. \tag{2}$$

(a) Express the Lagrangian $L = \int \mathrm{d}^3x\mathcal{L}$ in terms of the modes $\phi_{\mathbf{p}}(t)$

(b) Define the conjugate momentum $\pi_{\mathbf{p}}(t) = \frac{\partial L}{\partial\dot{\phi}_{\mathbf{p}}(t)}$

(c) Show that the Hamiltonian H is of the form:

$$H = N\sum_{\mathbf{p}}\left[\frac{1}{2}c^4\left(2/\ell\right)^6\pi^2_{\mathbf{p}} + \frac{1}{2}\omega^2_{\mathbf{p}}\phi^2_{\mathbf{p}}\right], \tag{3}$$

and determine the normalization constant N.

This implies that $\Psi[\phi, t]$ can be written as an infinite product $\prod_{\mathbf{p}}\Psi_{\mathbf{p}}[\phi_{\mathbf{p}}, t]$.

(d) Obtain the Schrödinger equation for $\Psi_{\mathbf{p}}[\phi_{\mathbf{p}}, t]$ from Eq. (1).

It looks like the Schrödinger equation for a harmonic oscillator.

(e) Use this knowledge to find the functional $\Psi_{\mathbf{p}}[\phi_{\mathbf{p}}, t]$ of the "ground state" in field space.

Problem 2:
Time Evolution of the Field Operator **3 Points**

The simplest example of a continuous system with infinitely many degrees of freedom is a scalar field $\phi(x)$, whereas the electromagnetic field is a bit more complicated. The value of the field in each point can be viewed as a coordinate of the system. In the operator formalism of quantum mechanics quantization means to impose the usual commutation relations onto the coordinates and their conjugate momenta. This means that the coordinates are now operators. For the scalar field, the *field operator* $\phi(x)$ and its conjugate $\dot{\phi}(x)$ in each point are the analogues of the operators x and p in usual single particle quantum mechanics.

(a) Use the decomposition of the field into ladder operators to show explicitly that $\phi(x)$ and $\dot{\phi}(x)$ fulfil canonical commutation relations.

(b) Show that the operator $\phi(t, \mathbf{x}) = U^{\dagger}(t)\phi(0, \mathbf{x})U(t)$ fulfills the Heisenberg equation of motion by explicit decompositions of the field operator, $U(t)$ and the Hamiltonian into ladder operators. Here $U(t)$ is the time evolution operator.