Exercises for Theoretical Particle Physics II

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Problem 1: Positronium annihilations **2.5 Points**

In the lectures we have discussed several scattering and pair-annihilation/production processes at tree level in QED. A common feature of all these processes is, that the incoming and outgoing states were considered as asymptotically free single particle states. The respective reactions refer to an experimental setup, where isolated particles collide with each other either in a head-on collision, or in a collision, where one particle is fired on a fixed target. Typically, the energies of the involved particles are relativistic.

This problem shall highlight some aspects of annihilation processes in a non-relativistic two-particle bound-state system: *positronium*.
Positronium is a bound state of a non-relativistic e^+e^- pair. For slowly moving particles, the Coulomb attraction of the e^+ and e^- becomes an important effect and leads to a distortion of the e^+ and e^- waves functions away from plane waves (that are considered in relativistic scattering processes of isolated single particles, as in the lecture, for example). The proper treatment of bound-states is a complicated topic. Here we will follow a simple heuristic approach, that however incorporates the basic features of annihilation reactions of non-relativistic bound states.[1]
Consider the center-of-mass system of the e^+e^- pair. In presence of a Coulomb potential interaction, the wave function $\psi(\vec{r})$, that is associated with the relative motion of the e^+e^- pair, is determined by the stationary Schroedinger equation

$$\left(-\frac{\vec{\nabla}^2}{2\mu} - \frac{\alpha}{r}\right)\psi(\vec{r}) \;=\; E\;\psi(\vec{r})\;, \tag{1}$$

where $\mu = m_e/2$ is the reduced mass of the two-particle system and $\vec{r}$ is its relative coordinate. $V(\vec{r}) = -\alpha/r$ represents the (attractive) Coulomb potential, where $\alpha = e^2/4\pi \approx 1/137$ is the fine-structure constant.
The state-vector of the bound system can be thought of as a linear superposition of the free-particle states $|e^+(\vec{p})\; e^-(-\vec{p})\;\rangle$ with three-momenta $\vec{p}$, weighted by the wave function $\psi(\vec{r})$ (or better, as we consider a superposition of momentum eigenstates, its Fourier-transform $\tilde{\psi}(\vec{p})$):

$$|\mathbf{Ps}\rangle \;=\; \sqrt{4m_e}\int\frac{d^3\vec{p}}{(2\pi)^3}\;\tilde{\psi}^*(\vec{p})\;\frac{1}{\sqrt{2m_e}\sqrt{2m_e}}\;|e^+(\vec{p})e^-(-\vec{p})\;\rangle\;. \tag{2}$$

a) Determine the wave function $\psi_0(\vec{r})$ corresponding to the positronium ground-state with energy eigenvalue $E_0 = -m_e\alpha^2/4$ by solving the Schroedinger equation (1). Make

[1] You might find chapter 5.3 in the textbook Peskin/Schroeder useful.

a factorization ansatz in spherical coordinates $\psi_0(\vec{r}) = R_{10}(r)Y_{00}(\theta,\phi)$ and determine $R_{10}(r)$. Recall, that the ground state is characterized by orbital angular momentum $l = 0$, hence the spherical harmonic $Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$ occurs in $\psi(\vec{r})$.

b) With the ansatz (2) for a positronium state vector, the matrix element for positronium annihilation into two photons can be written as

$$\mathcal{M}\left(\mathbf{Ps} \to \gamma\gamma\right) = \sqrt{4m_e} \int \frac{d^3\vec{p}}{(2\pi)^3}\, \tilde{\psi}^*(\vec{p})\, \frac{1}{\sqrt{2m_e}\sqrt{2m_e}}\, \mathcal{M}\left(e^+(\vec{p})\; e^-(-\vec{p}) \to\; \gamma\gamma\right) ,$$

where $\mathcal{M}\left(e^+(\vec{p})\; e^-(-\vec{p}) \to\; \gamma\gamma\right)$ is the perturbative, relativistic annihilation matrix element.

Using the results for the spin averaged annihilation matrix element derived in the lecture, determine the $\mathbf{Ps} \to \gamma\gamma$ annihilation rate $\Gamma_{\mathbf{Ps}\to\gamma\gamma}$ of positronium in the ground state in the extreme non-relativistic limit. You need to know the ground state wave function $\psi_0(\vec{r})$ derived under a) and the (spin-averaged) annihilation matrix element $\mathcal{M}$ derived in the lecture. In the latter keep only terms proportional to the zeroth order of the e^+ and e^- three momentum $\pm\vec{p}$. Under the assumption, that positronium in the ground state can only disintegrate into two photons, give the positronium life-time in seconds.

Problem 2: Positronium annihilations into $\gamma\gamma$ - refined analysis — 5 Points

As positronium is a bound system of two spin 1/2 particles, there exist positronium states with spin 0 called parapositronium and spin 1 states referred to as orthopositronium. Calculate the annihilation matrix element $\mathcal{M}(\mathbf{Ps} \to \gamma\gamma)$ separately for ortho- and parapositronium in the ground state at leading order in a non-relativistic expansion in the e^+ and e^- three momenta $\pm\vec{p}$. Note, that you cannot perform a spin-average here, as we explicitly want separate expressions for the spin 0 and spin 1 systems. Instead, you should consider the amplitude $\mathcal{M}(e^+(\vec{p},s)e^-(-\vec{p},s') \to \gamma\gamma)$ for a fixed spin configuration. You need an explicit form of the external spinors as a function of momenta and spins. It is convenient to work in the Dirac representation, where the spinors are given by

$$u(\vec{p},s) = \sqrt{p^0+m_e}\begin{pmatrix} \xi_s \\ \frac{\vec{p}\cdot\sigma}{p^0+m}\,\xi_s \end{pmatrix} , \qquad v(\vec{p},s) = \sqrt{p^0+m_e}\begin{pmatrix} \frac{\vec{p}\cdot\sigma}{p^0+m_e}\,\eta_s \\ \eta_s \end{pmatrix} , \tag{3}$$

with $\xi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\eta_{1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\eta_{-1/2} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and the γ matrices read

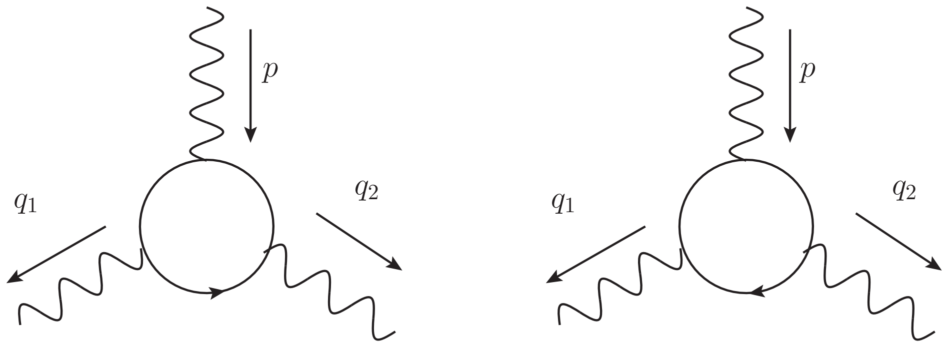
$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{4}$$

In order to simplify the expression for the amplitude $\mathcal{M}$, before doing the expansion in the non-relativistic regime, make use of the relations $\not{p}\, u(p) = m_e\, u(p)$ and $\epsilon(k)\cdot k = 0$. Further argue, that one can always choose the photon polarization vectors such, that in the non-relativistic regime $\epsilon(k)\cdot p \approx \epsilon(k)^0\, p^0 = 0$. Afterward you should perform an expansion in $\vec{p}$, keeping only the zeroth order. This implies for example $p^\mu_{e^+,e^-} \approx (m_e, \vec{0})^\mu$.

Determine the decay rates of para- and orthopositronium into two photons. Give the life-time of parapositronium, compare to your result in problem 1 above and discuss your result.

Problem 3: Furry's Theorem **2.5 Points**

In (spinor) QED a photon cannot split into an even number of photons. This statement is true to all orders of perturbation theory. Consider the process $\gamma(p) \to \gamma(q_1)\gamma(q_2)$; in leading order in the electromagnetic coupling there are two diagrams:



(a) Let us first define the matrix $N = \gamma^0\gamma^2$. On can explicitly show that $N\gamma^\mu N = -(\gamma^\mu)^T$. Use this to show that

$$Tr\{\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\cdots\} = Tr\{\cdots\gamma^{\mu_4}\gamma^{\mu_3}\gamma^{\mu_2}\gamma^{\mu_1}\}$$

(b) Proof that the total amplitude for $\gamma(p) \to \gamma(q_1)\gamma(q_2)$ vanishes at 1-loop level.