Exercises for Theoretical Particle Physics I

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This exercise deals with a very interesting problem: Neutrino oscillations. In the Standard Model, there are only left chiral (left handed) neutrinos ν_L, and they are exactly massless. The observed neutrino oscillations imply that neutrinos have masses. One possibility to explain neutrino masses is to assume that, in addition to ν_L, neutrinos ν_R, with right handed chirality exist (right handed neutrinos). Then, as for all other fermions, a mass term of the form

$$\bar{\nu}_\mathrm{L} m_D \nu_\mathrm{R} + \mathrm{h.c.} \tag{1}$$

can be generated via the Higgs mechanism from Yukawa couplings $\bar{\ell}_\mathrm{L} F \nu_\mathrm{R} \tilde{\phi}$. If we do not want to assume the existence of new particles ν_R, then the only mass term we can write down is a Majorana mass term of the form

$$\frac{1}{2}\bar{\nu}_\mathrm{L} m_\nu \nu_\mathrm{L}^c + \mathrm{h.c.}\,, \tag{2}$$

with $\nu_\mathrm{L}^c = C\bar{\nu}_\mathrm{L}^T$, where the charge conjugation matrix is $C = i\gamma_2\gamma_0$. This term, however, breaks gauge invariance unless it is generated by spontaneous symmetry breaking (Higgs mechanism) from a gauge invariant term like

$$\frac{1}{2}\bar{\ell}_\mathrm{L}\tilde{\phi}\frac{f}{\Lambda}\tilde{\phi}^T \ell_\mathrm{L}^c + \mathrm{h.c.}\,, \tag{3}$$

where f is a dimensionless flavour matrix and Λ an energy scale much larger than the scale of neutrino experiments. This dimension-5 operator is not renormalizable; in an effective field theory approach it can be understood as the low energy limit of renormalizable operators that is obtained after integrating out heavier degrees of freedom with masses $M \leq \Lambda$. At low energies $E \ll \Lambda$ one effectively observes only three massive neutrinos. At high energies new particles with masses $M \sim \Lambda$ appear. Therefore neutrino masses definitely imply the existence of new physics, although the Majorana mass term can be constructed from SM fields only. In fact, neutrino masses are the only definite sign of physics beyond the SM that has been seen in the laboratory to date. This makes them very interesting.

In the following we will assume that neutrinos are described by the following Lagrangian

$$-\frac{g}{\sqrt{2}}\nu_\mathrm{L}\gamma^\mu e_\mathrm{L} W_\mu^+ - \frac{g}{\sqrt{2}} e_\mathrm{L}\gamma^\mu \nu_\mathrm{L} W_\mu^- - \frac{g}{2\cos\theta_W}\nu_\mathrm{L}\gamma^\mu\nu_\mathrm{L} Z_\mu - \frac{1}{2}\bar{\nu}_\mathrm{L} m_\nu \nu_\mathrm{L}^c - \frac{1}{2}\overline{\nu^c}_\mathrm{L}(m_\nu)^\dagger \nu_\mathrm{L} \tag{4}$$

The interaction terms define the basis of weak interaction eigenstates (electron, muon and tau neutrino), which are also called flavour eigenstates. More precisely, if one considers the general form of the interaction $\frac{G_F}{2\sqrt{2}}\bar{\nu}_{\mathrm{L},\alpha}\gamma^\mu U_{\alpha\beta} e_{\mathrm{L},\beta} W_\mu$ in the basis where charged Yukawa couplings are diagonal, then the basis of weak interaction eigenstates for ν_L is the one where $U_{\alpha\beta} = \delta_{\alpha\beta}$. The matrix m_ν in general is not diagonal in that basis. It can be diagonalized by a transformation

$$m_\nu = U_\nu \mathrm{diag}(m_1, m_2, m_3) U_\nu^T\,. \tag{5}$$

In the mass base, the neutrino mixing matrix U_ν appears in the coupling to W_μ. Hence, the weak interaction eigenstates $\nu_{\mathrm{L},e}$, $\nu_{\mathrm{L},\mu}$ and $\nu_{\mathrm{L},\tau}$ are superpositions of three mass eigenstates $\nu_{\mathrm{L},i}$ of m_ν with masses m_i. For a given momentum, these have different energies if their masses are different, and their wave functions oscillate with different frequencies. Thus, the flavour decomposition of a neutrino state changes in time. This can explain the observed neutrino oscillations (see problem 3).

The matrix U_ν is commonly parametrized as

$$U_\nu = V^{(23)} U_\delta V^{(13)} U_{-\delta} V^{(12)} \mathrm{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)\,, \tag{6}$$

with $U_{\pm\delta} = \mathrm{diag}(e^{\mp i\delta/2}, 1, e^{\pm i\delta/2})$ and where the non-vanishing entries of $V^{(ab)}$ for $a = e, \mu, \tau$ are given by

$$V^{(ab)}_{aa} = V^{(ab)}_{ba} = \cos\theta_{ab}\ ,\ V^{(ab)}_{ab} = -V^{(ab)}_{ba} = \sin\theta_{ab}\ ,\ V^{(ab)}_{cc} = 1 \quad \text{for } c \neq a, b\,, \tag{7}$$

with θ_{ai} the neutrino mixing angles and $\alpha_{1,2}, \delta$ CP-violating phases. Many parameters of the mixing matrix U_ν have been measured. In particular, two mass square differences have been determined as $\Delta m^2_{\mathrm{sol}} = m_2^2 - m_1^2 \simeq 7.5 \times 10^{-5}\mathrm{eV}^2$ and $\Delta m^2_{\mathrm{atm}} = |m_3^2 - m_1^2| \simeq 2.5 \times 10^{-3}\mathrm{eV}^2$, the mixing angles are $\theta_{12} = 34°$, $\theta_{23} = 39°$ and $\theta_{13} = 9°$.
(arXiv:1303.6912 [hep-ph] by Marco Drewes might be useful for a deeper understanding.)

Problem 1: Lepton vs quark mixing **2 points**

The matrix U_ν is the lepton sector analogue to the CKM-matrix. What is the main difference between U_ν and the CKM-matrix? Why are there more free parameters? Can the number of physical parameters be reduced for some specific choice of m_ν?

Problem 2: Neutrino oscillations **4 points**

We will now consider a simplified system with only two neutrino flavours. We call the flavour eigenstates ν_e and ν_μ, i.e. electron and myon neutrino. We consider a plane wave of neutrinos with fixed momentum and represent this state by a vector $(\nu_e, \nu_\mu)^T$. The basis of mass eigenstates $(\nu_1, \nu_2)^T$ with masses m_1 and m_2 is rotated with respect to that by an angle θ. Show that the oscillations of these neutrinos in vacuum is governed by the Schroedinger equation

$$i\frac{d}{dt}\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix} = \underbrace{\frac{\Delta m^2}{4|\mathbf{p}|}\begin{pmatrix}-\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta\end{pmatrix}}_{H_{\mathrm{eff}}}\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix} \tag{8}$$

where $\Delta m^2 = m_1^2 - m_2^2$ is the mass square difference between the masses.

Hint: Begin with the Schroedinger equation in the mass basis

$$i\frac{d}{dt}\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix} = \frac{\Delta m^2}{4|\mathbf{p}|}\begin{pmatrix}\sqrt{|\mathbf{p}|^2 + m_1^2} & 0 \\ 0 & \sqrt{|\mathbf{p}|^2 + m_2^2}\end{pmatrix}\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix}\,, \tag{9}$$

with

$$\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix}\,, \tag{10}$$

expand the square roots in $m_i \ll |\mathbf{p}|$ and make use of the fact that adding and subtracting multiples of the unit matrix to H_{eff} does not affect the oscillations (it only gives an overall phase).

Problem 3: Neutrino oscillations in matter **4 points**

Let us consider neutrinos inside a star (e.g. the sun). There are many electrons in the stellar plasma (and almost no muons). ν_e can interact by charged and neutral current interactions with them, ν_μ only by neutral current interactions. That leads to an extra potential $V_e = \sqrt{2} G_F N_e$ for ν_e, where N_e is the electron density. The potential from neutral current interactions with the plasma can be neglected because all neutrinos feel it, hence it gives a contribution proportional to the unit matrix to H_{eff}. The effective Hamiltonian H_{eff} then reads

$$H_{\text{eff}} = \frac{\Delta m^2}{4|\mathbf{p}|} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} V_e(t) & 0 \\ 0 & 0 \end{pmatrix} . \tag{11}$$

In general $V_e(t)$ effectively is a function of time because the electron density can change along the neutrino trajectory. Show that this can be written in the form

$$H_{\text{eff}} = \frac{\Delta M^2(t)}{4|\mathbf{p}|} \begin{pmatrix} -\cos 2\Theta(t) & \sin 2\Theta(t) \\ \sin 2\Theta(t) & \cos 2\Theta(t) \end{pmatrix} , \tag{12}$$

where $2\Theta(t)$ and $\Delta M(t)^2$ are functions of Δm^2, θ and $V_e(t)$ (**Hint**: you may want to subtract $\mathbb{1} V_e/2$ first). This means that the mass- and flavour basis can rotate with respect to each other while the neutrinos oscillate. Consider the regimes the regimes $\Delta M(t) \gg \dot{\Theta}(t)$ and $\Delta M(t) \ll \dot{\Theta}(t)$, in which neutrino oscillations are either much faster or much slower than the rotation of the bases. In these two regimes, two qualitatively different mechanisms are at work that change the neutrino flavour (in the general case both mechanisms coexist). Find approximate solutions in these regimes. You may go into the time dependent mass base. If you cannot find explicit solutions, describe qualitatively what happens.