Exercises for Theoretical Particle Physics I

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Sheet 4

Problem 1: Scalar QED and Ward identities **12 points**

Consider the free Lagrangian for a complex scalar field

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi \tag{1}$$

with the propagator

$$\langle 0|T\Phi(x)\Phi^\dagger(y)|0\rangle = i\Delta^\Phi(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \ . \tag{2}$$

Note: Φ and $\Phi^\dagger$ are considered as "separate entities"; i.e. a functional derivative with respect to Φ will not hit $\Phi^\dagger$.

Let us now couple the scalar field to the electromagnetic field, i.e. we promote

$$\partial^\mu \to D^\mu = \partial^\mu + ieA^\mu \ .$$

The task will be to determine the Feynman rules for the vertices in momentum space, that is in the standard form.

To illustrate how one derives the rules we will first consider the well known case of (fermionic) QED. The vertex rule heuristically can be read off the action:

$$e^{iS_{\rm Int}} \approx 1 + iS_{\rm Int} = 1 + i\int d^4x \mathcal{L}_{\rm Int} = 1 + i\int d^4x \bar{\psi} i(+ie\not{A})\psi = 1 + \int d^4x \bar{\psi} (\underbrace{-ie\gamma^\mu}_{\text{vertex}}) A_\mu \psi$$

But of course this us a purely heuristic argument and things are not always this easy. Scalar QED is one of these cases.

Let us re-derive the QED fermion-photon vertex using a more stringent procedure. Consider:

$$\langle 0|T(\mathcal{O}(x_1,x_2,x_3,\ldots)\bar{\psi}_\gamma A^\mu(-ie\gamma_\mu)_{\gamma\delta}\psi_\delta)(z)|0\rangle = \ldots$$

where the operator $\mathcal{O}$ has the same field content[1] as the interaction term in iS. For standard QED we would choose

$$\mathcal{O} = \psi_\alpha(x_1)\bar{\psi}_\beta(x_2)A^\nu(x_3) \ .$$

Now perform **all possible** Wick contractions that connect the fields in $\mathcal{O}$ with the fields in the interaction term. In our case we find

$$\ldots = i\Delta^\psi_{\alpha\gamma}(x_1,z)\, i\Delta^\psi_{\delta\beta}(z,x_2)\, i\Delta^{\mu\nu}(x_3,z) \times [-ie\gamma_\mu]_{\gamma\delta} \ . \tag{3}$$

All left over operators that act on coordinates can be perform by considering the Fourier transformed propagators as we are only interested in the momentum space momentum space

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[1]With the fermions ordered in such a way that no minus signs arise from commutations.

Feynman rules. In the QED example there are no differential operator to take into account. Thus, the vertex is given by everything that is **not** part of a propagator:

$$[-ie\gamma_\mu]_{\gamma\delta} \ .$$

Here, it is important that the propagators must not share any common index (Dirac or Lorentz). All "connections" between propagators are part of the vertex, e.g. if we had found a product of the form

$$i\Delta^\psi_{\alpha\gamma}(x_1, z)\, i\Delta^\psi_{\gamma\beta}(z, x_2)$$

it would share a Dirac index. After rewriting it as

$$i\Delta^\psi_{\alpha\rho}(x_1, z)\, \delta_{\rho\sigma}\, i\Delta^\psi_{\sigma\beta}(z, x_2)$$

the $\delta_{\rho\sigma}$ must be associated with the vertex.

(a) Derive, using the same method as described above, the vertex Feynman rules of scalar QED.
As the action contains a product of two covariant derivatives there will be terms with one and two powers of the photon field A^μ; thus, there are two different vertices (with one and two photon fields). It is convenient to start with the two-photon vertex as its derivation is fairly simple.
The single photon vertex is more complicated. You should find that there is a derivative involved. Since we are interested in the momentum space Feynman rules we always assume that incoming and outgoing scalars have definite momentum.This can be used to convert the derivative into a factor of a momentum.

(b) Ward identities provide a powerful check for the vertex rules and can even be used to derive a missing vertex. Write down the full amplitude $\epsilon^\star_\mu \epsilon_\nu M^{\mu\nu}$ for Compton scattering in scalar QED. (This is very similar to what was done in the lecture, but you should find three diagrams that contribute.) Verify that the amplitude fulfils the QED Ward identity. If you could not derive the Feynman rules in part (a), you can use the Feynman rules given in textbooks, e.g. the appendix of "Gauge Theory of elementary particle physics" by Cheng & Li. The book is notorious for typos but the scalar QED rules are fine (this cannot be said about the scalar QCD rules). Please indicate from where you took the vertex rules if you did not derive them.

Problem 2: Gordon Identity — 4 points

The Gordon identity is a very useful property of QED on-shell fermion-photon vertex. It has the form:

$$\bar{u}(p')i\sigma^{\mu\nu}(p'-p)_\nu u(p) = 2m \cdot \bar{u}(p')\gamma^\mu u(p) - (p'+p)^\mu \cdot \bar{u}(p')u(p) \ ,$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$ and m is the mass of both in- and out-going fermion. The derivation of the Gordon identity is straightforward and can be found in almost all text books. Gordon's identity will be important later for the discussion of the anomalous magnetic moment of the electron and the Rosenbluth formula.
Now consider the bilinear

$$\bar{u}_1(p_1)i\sigma^{\mu\nu}\gamma_5(p_1-p_2)_\nu u_2(p_2) = \dots \ , \qquad (4)$$

where particle '1' has momentum p_1 and mass m_1 and particle '2' has momentum p_2 and mass m_2. Derive the analogue of Gordon's identity for (4).