3. The Standard Model of Particle Physics

By the 1960s, the QFT description of electrodynamics had been confirmed with great success. However, strong and weak interactions exibit a rather different phenomenology and it was not clear that these are to be described by QFT as well. Within two decades, many theoretical and experimental concepts had come together to form the Standard Model of Particle Physics:

1961: "Eightfold Way" — Classification of Mesons and Spin-$\frac{1}{2}$ Baryons into $SU(3)$ flavour octets, spin-$\frac{3}{2}$ Baryons into a decuplet with the prediction of Ω^- (1962)

1964: Discovery of the Ω^-

1964: Quark model (Gell-Mann & Zweig)

1964: Spontaneous symmetry breaking in gauge field theories with pertaining Higgs bosons (Brout, Englert; Higgs; Guralnik, Hagen, Kibble)

1968: Electroweak unification and symmetry breaking (Glashow, Weinberg, Salam), prediction of $W^\pm$ and in particular Z_0

1968: Experiments on deep inelastic scattering of electrons off nucleons @ SLAC

1970: Glashow, Iliopoulos, Maiani (GIM) mechanism of suppressing flavour-changing neutral currents (FCNCs), prediction of the charm quark (in addition to up, down, strange)

1971: Colour as a quantum number (Fritzsch & Gell-Mann)

1972: Renormalization of the electroweak theory by 't Hooft & Veltman

1973: Asymptotic freedom of QCD (Politzer, Wilczek & Gross)

1973: Neutral currents in the scattering of neutrinos off nucleons @ Gargamelle (CERN)

1974: Discovery of J/ψ (charmonium) by Richter/Ting @ SLAC/BNL

1977: b-quark @ Fermilab (t-quark only in 1995)

1983: Direct discovery of $W^{\pm}$ and Z^0 @ CERN (SPS accelerator, Rubbia and van Meer)

It is needless to say that this list is selective. A main goal of this course is to present the basic QFT behind these points. The keystone of the Standard Model, the Higgs boson, was eventually discovered in 2012.

The two main features that account for the difference of weak and strong interactions from electrodynamics are their non-Abelian structure and spontaneous symmetry breaking. We will therefore next review these aspects.

3.1 Non-Abelian Gauge Symmetry

Let us assume that the generators of a Lie group are given by t^a. The group is characterised by the Lie algebra

$$[t^a, t^b] = i f^{abc} t^c$$

where f^{abc} are the structure constants. One may choose a basis for t^a such that f^{abc} is totally antisymmetric. Recall that we have encountered a Lie-algebra and its structure constants already for the Lorentz group, which is not gauged, however. For $SU(2)$, $t^i = \frac{\sigma^i}{2}$ and $f^{abc} = \varepsilon^{abc}$.

Now, we consider the transformation

$$\psi(x) \longmapsto e^{i\alpha^a(x) t^a} \psi(x)$$

In order to construct a covariant derivative, define a matrix-valued function with $U(x,x) = 1$ and that transforms as

$$U(x,y) \longmapsto e^{i\alpha^a(x) t^a} U(x,y) e^{-i\alpha^a(y) t^a}$$

This operator undoes the effect of different gauge transformation parameters at different space-time points. We use it to define the covariant derivative as

$$n^\mu D_\mu \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x) \right]$$

Now consider a small gauge transformation α:

$$U(x+\varepsilon n, x) = \mathbb{1} + i g \varepsilon n^\mu t^a A_\mu^a(x) + \mathcal{O}(\varepsilon^2)$$

$$\longmapsto \mathbb{1} + i g \varepsilon n^\mu t^a \left[A_\mu^a(x) + \Delta A_\mu^a(x)\right] + \mathcal{O}(\varepsilon^2)$$

$$\overset{!}{=} e^{i\alpha^a(x+\varepsilon n) t^a} \left[\mathbb{1} + i g \varepsilon n^\mu t^b A_\mu^b(x) + \mathcal{O}(\varepsilon^2)\right] e^{-i\alpha^c(x) t^c}$$

$$= \left[\mathbb{1} + i\alpha^a(x) t^a + i\varepsilon \frac{\partial \alpha^a(x)}{\partial x^\mu} n^\mu t^a \mathcal{O}(\varepsilon^2, \alpha^2)\right]$$

$$* \left[\mathbb{1} + i g \varepsilon n^\mu t^b A_\mu^b(x) + \mathcal{O}(\varepsilon^2)\right] * \left[\mathbb{1} - i\alpha^c(x) t^c + \mathcal{O}(\alpha^2)\right]$$

$$\Longrightarrow i g \Delta A_\mu^a(x) t^a = i \frac{\partial \alpha^a(x)}{\partial x^\mu} t^a - g \alpha^a(x) [t^a, t^b] A_\mu^b(x)$$

(note that we sum over double indices in each term)

$$\Longrightarrow A_\mu^a(x) t^a \longmapsto \left[A_\mu^a(x) + \Delta A_\mu^a(x)\right] t^a$$

$$= A_\mu^a(x) t^a + \frac{1}{g} \frac{\partial \alpha^a(x)}{\partial x^\mu} t^a - f^{abc} \alpha^a(x) A_\mu^b(x) t^c$$

$$\Longrightarrow A_\mu^a(x) \longmapsto A_\mu^a(x) + \frac{1}{g} \frac{\partial \alpha^a(x)}{\partial x^\mu} + f^{abc} A_\mu^b(x) \alpha^c(x)$$

When α is finite, we can more formally write:

$$i g \varepsilon n^\mu t^a \left[A_\mu^a(x) + \Delta A_\mu^a(x)\right]$$

$$= e^{i\alpha^a(x) t^a} \left[i g \varepsilon n^\mu t^b A_\mu^b(x) + \mathcal{O}(\varepsilon^2)\right] e^{-i\alpha^c(x) t^c}$$

$$- e^{i\alpha(x) t^a} \varepsilon n^\mu \frac{\partial}{\partial x^\mu} e^{-i\alpha^c(x) t^c}$$

$$\Longrightarrow A_\mu^a t^a \longmapsto e^{i\alpha^a(x) t^a} \left(A_\mu^b t^b + i g \frac{\partial}{\partial x^\mu}\right) e^{-i\alpha^c(x) t^c}$$

With these results, the covariant derivative is given by

$$D_\mu = \partial_\mu - i g A_\mu^a t^a$$

In analogy with QED, we define the field strength tensor as

$$F^a_{\mu\nu}\, t^a = \frac{i}{g}\left[D_\mu , D_\nu\right] = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) t^a + g\, f^{abc} A^b_\mu A^c_\nu\, t^a$$

Note that this quantity is not gauge-invariant, as it transforms (for infinitesimal α)

$$F^a_{\mu\nu} t^a \longmapsto F^a_{\mu\nu}\, t^a + \frac{1}{g}\cancel{\partial_\mu \frac{\partial \alpha^a(x)}{\partial x^\nu}}\, t^a - \frac{1}{g}\cancel{\partial_\nu \frac{\partial \alpha^a(x)}{\partial x^\mu}}\, t^a$$

$$+ \partial_\mu f^{abc} A^b_\nu\, \alpha^c(x) t^a - \partial_\nu f^{abc} A^b_\mu(x)\alpha^c(x)\, t^a$$

$$+ g\, f^{abc}\left[\underline{\frac{1}{g}\frac{\partial \alpha^b}{\partial x^\mu}} + \underline{f^{bde} A^d_\mu(x)\alpha^e(x)}\right] A^c_\nu\, t^a$$

$$+ g\, f^{abc} A^b_\mu \left[\underline{\frac{1}{g}\frac{\partial \alpha^c}{\partial x^\nu}} + \underline{f^{cfg} A^f_\nu(x)\,\alpha^g(x)}\right] t^a$$

$$f^{abc} f^{bde} A^d_\mu \alpha^e A^c_\nu + f^{abc} f^{cfg} A^b_\mu \alpha^g A^f_\nu$$

$$= f^{aeg} f^{efb} A^f_\mu \alpha^b A^g_\nu + f^{afc} f^{cgb} A^f_\mu \alpha^b A^g_\nu$$

$$= - f^{abc} \alpha^b f^{cfg} A^f_\mu A^g_\nu$$

Where we have made use of the Jacobi-identity:

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

$$- f^{aed} f^{bcd} + f^{bed} f^{acd} - f^{ced} f^{abd} = 0$$

$$\Longrightarrow - f^{age} f^{bfe} - f^{afc} f^{bgc} = + f^{abc} f^{gfc}$$

$$\Longrightarrow F^a_{\mu\nu} \longmapsto F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}$$

For totally anti-symmetric structure constants however, we easily see that $F^a_{\mu\nu} F^{a\mu\nu}$ is gauge invariant.

The fundamental representation of $SU(3)$ is given by unitary 3×3 matrices with unit determinant. These can be generated by exponentiating i times Hermitian, traceless 3×3 matrices. A basis of these matrices is eight dimensional (in general, for $SU(N)$, N^2-1 dimensional). We also choose the convention that $\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$. The standard choice for QCD is $t^a = \lambda^a$, with λ^a being the <u>Gell-Mann matrices</u>

$$\lambda^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The gauge particles of QCD, that are quanta of the fields A^a_μ $(a=1,\dots,8)$, are called <u>gluons.</u>

Strongly interacting fields (i.e. fields that interact with gluons) can live in any non-trivial representation of $SU(3)$. A representation is any set of operators that satisfy the same multiplication table as the original group.

Each representation can be defined by eight representation matrices that satisfy the Lie Algebra. The quarks of QCD are Dirac fermions that live in the fundamental representation, which is obviously three-dimensional.

We summarise our consideration within the

$$\boxed{\begin{array}{c}\underline{\text{QCD Lagrangian}}\\ \mathcal{L} = \bar{\psi}\,(i\not{D})\psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - m\,\bar{\psi}\psi\end{array}}$$

There are several flavours of quarks $f = \{u, d, c, s, t, b\}$. In the above Lagrangian, we have suppressed the flavour indices. We will make the flavour indices explicit whenever this is notationally convenient.

Before stating the Feynman rules, we should make two remarks. First, unlike for QED, the term $\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$ is not quadratic in A^a_μ, i.e. it contains interactions of three and four gauge bosons. It is an essential feature of non-Abelian gauge theories, that the pure (gauge fields only) gauge sector is self-interacting already at tree-level. Second, which is related to the first remark, the gauge coupling g is the same for all strongly interacting particles. This is because due to the non-linear term in the gauge transformation of A^a_μ, different values of g for different fields cannot be compensated by redefinitions of $\alpha(x)$. A natural and rather elegant explanation for the fractional integer values for the electric charges of elementary particles ($q(e^-) = -1$, $q(u) = \frac{1}{3}$, $q(d) = -\frac{2}{3}$) would be if the $U(1)$ symmetry of QED was emerging from a higher non-Abelian gauge group.

Feynman Rules for QCD

Our Feynman rules for QED are based on the so called Gupta-Bleuler method: After gauge fixing, canonical quantisation is performed and a propagator obtained. The unphysical, longitudinally polarised, negative norm states are then weeded out by hand. This works, because provided all incoming states within a Feynman graph are physical, the same applies to the outgoing ones.
For QCD, it turns out that this is no longer true when simply adapting the Feynman rules of QED: negative norm states may be produced. Within the functional formalism, the gauge fixing of the quantum theory can be performed more systematically by means of the Faddeev-Popov method. It turns out that the effects from the unphysical polarisations are cancelled by unphysical, fermionic but spin-0 "ghost fields". Neither the unphysical polarisations nor the ghosts will appear as external particles. While ghosts in principle are present in QED as well, they decouple due to the Abelian nature of the theory. In QCD, they couple to gluons and may appear in loops due to the self-interacting nature of the gauge sector.
We encourage to study the Faddeev-Popov method. However, as it is covered in the lectures on Quantum Field Theory and as the focus of the present lectures should be less theoretical, we omit their derivation and directly present the gauge-fixed

Lagrangian of QCD:

in fundamental representation　　　　　in adjoint representation

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) + \bar\psi(i\not{D} - m)\psi + \bar{c}\,(-\partial^\mu D_\mu)\,c$$

The anti-commuting scalar field c lives in the adjoint representation (the same that the gluons live in). Hence, there are eight ghosts c^a, $a = 1, \dots, 8$.

As defined above, the covariant derivative is valid for any representation. For the adjoint representation, it is possible to express the representation matrices in terms of the structure constants:

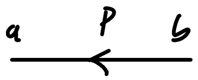
$$[t^b_{adj}]_{ac} = i f^{abc}$$

such that

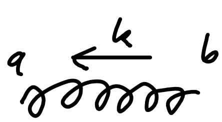
$$D_\mu = \partial_\mu \mathbb{1}_{8\times 8} - ig A^b_\mu [t^b_{adj}] \Rightarrow D^{ab}_\mu = \mathbb{1}\,\delta^{ab} + g A^c_\mu f^{acb}$$

We can now state the

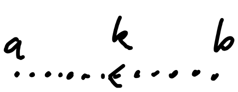
<u>Feynman Rules for QCD:</u>



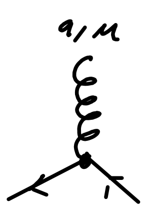
$$= \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}\,\delta^{ab}$$ quark propagator



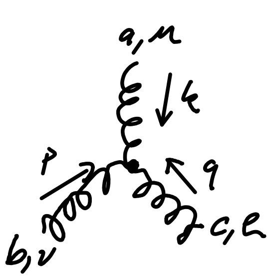
$$= \frac{-i}{k^2 + i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}$$ gluon propagator



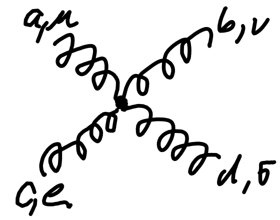
$$= \frac{i}{k^2 + i\varepsilon}\,\delta^{ab}$$ ghost propagator



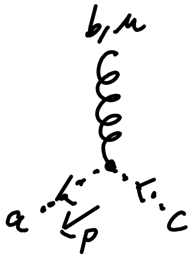
$$= i g^\mu \lambda^a$$



$$= g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$



$$= -ig^2\left[f^{abe} f^{cde}\left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) + f^{ace} f^{bde}\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) + f^{ade} f^{bce}\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right)\right]$$

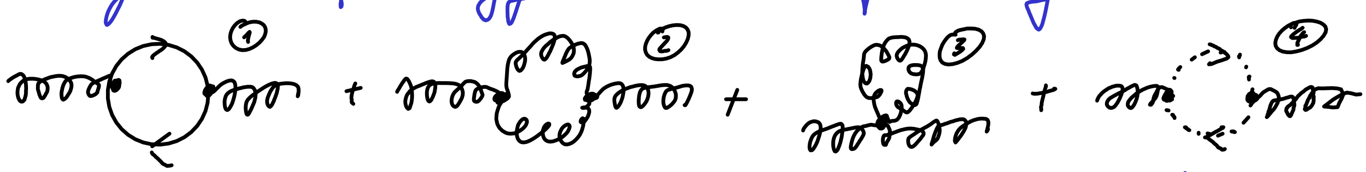


$$= -g\, f^{abc} p^{\mu}$$

The β-Function of QCD - Asymptotic Freedom

We now determine the one-loop corrections to QCD-gauge boson self-energy, quark self-energy & the vertex correction. The goal is to determine how the coupling strength varies with the energy scale of the interaction.

The gluon self-energy receives the following contributions:



Before we start, we remark that the Ward-Takahashi identities generalise in the non-Abelian case to the so-called Slavnov-Taylor identities. We do not discuss these intricate identities here, but note without proof that the Ward identity

$$k_\mu\, i\Pi^{ab\mu\nu}(k) = 0$$

still holds. In the following, we will see this explicitly.

For the first diagram, , we remember an old friend. It is the same diagram as in QED, except for a few proportionality factors, that we now explain

The SU(3) colour algebra gives

$$[\lambda^a]_{ef}\,\delta_{fg}\,[\lambda^b]_{gh}\,\delta_{he} = \mathrm{tr}[\lambda^a\lambda^b]$$

from the fermion propagators

from the vertices

Traces over representation matrices t^a (here in the concrete form λ^a) are frequently encountered within non-Abelian gauge theories. It is therefore useful to treat these in a bit more generality. For a given representation R, the quadratic Casimir operator is defined as

$$t^a_R t^a_R = C_2(R)\, \mathbb{1}_{d(R)\times d(R)}$$

where $d(R)$ is the dimension of the representation (i.e. the dimension of the vector space that the t^a_R are acting on. For $R=F$ the fundamental representation of $SU(N)$, $d(F)=N$, for the adjoint representation $R=G$, $d(G)=N^2-1$).

It is also customary to identify the representations by their dimension, such that $F\equiv N$ and $G\equiv N^2-1$ for $SU(N)$.

In the trace of the self-energy diagram, there appears

$$\mathrm{tr}[t^a_R t^b_R] = C(R)\,\delta^{ab}$$

(The generators are chosen by convention in such a way that this trace is proportional to δ^{ab}, and it may be proved that indeed, it is possible to do so).

Recall that $a,b = 1,\dots,d(G)$. Therefore, there is the relation

$$d(R)\,C_2(R) = d(G)\,C(R)$$

For $SU(N)$,

$$C(N) = \frac{1}{2},\quad C_2(N) = \frac{N^2-1}{2N},\quad C(N^2-1) = C_2(N^2-1) = N,\quad f^{acd}f^{bcd} = C_2(G)\,\delta^{ab}.$$

Finally, we account for the fact that n_f flavours of quarks may propagate within the loop, which gives a simple factor. The vacuum polarisation diagram therefore generalises to

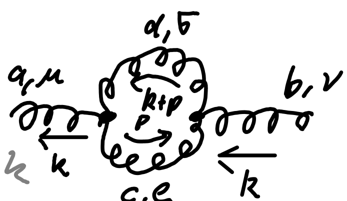
$$i\Pi^{(1)ab}_{\mu\nu}(k) = -\mathrm{tr}[\lambda^a\lambda^b](g_{\mu\nu}k^2 - k_\mu k_\nu)\frac{ig^2 n_f}{2\pi^2}\int_0^1 dx\; x(1-x)\left[\Delta_\varepsilon - \log\frac{m^2 - x(1-x)k^2}{\mu^2}\right]$$

$$\Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi$$

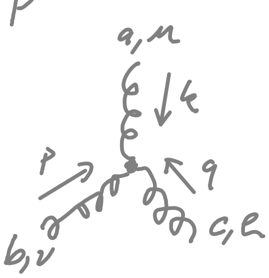
We are interested here in the scale-dependence of the strong coupling constant. For this purpose, we only need to keep the divergent contributions and the dependence on the scale μ, such that

$$i\Pi^{(1)ab}_{\mu\nu}(k) = i(g_{\mu\nu}k^2 - k_\mu k_\nu)\,\delta^{ab}\left[-\frac{g^2}{(4\pi)^2}\,\frac{4}{3}\,n_f\,C(R)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)\right]$$

where # stands for a dimensionful constant that is irrelevant for the determination of the β-function.
For the remaining contributions to the gluon self-energy, we quote the expressions for the Feynman diagrams and the final result for the μ-dependence.



$$= i\Pi^{(2)ab}_{\mu\nu}(k) = \frac{1}{2}\mu^\varepsilon g^2\int\frac{d^dp}{(2\pi)^d}\,\frac{-i}{p^2+i\varepsilon}\,\frac{-i}{(p+k)^2+i\varepsilon}\,f^{acd}f^{bcd}N^{\mu\nu}$$



$$= g f^{abc}\left[g^{\mu\nu}(k-p)^e + g^{\nu e}(p-q)^\mu + g^{e\mu}(q-k)^\nu\right]$$

$$N^{\mu\nu} = \left[-g^{\mu e}(k-p)^\sigma - g^{e\sigma}(2p+k)^\mu + g^{\sigma\mu}(2k+p)^\nu\right]$$

$$*\left[\delta^\nu{}_e(k-p)_\sigma + g_{e\sigma}(2p+k)^\nu - \delta_\sigma{}^\nu(2k+p)_e\right]$$

$k \to -k$ $\quad \mu \to \mu$ $\quad a \to a$
$p \to -p$ $\quad \nu \to e$ $\quad b \to c$
$q \to k+p$ $\quad e \to \sigma$ $\quad c \to d$

$k \to k$ $\quad \mu \to \nu$ $\quad a \to b$
$p \to p$ $\quad \nu \to e$ $\quad b \to c$
$q \to -k-p$ $\quad e \to \sigma$ $\quad c \to d$

Introducing a Feynman parameter

$$\frac{1}{p^2+i\varepsilon}\,\frac{1}{(p+k)^2+i\varepsilon} = \int_0^1 dx\, \frac{1}{\left[(1-x)p^2 + x\,(p+k)^2 + i\varepsilon\right]^2}$$

and performing a Wick rotation, one obtains

$$i\Pi^{② \, ab}_{\mu\nu}(k) = i\,\frac{g^2}{(4\pi)^{\frac{d}{2}}}\,\mu^{\varepsilon}\, C_2(G)\,\delta^{ab}\int_0^1 dx\,\frac{1}{\left[-x(1-x)k^2+i\varepsilon\right]^{2-\frac{d}{2}}}$$

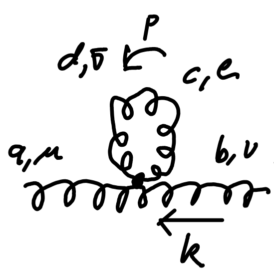
$$* \Big(\Gamma\left(1-\tfrac{d}{2}\right) g^{\mu\nu} k^2 \left[\tfrac{3}{2}(d-1)\,x\,(1-x)\right]$$

$$+ \Gamma\left(2-\tfrac{d}{2}\right) g^{\mu\nu} k^2 \left[\tfrac{1}{2}(2-x)^2 + \tfrac{1}{2}(1+x)^2\right]$$

$$- \Gamma\left(2-\tfrac{d}{2}\right) k^{\mu} k^{\nu} \left[\left(1-\tfrac{d}{2}\right)(1-2x)^2 + (1+x)(2-x)\right]\Big)$$

Notice that the first term is singular for $d \to 2$, due to a quadratic divergence. However, after cancellation with additional loops, we will find below that just like QED, QCD is free of quadratic divergences.

The next diagram is



$$= i\Pi^{③ \, ab}_{\mu\nu} = \frac{1}{2}\,\mu^{\varepsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{-i\,g_{e\sigma}}{p^2+i\varepsilon}\,\delta^{cd}\,(-ig^2)$$

$$* \left[f^{abe} f^{cde}\left(g^{\mu e} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu e}\right) \right.$$

$$+ f^{ace} f^{bde}\left(g^{\mu\nu} g^{e\sigma} - g^{\mu\sigma} g^{\nu e}\right)$$

$$\left. + f^{ade} f^{bce}\left(g^{\mu\nu} g^{e\sigma} - g^{\mu e} g^{\nu\sigma}\right)\right]$$

The first term vanishes due to the antisymmetry of f^{cde}. The last two simplify using the identity $f^{acd} f^{bcd} = C_2(G)\,\delta^{ab}$, such that

$$i\Pi^{③ \, ab}_{\mu\nu} = -\mu^{\varepsilon} g^2 C_2(G)\,\delta^{ab}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+i\varepsilon}\,g^{\mu\nu}(d-1)$$

Recall that

$$I_{r,m} = \int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^r}{[l^2 - \Delta + i\varepsilon]^m}$$

$$= i(-1)^{r-m} \frac{1}{(4\pi)^{2-\frac{\varepsilon}{2}}} \left(\frac{1}{\Delta}\right)^{-m+r+2-\frac{\varepsilon}{2}} \frac{\Gamma(r+2-\frac{\varepsilon}{2})\Gamma(m-r-2+\frac{\varepsilon}{2})}{\Gamma(2-\frac{\varepsilon}{2})\Gamma(m)}$$

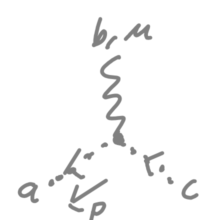
Above integral is therefore $x^\varepsilon = e^{\varepsilon \log x} \approx 1 + \varepsilon \log x$

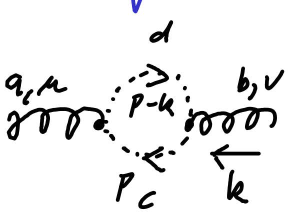
$$(d-1) I_{0,1} = -i \frac{1}{(4\pi)^d} (0)^{-1+\frac{d}{2}} \Gamma(1-\tfrac{d}{2})$$

This vanishes for $d \to 4$, but is singular for $d=2$. We therefore need to account for this diagram in order to demonstrate the cancellation of the quadratic divergences. In order to make this diagram to match the form of the previous one in terms of the Feynman parameter, we multiply the integrand by $1 = \frac{(p+k)^2}{(p+k)^2}$. This way, one finds

$$i\Pi^{③\, ab}_{\mu\nu}(k) = \frac{i g^2 \mu^\varepsilon}{(4\pi)^{\frac{d}{2}}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{[-x(1-x)k^2 + i\varepsilon]^{2-\frac{d}{2}}}$$

$$* \Big(-\Gamma(1-\tfrac{d}{2})\, g^{\mu\nu} k^2 \left[\tfrac{1}{2} d(d-1)\, x(1-x)\right] - \Gamma(2-\tfrac{d}{2})\, g^{\mu\nu} k^2 \left[(d-1)(1-x)^2\right]\Big)$$

The ghost loop is  $= -g f^{abc} p^\mu$



$$= i\Pi^{④}_{\mu\nu}(k) = -\mu^\varepsilon g^2 \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 + i\varepsilon} \frac{i}{(p-k)^2 + i\varepsilon}$$

from the ghost loop

$$* f^{dac} (p-k)^\mu f^{cbd} p^\nu$$

$$= \frac{i\mu^{\varepsilon} g^2}{(4\pi)^{\frac{d}{2}}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{\left[-x(1-x)k^2 + i\varepsilon\right]^{2-\frac{d}{2}}}$$

$$* \left(-\Gamma(1-\tfrac{d}{2})\, g^{\mu\nu} k^2 \left[\tfrac{1}{2} x(1-x)\right] + \Gamma(2-\tfrac{d}{2})\, k^\mu k^\nu \left[x(1-x)\right]\right)$$

The coefficient of the quadratic divergence $\Gamma(1-\frac{d}{2}) g^{\mu\nu} k^2 x(1-x)$ from the diagrams 2, 3, 4 is

$$\underbrace{\frac{3}{2}(d-1)}_{2} \underbrace{- \frac{1}{2} d(d-1)}_{3} \underbrace{- \frac{1}{2}}_{4} = -\frac{1}{2}d^2 + 2d - 2 = (1-\tfrac{d}{2})(d-2)$$

This coefficient renders the terms $\propto \Gamma(1-\frac{d}{2})$ as vanishing when $d \to 2$, thus there are no quadratic divergences in QCD.

Yet, these terms are divergent for $d \to 4$. In order to bring them to the form of the purely logarithmic divergences, write

$$(1-\tfrac{d}{2})\Gamma(1-\tfrac{d}{2}) = \Gamma(2-\tfrac{d}{2})$$

The coefficient of the term $\Gamma(2-\frac{d}{2}) g^{\mu\nu} k^2$ is thus

$$\underbrace{(d-2)x(1-x)}_{2,\,3,\,4} + \underbrace{\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2}_{2} \underbrace{- (d-1)(1-x)^2}_{3}$$

$$= (-2d+4)x^2 + (3d-5)x + \frac{7}{2} - d$$

The integrand is symmetric in $x \longleftrightarrow (1-x)$. Therefore, under the integral, we may replace

$$x \longrightarrow \frac{1}{2}x + \frac{1}{2}(1-x) = \frac{1}{2},$$

such that the coefficient becomes

$$4(1-\tfrac{d}{2})x^2 + 2dx + \frac{d}{2} - 4x - \frac{1}{2} + \frac{7}{2} - d$$

$$= 4\left(1-\frac{d}{2}\right)x^2 - 4\left(1-\frac{d}{2}\right)x + \left(1-\frac{d}{2}\right) + 2$$

$$= \left(1-\frac{d}{2}\right)(1-2x)^2 + 2$$

Now, we collect the coefficients of $\Gamma(2-\frac{d}{2})k^\mu k^\nu$:

$$\underbrace{-\left(1-\frac{d}{2}\right)(1-2x)^2 - (1+x)(2-x)}_{2} + \underbrace{x(1-x)}_{4} = -\left(1-\frac{d}{2}\right)(1-2x)^2 - 2$$

Putting these results together, one obtains

$$i\Pi_{\mu\nu}^{\overset{\text{pure gauge}}{PG}\,ab}(k) = i\Pi_{\mu\nu}^{2\,ab}(k) + i\Pi_{\mu\nu}^{3\,ab}(k) + i\Pi_{\mu\nu}^{4\,ab}(k)$$

$$= \frac{ig^2\mu^\varepsilon}{(4\pi)^{\frac{d}{2}}} C_2(G)\delta^{ab}\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\left[-x(1-x)k^2 + i\varepsilon\right]^{2-\frac{d}{2}}}(k^2 g^{\mu\nu} - k^\mu k^\nu)$$
$$* \left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right]$$

This is manifestly transversal, as required by the Ward identity. In view of our goal to obtain the β-function, we extract the divergence and the dependence on the renormalisation scale μ as:

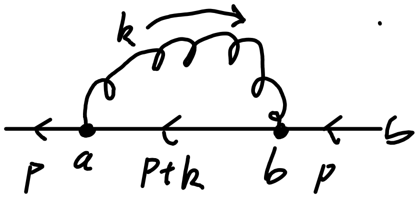
$$i\Pi_{\mu\nu}^{PG\,ab}(k) = i(k^2 g^{\mu\nu} - k^\mu k^\nu)\delta^{ab}\left[\frac{-g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)C_2(G)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \ldots\right)\right]$$

By imposing renormalisation conditions, we require that the corrections to the gluon self-energy are cancelled by the counterterm

$$\delta Z_3 = \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f\, C(R)\right)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#}\right)$$

In QED, the vacuum polarisation is sufficient to compute the running coupling, as the contributions from the fermion self-energy and the vertex correction neutralise due to the Ward-Takahashi identity. The latter does not hold for QCD, such that we now calculate these

two corrections as well.
We know the fermionic wave function correction already. It remains to determine the colour factor



$$-i\Sigma(p) = -\mu^{\varepsilon} g^2 \int \frac{d^dk}{(2\pi)^d}\, t^a \frac{\delta^{ab}}{k^2-\lambda^2+i\varepsilon}\, t^b \gamma^\mu \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\varepsilon}\gamma^\mu$$

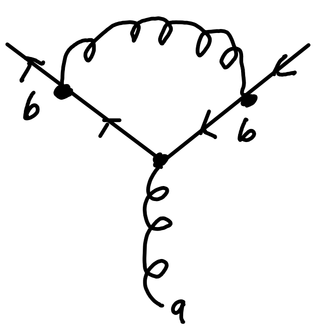
Compared to the QED case, there is an additional factor of a $d(R)\times d(R)$ matrix, $t^a_R t^a_R = C_2(R)\,\mathbb{1}_{d(R)\times d(R)}$. In the following, we suppress the explicit notation of the matrix structure. For a massless quark, we obtain

$$-i\Sigma(p) = i\not{p}\,\frac{g^2}{(4\pi)^2}C_2(R)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)$$

The corresponding counterterm is

$$\delta Z_2 = -\frac{g^2}{(4\pi)^2}C_2(R)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)$$

For the vertex correction, there are two contributions. The first one is again familiar from QED, up to a colour factor:



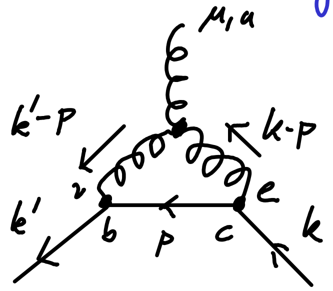
The new feature is here the product of representation matrices

$$t^b t^a t^b = t^b t^b t^a + t^b[t^a, t^b] = C_2(R)t^a + i t^b f^{abc}t^c$$
$$= C_2(R)t^a + \frac{i}{2}f^{abc}[t^b, t^c] = C_2(R)t^a - \frac{1}{2}f^{abc}f^{bcd}t^d$$
$$= \left[C_2(R) - \frac{1}{2}C_2(G)\right]t^a$$

Hence, this diagram evaluates to

$$i\frac{g^3}{(4\pi)^2}\left[C_2(R) - \frac{1}{2}C_2(G)\right]t^a\gamma^\mu\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)$$

The second diagram is



$$= \int \frac{d^4p}{(2\pi)^4} (ig\gamma_\nu t^b) \frac{i\not{p}}{p^2+i\varepsilon} (ig\gamma_e t^c) \frac{-i}{(k'-p)^2+i\varepsilon} \frac{-i}{(k-p)^2+i\varepsilon}$$
$$* g f^{abc}\left[g^{\mu\nu}(2k'-k-p)^e + g^{\nu e}(-k'+k+2p)^\mu + g^{e\mu}(2k-k'-p^\nu)\right]$$

The colour factor is

$$f^{abc} t^b t^c = \frac{1}{2} f^{abc}[t^b, t^c] = \frac{i}{2} f^{abc} f^{bcd} t^d = \frac{i}{2} C_2(G) t^a$$

The result for this diagram is

$$\frac{ig^3}{(4\pi)^2} \frac{3}{2} C_2(G) t^a \gamma^\mu \left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)$$

The divergences in the vertex correction are then cancelled by the counterterm

$$\delta Z_1 = -\frac{g^2}{(4\pi)^2}\left[C_2(R) + C_2(G)\right]\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)$$

As for QED, we now take the point of view that $\delta Z_{1,2,3}$ are parameters in a fundamental Lagrangian, while g and μ may be chosen arbitrarily, with the constraint that they reproduce the observed coupling g_{eff}, where

$$g_{eff} = g\left[1 - \frac{g^2}{(4\pi)^2}\left[C_2(R) + C_2(G)\right]\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right) - \delta Z_1\right]^{-1}$$
$$* \left[1 - \frac{g^2}{(4\pi)^2} C_2(R)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right) - \delta Z_2\right]$$
$$* \left[1 + \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(R)\right)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#}\right) - \delta Z_3\right]^{\frac{1}{2}}$$

When g_{eff} is independent of μ, $g(\mu)$ must have the property

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} = \frac{g^3}{(4\pi)^2}\left[-2\left(C_2(R) + C_2(G)\right) + 2C_2(R) - \frac{5}{3}C_2(G) + \frac{4}{3}n_f C(R)\right]$$

$$= -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(R)\right]$$

Recall that in the Chapter on QED, we noted that provided we may expand
$\beta(g) = b\, g^n + \dots$, it follows that

$$g^{n-1}(\mu) = \frac{g^{n-1}(\mu_0)}{1 - b(n-1)g^{n-1}(\mu_0)\log\frac{\mu}{\mu_0}}$$

For the present case, SU(N) gauge theory with n_f fundamental Dirac fermions,

$$C_2(G = N^2-1) = N, \quad C(R = N) = \frac{1}{2},$$

$$b = -\frac{1}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2}{3}n_f\right], \quad n = 3$$

such that we obtain the <u>running coupling of QCD</u>

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{1}{2}\alpha(\mu_0)\left[\frac{11}{3}N - \frac{2}{3}n_f\right]\log\frac{\mu}{\mu_0}}$$

A striking feature is that provided $N > \frac{2}{11}n_f$, the coupling decreases with larger μ (recall that $\alpha(\mu)$ may be interpreted as the measured coupling at momentum transfer $-q^2 \sim \mu^2$). That is, the theory is <u>asymptotically free</u> and perturbatively well-defined up to arbitrarily high energies.
Asymptotic freedom would therefore be an attractive possibility of UV-completion of field theory.
Unfortunately however, gravitational interactions have a rather different behaviour under loop corrections than

QED and QCD and exhibit in particular no asymptotic freedom.

In order to give a quantitative meaning to above relation, we have to fix $\alpha(\mu_0)$ at a certain scale μ_0. Usually, the reference scale is chosen to be the rather well-known Z_0-boson mass

$M_Z = (91,1876 \pm 0,0021)$ GeV (PDG booklet 2012, p. 9).

(In the following, we will refer to this reference in short by "PDG"). The observed value is

$\alpha(M_Z) = 0,1184 \pm 0,0007$

The running of the coupling is well-confirmed experimentally, cf. PDG, Fig. 9.4.

Another interesting number to keep in mind is the value of $\Lambda_{QCD} = \mu$, for which the Landau pole (vanishing denominator) of the running coupling occurs. This celebrated quantity roughly takes the value

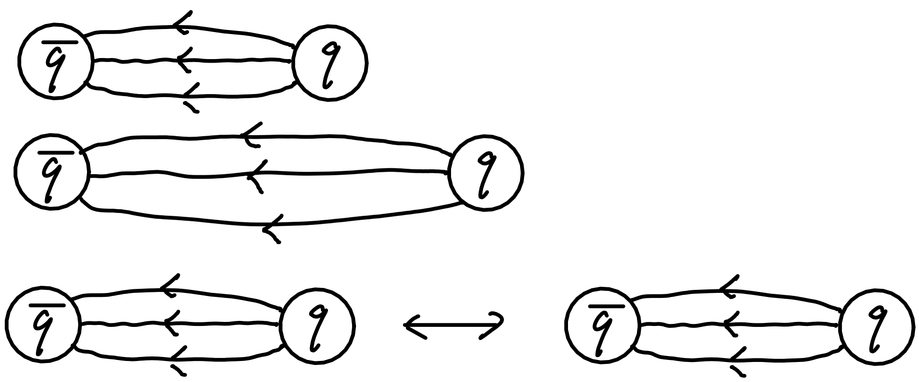
$$\Lambda_{QCD} = (250 \pm 100) \text{ MeV}$$

From above formula with $n_f = 4$ or 5 (accounting for the quarks u, d, c, s and partly b, while t is way above the Z mass), we would obtain a larger value, around 10 GeV. In order to obtain Λ_{QCD} theoretically, one indeed needs to include next-to-next-to leading order (NNLO) corrections, as one extends to regions where perturbation theory converges slowly and eventually breaks down. In summary, our formula for the running coupling should only be applied when $\mu \gg \Lambda_{QCD}$

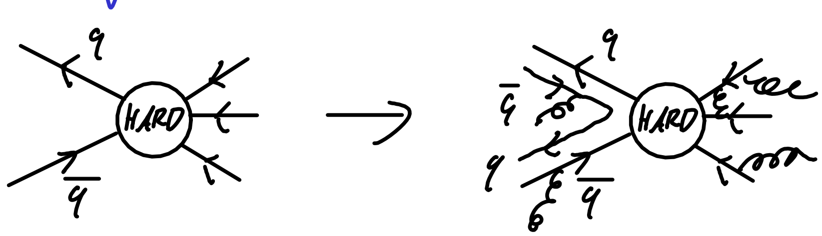
and be loop-improved otherwise.

Now, what happens below Λ_{QCD}? Way above this energy scale, our analytic calculation exhibits asymptotic freedom, below, there is <u>confinement</u>. This is the phenomenon that there are no isolated particles that carry colour charge (quarks and gluons). So far, there is no analytic proof, but overwhelming experimental evidence as well as theoretical evidence from lattice simulations of QCD.

Typical pictures of confinement include that upon separation of two quarks, a flux tube of constant force forms. At some point, the formation of a second quark anti-quark pair becomes more favourable:



In high energy scatterings, colour-non singlet states interact at the elementary level, but at large distances, they are accompanied by additional particles, such that there are only colour-singlets. A simple process of this kind may be pictured as



Factorisation states that the amplitudes can be obtained by convoluting the hard amplitudes with functions that encompass the soft, non-perturbative properties that should be obtained experimentally or on the lattice. The proof is technically involved and could be the content of a seperate lecture course. Nonetheless, the concept is somewhat intuitive and we will use it in the following discussion on deep inelastic scattering.

Before getting into that, we should state that at low energy, the strongly interacting particles are bound states, that are colour singlets and only participate in strong interactions through their constituents. These particles are called hadrons. Mesons are quark-antiquark pairs, whereas baryons consist (after subtracting antiquarks from quarks) of three valence-quark. The two most common specimen are the nucleons, the proton (uud) and the neutron (udd), with total spin $\frac{1}{2}$. Besides up and down, there are the quark flavours

		charge Q_i	mass
u	"up"	$\frac{2}{3}$	1,7–3,1 MeV
d	"down"	$-\frac{1}{3}$	4,1–5,7 MeV
c	"charm"	$\frac{2}{3}$	1,3 GeV
s	"strange"	$-\frac{1}{3}$	100 MeV
t	"top"	$\frac{2}{3}$	173 GeV
b	"bottom"	$-\frac{1}{3}$	4,2 GeV

In addition to the valence quarks, there are pairs (quark & antiquarks) of sea quarks in the nucleon. These may be from each flavour, but in practice, only c and s are of importance next to u and d. (Recall that $m_p = 938,3$ MeV, $m_n = 939,6$ MeV.) Of importance for the structure of hadrons are of course also the gluons. Unlike from QED, where the wave-functions of e.g. atoms or positronium can be calculated, the non-perturbative nature of QCD at low energies has so far prevented us from achieving an analytic understanding of the structure of hadrons. Fortunately, in order to make predictions for high-energy interactions, only a course knowledge in terms of structure functions is needed. In the following Section, we are concerned with the properties of the proton in high-energy interactions. A more complete characterisation of hadrons will be given once we have obtained a more detailed understanding of flavour in the light of Electroweak unification, i.e. in the next semester.

The values of the lighter quark masses are determined from the observation of their bound states. In the next semester, we discuss how the u, d, s masses are related to the masses of the π and K mesons due to chiral symmetry breaking, while the c, d masses follow from more detailed considerations of the potential of their mesonic (quark-antiquark pairs) bound states.

Nonetheless, it is interesting to demonstrate the reality of quarks from pair production in e^+e^- collisions. Recall that for the total cross section for muon production, we obtained:

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{\pi\alpha^2}{3}\frac{\sqrt{\vec{p}^{\,2}-m_\mu^2}}{|\vec{p}|^3}\left(1+\frac{m_\mu^2}{\vec{p}^{\,2}}\right) \underset{m_\mu \ll |\vec{p}|}{\approx} \frac{4\pi\alpha^2}{3E_{CMS}}$$

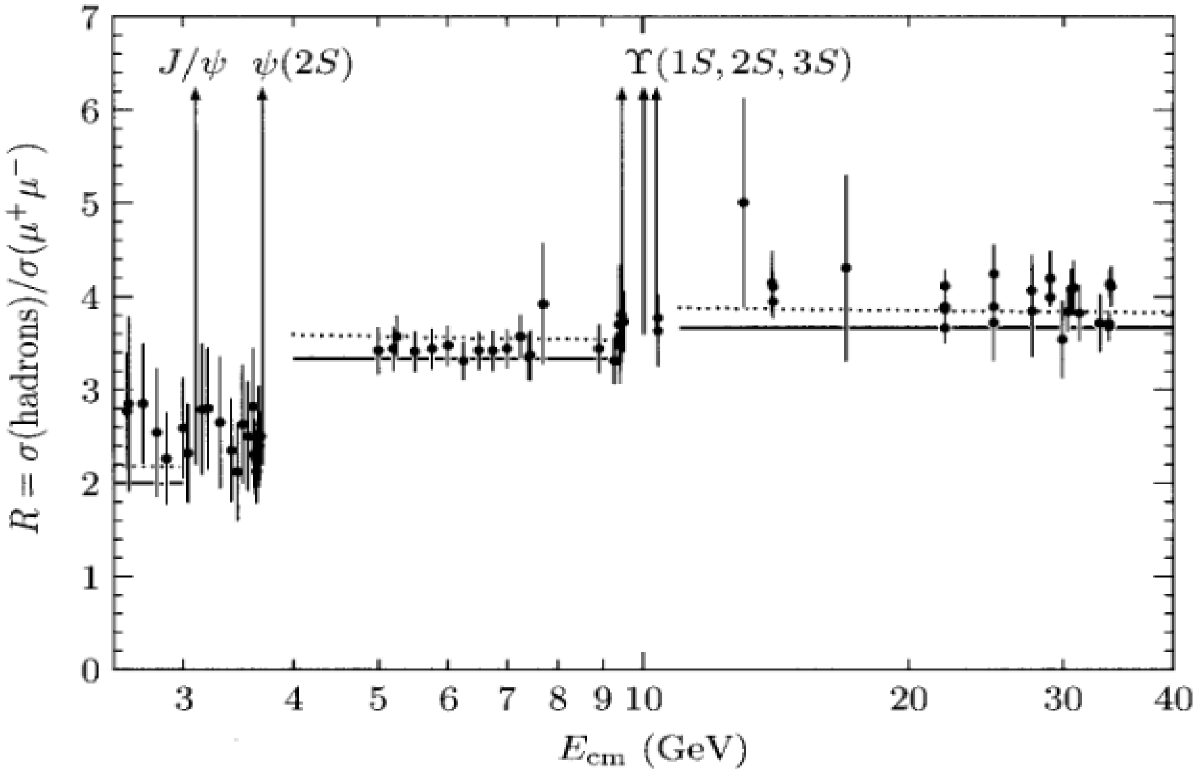


Figure 5.3. Experimental measurements of the total cross section for the reaction $e^+e^- \to$ hadrons, from the data compilation of M. Swartz, *Phys. Rev.* **D53**, 5268 (1996). Complete references to the various experiments are given there. The measurements are compared to theoretical predictions from Quantum Chromodynamics, as explained in the text. The solid line is the simple prediction (5.16).

from Peskin & Schroeder

We can directly infer that the cross section for hadron (particles containing strongly interacting constituents) is given by

$$\sigma_{e^+e^- \to \text{hadrons}} = \underset{\text{number of colours}}{3} \left(\sum_i \underset{\text{quark charge}}{Q_i^2} \right) \sigma_{e^+e^- \to \mu^+\mu^-}$$

Of course, not all quarks are produced at low energies, but the heavier flavours only kick in when E_{CMS} exceeds twice their mass. This behaviour is neatly demonstrated in Figure 5.3 of Peskin & Schroeder. The height of the first step is $3 * \left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right) = 2$, of the second step $2 + 3 * \left(\frac{2}{3}\right)^2 = 3\frac{1}{3}$ and of the third step $3\frac{1}{3} + 3 * \left(\frac{1}{3}\right)^2 = 3\frac{2}{3}$.

3.2 Spontaneous Symmetry Breaking & Electroweak Unification

Abelian Higgs Model

We have seen that adding a mass term for a gauge boson breaks the gauge invariance of a Lagrangian. In turn, gauge symmetry protects the vanishing of the gauge boson masses from radiative corrections. Nonetheless, there are massive spin-1 bosons in Nature. Some of these have been identified as bound states of a quark and an anti-quark with angular momentum one. (It was for these particles, that the Higgs mechanism was proposed originally, even though it turns out that it does not apply there.) The important specimen in this Chapter are the force carriers of the weak interactions, $W^\pm$ and Z^0. As force carriers, these should be categorised as gauge particles, however, they are rather massive (80,39 GeV and 91,188 GeV).

The <u>Higgs mechanism</u> is a way to reconcile masses with gauge invariance which is based on <u>spontaneous symmetry breaking</u>. We explain the salient features on the Abelian $U(1)$ model and then adapt these to the relevant case of $SU(2)\times U(1)$.

Consider therefore the $U(1)$ symmetric Lagrangian for a complex scalar field Φ:

$$\mathcal{L} = (\partial_\mu \Phi)(\partial^\mu \Phi)^* + \mu^2 |\Phi|^2 - \lambda\, |\Phi|^4$$

The potential is of the "Mexican Hat" type and has minima for

$$2\mu^2 - 4\lambda\, |\Phi|^2 = 0 \Rightarrow \langle |\Phi| \rangle = \frac{v}{\sqrt{2}} = \sqrt{\frac{\mu^2}{2\lambda}}$$

that is, Φ acquires a vacuum expectation value (VEV).

Now decompose $\Phi = \frac{1}{\sqrt{2}}(\Phi_R + i\,\Phi_i)$ in two real degrees of freedom. Then,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_R)(\partial^\mu \Phi_R) + \frac{1}{2}(\partial_\mu \Phi_I)(\partial^\mu \Phi_I) + \frac{1}{2}\mu^2(\Phi_R^2 + \Phi_I^2) - \frac{\lambda}{4}(\Phi_R^2 + \Phi_I^2)^2$$

Next, we make use of the $U(1)$ symmetry to have Φ pointing into the real direction:

$$\sqrt{2}\langle \Phi \rangle = \langle \Phi_R \rangle = v$$

$$0 = \langle \Phi_I \rangle$$

Since v is a large classical vacuum expectation value, we should expand around it in order to identify the quantum particle excitations:

$$\Phi = \frac{1}{\sqrt{2}}(v + \phi_R + i\,\phi_I)$$

The Lagrangian then becomes:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_R)(\partial^\mu \phi_R) + \frac{1}{2}(\partial_\mu \phi_I)(\partial^\mu \phi_I) - \mu^2 \phi_R^2 - \sqrt{\lambda}\,\mu\,\phi_R(\phi_R^2 + \phi_I^2) - \frac{\lambda}{4}(\phi_R^2 + \phi_I^2)^2 + \frac{1}{4\lambda}\mu^4$$

When a Lagrangian has a symmetry that is broken by a particular field configuration, one speaks of spontaneous symmetry breaking. In the present case, a global $U(1)$ symmetry is spontaneously broken. The real field ϕ_R with mass $\sqrt{2}\,\mu$ is called a

Higgs boson, whereas the massless field ϕ_I is a Goldstone boson. One can prove that the number of Goldstone bosons equals the number of broken symmetries and that furthermore, the vanishing of the Goldstone boson mass is protected against radiative corrections.

Now, we promote the global $U(1)$ to a local one by gauging it, i.e. we add the gauge kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and replace $\partial_\mu \longmapsto D_\mu = \partial_\mu + ie A_\mu$.
Expanding around the VEV, the kinetic term becomes

$$\frac{1}{2}\left[(\partial_\mu + ieA_\mu)(v+\phi_R+i\phi_I)\right]\left[(\partial^\mu - ieA^\mu)(v+\phi_R-i\phi_I)\right]$$

$$= \frac{1}{2}(\partial_\mu\phi_R)(\partial^\mu\phi_R) + \frac{1}{2}(\partial_\mu\phi_I)(\partial^\mu\phi_I) + evA_\mu\partial^\mu\phi_I + \frac{1}{2}e^2v^2A_\mu A^\mu$$

$$+ \text{interaction terms}$$

$$= \frac{1}{2}(\partial_\mu\phi_R)(\partial^\mu\phi_R) + \frac{1}{2}(\partial_\mu\phi_I)(\partial^\mu\phi_I) + \frac{1}{2}(\partial_\mu\phi_I + evA_\mu)(\partial_\mu\phi_I + evA^\mu)$$

We can now fix a gauge by choosing Φ to point in the real direction, which would amount to $\phi_I \equiv 0$. This is called <u>unitary gauge</u>. Apparently, the same can be achieved by $A_\mu \longmapsto A_\mu - \frac{1}{ev}\partial_\mu\phi_I(x)$.
Note that the resulting Lagrangian is no longer gauge invariant due to this gauge fixing.

We have thus removed the Goldstone-boson entirely from the theory! However, a remnant is still present, because the number of field degrees of freedom is conserved and there is a new longitudinal polarisation state of the gauge boson. One colloquially speaks of the gauge boson haven "eaten" the Goldstone boson.

The disappearance of the Goldstone boson relies on unitary gauge. We it comes to loop calculations, this is no longer convenient as it obscures the renormalisability of the theory. For the Electroweak gauge interactions, a systematic gauge fixing may again be achieved by the means of the Faddeev Popov gauge fixing procedure, but for the tree-level effects that we will discuss, the unitary gauge will often prove applicable.

Electroweak Gauge Theory

Electroweak theory is based on the gauge group $SU(2)_L \times U(1)_Y$, where L indicates that only left-handed quarks and leptons form $SU(2)_L$-doublets and Y denotes the weak hypercharge of matter particles. The gauge boson Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \qquad (i=1,2,3)$$

where

$$W^i_{\mu\nu} = \partial_\mu W^i_\mu - \partial_\nu W^i_\mu - g_W \varepsilon^{ijk} W^j_\mu W^k_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

and where g_W is the $SU(2)_L$ gauge coupling.
Matter couples via the covariant derivative

$$D^\mu_{ij} = \delta_{ij} \partial^\mu + i g_W (T \cdot W^\mu)_{ij} + i Y \delta_{ij} g_W' B^\mu$$

The representation matrices of the representation R of $SU(2)_L$ are given by T, and $i,j = 1, \dots, d(R)$. Note that for $SU(2)_L$, the structure constants are given by the ε-tensor:

$$[T^i, T^j] = i\,\varepsilon^{ijk}\,T^k$$

Now, define

$$W_\mu^{\pm} = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2) \quad \text{and} \quad T^{\pm} = T^1 \pm i T^2$$

$$\Longrightarrow W_\mu \cdot T = W_\mu^3 T^3 + \frac{1}{\sqrt{2}} W_\mu^+ T^+ + \frac{1}{\sqrt{2}} W_\mu^- T^-$$

$$T^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[T^+, T^-] = 2T^3, \qquad [T^3, T^{\pm}] = \pm T^{\pm}$$

The matrix T^3 is called the weak isospin operator.

Next, we aim to spontaneously break the $SU(2)_L \times U(1)_Y$ symmetry such that there are three massive gauge bosons ($W^{\pm}, Z^0$) and one massless (the photon A).

Now we introduce the Higgs field, which is an $SU(2)_L$ doublet (i.e. it is in the fundamental representation) with weak hypercharge $Y = +\frac{1}{2}$:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

The superscripts of the components denote the electric charge which we will identify below.

Explicitly, the Lagrangian for this field with a potential that leads to spontaneous symmetry breaking is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger - i g_w \phi^\dagger_\mu W \cdot T - \frac{1}{2} i g'_w B_\mu \phi^\dagger)$$
$$* (\partial^\mu \phi + i g_w W^\mu \cdot T \phi + \frac{1}{2} i g'_w B^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

We use the freedom of $SU(2)_L \times U(1)_Y$ rotations to express the VEV as

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

We note that transformations generated by T^3+Y leave this VEV invariant:

$$(T^3+Y)\langle\phi\rangle = \left[\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}0\\ v\end{pmatrix} = 0$$

This combination of the symmetry remains therefore unbroken. There should be a massless photon and we identify the electric charge

$$Q = T^3 + Y$$

Now we expand ϕ around the minimum of the potential as

$$\phi = U^{-1}(\xi)\begin{pmatrix}0\\ \frac{H+v}{\sqrt{2}}\end{pmatrix} \quad \text{where} \quad U(\xi) = e^{-i\frac{T\cdot\xi}{v}}$$

There are still four real degrees of freedom, the three Goldstone modes ξ and the Higgs field H. The gauge transformation

$$\phi \longmapsto U(\xi)\phi$$
$$T\cdot W^\mu \longmapsto U\, T\cdot W^\mu U^{-1} + \frac{i}{g_W}(\partial^\mu U)U^{-1}$$

brings us to unitary gauge and removes the ξ from the game. We define $\chi = \begin{pmatrix}0\\1\end{pmatrix}$ and are left with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}\mu^2(H+v)^2 - \frac{\lambda}{4}(H+v)^4$$
$$+ \frac{(v+H)^2}{8}\chi^\dagger(2g_W T\cdot W_\mu + g_W' B_\mu)(2g_W T\cdot W^\mu + g_W' B^\mu)\chi$$

The last term is the mass term for the gauge bosons. We expect three massive modes and find them in

$$\frac{v^2}{8}\left[(g_W W^3_\mu - g_W' B_\mu)(g_W W^{3\mu} - g_W' B^\mu) + 2g_W^2 W^-_\mu W^{+\mu}\right]$$

We denote the massive neutral field by

$$Z^{0\mu} = \frac{g_W W^{3\mu} - g_W' B^\mu}{\sqrt{g_W^2 + g_W'^2}}$$ and identify the massless photon with $$A^\mu = \frac{g_W' W^{3\mu} + g_W B^\mu}{\sqrt{g_W^2 + g_W'^2}}$$

Conversely, $$W^{3\mu} = \frac{g_W Z^{0\mu} + g_W' A^\mu}{\sqrt{g_W^2 + g_W'^2}}, \quad B^\mu = -\frac{g_W' Z^{0\mu} - g_W A^\mu}{\sqrt{g_W^2 + g_W'^2}}$$

With these new fields, the covariant derivative is

$$D_\mu = \partial_\mu - i\frac{g_W}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{1}{\sqrt{g_W^2 + g_W'^2}} Z_\mu (g_W^2 T^3 - g_W'^2 Y) - i\frac{g_W g_W'}{\sqrt{g_W^2 + g_W'^2}} A_\mu (T^3 + Y)$$

This suggests the definition of the <u>Weinberg angle</u> or <u>electroweak mixing angle</u>

$$\sin^2\vartheta_W = \frac{g_W'^2}{g_W^2 + g_W'^2} \simeq 0{,}23$$ and $$|e| = \frac{g_W g_W'}{\sqrt{g_W^2 + g_W'^2}} = g_W \sin\vartheta_W$$

$$\Rightarrow \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_W & \sin\vartheta_W \\ -\sin\vartheta_W & \cos\vartheta_W \end{pmatrix} \begin{pmatrix} Z^{0\mu} \\ A^\mu \end{pmatrix}$$

The mass terms then read

$$\frac{g_W^2 v^2}{4} W_\mu^+ W^{-\mu} + \frac{(g_W^2 + g_W'^2) v^2}{8} Z_\mu Z^\mu$$

such that we may identify

$$M_W = \frac{1}{2} v g_W \qquad M_Z = \frac{1}{2} v \sqrt{g_W^2 + g_W'^2} = \frac{M_W}{\cos\vartheta_W}$$

This constraint is usually expressed as the <u>ρ parameter</u>

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\vartheta_W}$$

At tree-level, $\rho = 1$. With the observed

$M_Z = 91{,}884 \pm 0{,}0022$ GeV $\qquad M_W = 80{,}22 \pm 0{,}26$ GeV

$\sin^2\vartheta_W = 0{,}2325 \pm 0{,}0013$

it follows

$\varrho = 1{,}008 \pm 0{,}007$

We therefore see that the coupling strengths or the mixing angle determine the ratios of the gauge boson masses, which is a distinctive feature of the Higgs mechanism. Moreover, the observed vector boson masses allow to infer the Higgs VEV $v = 246$ GeV.

The self-couplings of the Higgs boson are

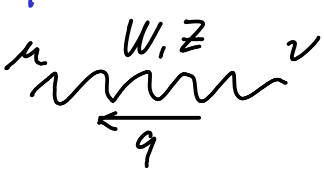
$$\mathcal{L}_H = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}\mu^2 (H+v)^2 - \frac{\lambda}{4}(H+v)^4$$

$$\underset{\substack{\uparrow \\ v=\sqrt{\frac{\mu^2}{\lambda}}}}{=} -\mu^2 H^2 - \lambda v H^3 - \frac{\lambda}{4} H^4 + \frac{1}{4}\frac{\mu^4}{\lambda}$$

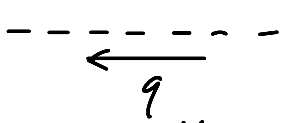
such that the Higgs mass is $M_H = \sqrt{2}\,\mu = \sqrt{2\lambda}\,v$

A measurement of these self-couplings would be an important check of the theory.

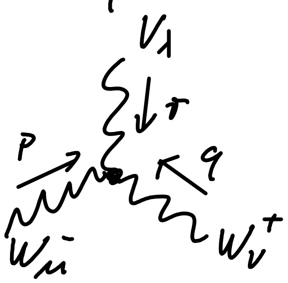
The unitary gauge is good enough to calculate tree-level processes. The Feynman rules are



$$= \left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right]\frac{i}{q^2 - M^2 + i\varepsilon}$$

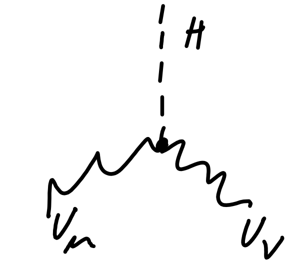


$$= \frac{i}{q^2 - M_H^2 + i\varepsilon}$$



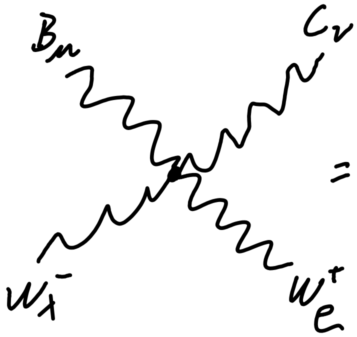
$$= i g_V \left[(p-q)_\lambda g_{\mu\nu} + (q-r)_\mu g_{\nu\lambda} + (r-p)_\nu g_{\lambda\mu}\right]$$

$g_A = -e, \quad g_Z = g_W \cos\vartheta_W$



$$= i g_{VH} M_W g_{\mu\nu}$$

$$g_{WH} = g_W \quad g_{ZH} = \frac{g_W}{\cos^2\vartheta_W}$$



$$= i\left[2 g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}\right] \begin{cases} g_W^2 & \text{for } (B_\mu, C_\nu) = (W_\mu^+, W_\nu^-) \\ -g_W^2 \cos\vartheta_W & \text{for } (B_\mu, C_\nu) = (Z_\mu, Z_\nu) \\ -e^2 & \text{for } (B_\mu, C_\nu) = (A_\mu, A_\nu) \\ -e g_W & \text{for } (B_\mu, C_\nu) = (A_\mu, Z_\nu) \end{cases}$$

In nature, we observe that $SU(2)_L$ couples only to the left handed components of fermions and that these fermions can be grouped in doublets (fundamental representation). Furthermore, left- and right handed fermions have different weak hypercharges. Interactions with that property can be built with the help of the chiral projection operators

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}$$

$$Q_L^{1,2,3} = P_L\begin{pmatrix} u \\ d \end{pmatrix}, P_L\begin{pmatrix} c \\ s \end{pmatrix}, P_L\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

$$u_R^{1,2,3} = P_R u, P_R c, P_R t = u_R, c_R, t_R$$

$$d_R^{1,2,3} = P_R d, P_R s, P_R b = d_R, s_R, b_R$$

$$L_L^{1,2,3} = P_L\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, P_L\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, P_L\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} = \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L^- \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L^- \end{pmatrix}$$

$$e_R^{1,2,3} = P_R e, P_R \mu, P_R \tau = e_R, \mu_R, \tau_R$$

$$\psi_L = L_L, Q_L$$

$$\psi_R = e_R, u_R, d_R$$

We observe that according to this scheme, the matter fermions fall into three <u>generations.</u>

The Lagrangian for the coupling of the Electroweak interactions to matter is given by

$$\mathcal{L} = \overline{\psi_R^i}\, i(\partial\!\!\!/ + i g_W' Y_i \, B\!\!\!/) \psi_R^i + \overline{\psi_L^j}\, i(\partial\!\!\!/ + i g_W T\cdot W\!\!\!\!/ + i g_W' Y_j B\!\!\!/) \psi_L^j$$

The quantum numbers of matter and Higgs fields are given by

	$SU(3) \times SU(2)_L \times U(1)_Y$
Q_L	$(3, 2, \frac{1}{6})$
u_R	$(3, 1, \frac{2}{3})$
d_R	$(3, 1, -\frac{1}{3})$
L_L	$(1, 2, -\frac{1}{2})$
e_R	$(1, 1, -1)$
H	$(1, 2, \frac{1}{2})$

For the groups SU, we are giving here the dimension of the representation, whereas for $U(1)$, the weak hypercharge Y.

Recall that the electric charge operator is $Q^i = T^3 + Y^i$ and verify that these assignment are in consistency with the electric charges.

Now for the fermion masses. Mass terms violate chirality, i.e. they mix left- and right-handed components. Writing down terms of the form

$$m_u \bar{u} u + m_d \bar{d} d$$

does not fly, because these are neither $SU(2)_L$ invariants (singlets) nor $U(1)_Y$ invariant (weak hypercharge does not add up to zero).

What works however are Lagrangian terms of the form

$$\mathcal{L} = -y_d^{ij}\,\overline{Q}^i_{La}\,\phi_a\,d^j_R - y_u^{ij}\,\varepsilon^{ab}\,\overline{Q}^i_{La}\,\phi^+_b\,u^j_R - y_e^{ij}\,\overline{L}^i_{La}\,\phi_a\,e^j_R + h.c.$$

The y are matrices of Yukawa couplings. Notice that δ^{ab} and ε^{ab} are $SU(2)$ invariant tensors and that furthermore, in each term, the weak hypercharge adds up to zero. The Yukawa matrices are in general non-diagonal. They can however be diagonalised through the biunitary transformation

$y_{u,d,e} = U_{u,d,e}\, y^D_{u,d,e}\, W^+_{u,d,e}$ where y^D is diagonal and where U, W diagonalise $y y^+ = U y^{D2} U^+$, $y^+ y = W y^{D2} W^+$. The Lagrangian terms read (suppressing ε^{ab})

$$-\phi\,\overline{\psi_L}\,y\,\psi_R = -\phi\,\overline{\psi_L}\,U\,y^D\,W^+\,\psi_R\,,$$

such that we can eliminate W by redefining

$$\psi_R \longmapsto W\psi_R$$

In addition, we get rid of U by replacing

$$u_L \longmapsto U_u u_L \qquad d_L \longmapsto U_d\, d_L$$
$$e_L \longmapsto U_e\, e_L$$

The interactions with the gauge bosons B and W^3 (and therefore A and Z) are invariant under these transformations.

However

$$\frac{1}{\sqrt{2}}\,\overline{u}_L\,\gamma^\mu d_L \longmapsto \frac{1}{\sqrt{2}}\,\overline{u}_L\,\gamma^\mu\, U_u^+ U_d\, d_L$$

We define the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V = U_u^+ U_d$.

The u and d quarks are now defined such that the Yukawa couplings are diagonal, such that the quarks are mass eigenstates, but they are no longer eigenstates of the exchange of charged W bosons. This has the important consequence that W bosons can mediate between the different generations. Now back to the question of fermion masses. Since $\langle \phi^0 \rangle = \frac{v}{\sqrt{2}}$, we find that these are given by

$m_{u,d,e} = y^D_{u,d,e} \frac{v}{\sqrt{2}}$. Spontaneous symmetry breaking therefore gives rise to mass terms mixing fermions in different representations of $SU(2)_L \times U(1)_Y$ by gauge invariant Lagrangian terms.

The model gives rise to the weak interactions at low energies. Calculating an "effective action" (in Quantum Field Theory, there is a well-defined method for this), and "integrating" out the W bosons from the path integral, we obtain the effective interaction:

$$\sum_{\pm} i \frac{1}{\sqrt{2}} \overline{\psi_L^j} g_W T^{\mp} \psi_L^j \frac{-i}{M_W^2} \frac{1}{\sqrt{2}} \overline{\psi_L}^i g_W T^{\pm} \psi_L^i$$

$$= \sum_{\pm} \frac{4}{\sqrt{2}} G_F \overline{\psi_L^j} T^{\mp} \psi_L^j \overline{\psi_L}^i T^{\pm} \psi_L^i$$

Where $G_F = \frac{\sqrt{2}\, g_W^2}{8 M_W^2}$ is the Fermi constant, that can be observed from radioactive decay.

This is valid, provided the fermion momenta are much below M_W. Notice that it is then of the form that we had for the interactions between leptons and quarks in DIS.

Above interaction explains weak radioactive decays:

$d \longrightarrow u + e + \overline{\nu}_e$

Besides, due to the replacement $d_L \longmapsto V d_L$, it mediates between the different generations of quarks, as we will discuss later in these lectures.

Neutral currents mediated by the Z boson were first observed in $\nu N \longrightarrow \nu +$ hadron scatterings (N stands for nucleon) in 1973 by the Gargamelle bubble chamber at CERN. This way, the weak mixing angle could be determined and the strength of the weak interactions through $|e| = g_W \sin\vartheta_W$. Given these observations, the theory of Electroweak unification not only predicts the existence of $W^\pm$ and Z but also their masses. Eventually, $W^\pm$ and Z were first directly produced in 1983 at SPS (Super Proton Synchrotron) at CERN and their masses were found to be in agreement with low-energy observations.

Hence, Electroweak symmetry breaking (EWSB) is a satisfactory model to explain weak interactions at low energy in terms of a gauge theory. In addition, it <u>predicts</u> the existence of the $W^\pm$, Z and the Higgs boson. Besides the existence of these particles, there are also consistency checks, such as the ρ-parameter.

3.3 Discrete Symmetries

The chiral nature of the weak interactions together with flavour mixing give rise to the possibility of charge-parity (CP) violation. The circumstances under which this may occur are however very restrictive and thus, CP violation is both a hallmark feature and testing ground of the Standard Model.

Meaningful definitions of the discrete symmetries are often ambiguous because they may or may not include certain internal symmetries that leave the theory (approximately) invariant otherwise.

Furthermore, since QED and QCD are parity (P) and charge (C) symmetric while weak interactions are not, we must apply a strategy that defines even and odd states in QED & QCD and then extends this concept to weak interactions.

We first consider the action of a certain symmetry transformation through an operator $\mathcal{Q}$:

$$|\psi_\mathcal{Q}\rangle = \mathcal{Q}|\psi\rangle, \quad |\chi_\mathcal{Q}\rangle = \mathcal{Q}|\chi\rangle$$

One can prove (Wigner's theorem) that observables are invariant if <u>either</u>

$$\langle\chi_\mathcal{Q}|\psi_\mathcal{Q}\rangle = \langle\chi|\mathcal{Q}^\dagger\mathcal{Q}|\psi\rangle = \langle\chi|\psi\rangle,$$

i.e. $\mathcal{Q}$ is unitary <u>or</u>

$$\langle\chi_\mathcal{Q}|\psi_\mathcal{Q}\rangle = \langle\chi|\mathcal{Q}^\dagger\mathcal{Q}|\psi\rangle = \langle\chi|\psi\rangle^* = \langle\psi|\chi\rangle$$

and $\mathcal{I}(\alpha|\psi\rangle + \beta|\chi\rangle) = \alpha^* \mathcal{I}|\psi\rangle + \beta^* \mathcal{I}|\chi\rangle$

i.e. $\mathcal{I}$ is antiunitary. Note in particular that it leads to complex conjugation of $\mathbb{C}$-number factors.

Let $a(\vec{p},h)$ and $b(\vec{p},h)$ be particle and antiparticle annihilation operators of a state with momentum $\vec{p}$ and helicity h. Then

$$C\, a^+(\vec{p},h)\, C^+ = e^{i\vartheta}\, b^+(\vec{p},h)$$

$$C\, b^+(\vec{p},h)\, C^+ = e^{-i\vartheta}\, a^+(\vec{p},h)$$

As for time reversal, consider the quantum mechanical commutator

$$[x_i, p_k] = i\,\delta_{ik}$$

Under time reversal, this does not hold unless

$$i \longmapsto (i)^* = -i$$

Similarly, the Schrödinger equation reads

$$i\,\frac{\partial \psi(\vec{x},t)}{\partial t} = \left[-\frac{\vec{\nabla}^2}{2m} + V\right]\psi(\vec{x},t)$$

which also requires a complex conjugation of i and $\psi \longmapsto \psi^*$ if the theory is to be time-reversal invariant:

$$T\, i\,\frac{\partial \psi(\vec{x},t)}{\partial t}\, T^{-1} = (-i)\,\frac{\partial \psi^*(\vec{x},t')}{\partial t'}\bigg|_{t'=-t} = i\,\frac{\partial \psi^*(\vec{x},-t)}{\partial t}$$

$$= T\left[-\frac{\vec{\nabla}^2}{2m} + V\right]\psi(\vec{x},t)\, T^{-1} = \left[-\frac{\vec{\nabla}^2}{2m} + V\right]\psi^*(\vec{x},-t)$$

Now we make use of the fact that $\mathcal{T} = U_{\mathcal{T}} K$, where $U_{\mathcal{T}}$ is unitary and K performs complex conjugations to its right. $\Longrightarrow$

$$\langle \mathcal{T}\chi | \mathcal{T}\psi \rangle = \langle U_{\mathcal{T}} K \chi | U_{\mathcal{T}} K \psi \rangle = \langle \chi | K^{+} \underbrace{U_{\mathcal{T}}^{+} U_{\mathcal{T}}}_{=1} K | \psi \rangle$$

$$= \langle \chi | K^{+} K | \psi \rangle \overset{!}{=} \langle \chi | \psi \rangle^{*}$$

Hence, K^{+} performs complex conjugations to its right. Therefore, $\mathcal{T}^{+}\mathcal{T} = K^{+}K \neq 1$.

Apparently, $K^{2} = 1$ such that

$$U_{\mathcal{T}} K K U_{\mathcal{T}}^{+} = U_{\mathcal{T}} U_{\mathcal{T}}^{+} = 1 \Longrightarrow \mathcal{T}^{-1} = K U_{\mathcal{T}}^{+}$$

A field-theoretical Lagrangian invariant under the discrete symmetries must satisfy

$P: (t, \vec{x}) \longmapsto (t', \vec{x}') = (t, -\vec{x}) \Longrightarrow P \mathcal{L}(t, \vec{x}) P^{+} = \mathcal{L}(t', \vec{x}') = \mathcal{L}(t, -\vec{x})$

$T: (t, \vec{x}) \longmapsto (t', \vec{x}') = (-t, \vec{x}) \Longrightarrow T \mathcal{L}(t, \vec{x}) T^{-1} = \mathcal{L}(t', \vec{x}') = \mathcal{L}(-t, \vec{x})$

$C \mathcal{L}(t, \vec{x}) C^{+} = \mathcal{L}(t, \vec{x})$

We now move to QED. We can write

$$P A_\mu(t,\vec{x}) P^\dagger = A^\mu(t,-\vec{x}),$$

and since $x_\mu \longmapsto x'_\mu = x^\mu$ also $\partial^\mu \longmapsto \partial'_\mu$

Therefore, $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$ is parity invariant.

As for the charge, we demand

$$C j^\mu C^\dagger = -j^\mu$$

and we hence define

$$C A_\mu(t,\vec{x}) C^\dagger = -A_\mu(t,\vec{x})$$

such that the coupling $j_\mu A^\mu$ is charge conjugation invariant since this is what we observe in QED.

Similarly, $T x^\mu T^{-1} = x'^\mu = -x_\mu$

$$T j^\mu T^{-1} = j_\mu$$

i.e. the spatial current changes its sign, such that we require

$$T A^\mu T^{-1} = A_\mu$$

Next, add a scalar field through the Lagrangian terms

$$\mathcal{L} = (\partial_\mu \phi^\dagger - i q A_\mu \phi^\dagger)(\partial^\mu \phi + i q A^\mu \phi) - m^2 |\phi|^2$$

Under parity, the field operator transforms as follows:

$$P \phi(t,\vec{x}) P^\dagger = e^{i\alpha} \phi(t,-\vec{x}) \overset{\text{h.c.}}{\Longleftrightarrow} P \phi^\dagger(t,\vec{x}) P^\dagger = e^{-i\alpha} \phi^\dagger(t,-\vec{x})$$

Together with the transformation properties of ∂_μ and A_μ, it follows

$$\partial_\mu \phi(t,\vec{x}) \longmapsto e^{i\alpha}\partial'^\mu \phi(t',\vec{x}')\Big|_{\substack{t'=t\\ \vec{x}'=-\vec{x}}}$$

$$A_\mu(t,\vec{x}) \longmapsto A^\mu(t',\vec{x}')\Big|_{\substack{t'=t\\ \vec{x}'=-\vec{x}}}$$

$$P\mathcal{L}(t,\vec{x})P^\dagger = \mathcal{L}(t,-\vec{x})$$

Under time reversal, there should be the transformation

$$T\phi(t,\vec{x})T^{-1} = e^{i\beta}\phi(-t,\vec{x}) \overset{h.c.}{\Longleftrightarrow} T\phi^\dagger(t,\vec{x})T^{-1} = e^{-i\beta}\phi^\dagger(-t,\vec{x})$$

In particular

$$T(\partial^\mu\phi(t,\vec{x}) + iqA^\mu(t,\vec{x})\phi(t,\vec{x}))T^{-1}$$

c.c. of $\mathbb{C}$ number factor

$$= e^{i\beta}(\partial'^\mu\phi(t',\vec{x}') - iqA_\mu(t',\vec{x}')\phi(t',\vec{x}'))\Big|_{\substack{t'=-t\\ \vec{x}'=\vec{x}}}$$

$$= e^{i\beta}(-\partial_\mu\phi(-t,\vec{x}) - iqA_\mu(-t,\vec{x})\phi(-t,\vec{x}))$$

Such that again,

$$T\mathcal{L}(t,\vec{x})T^{-1} = \mathcal{L}(t',\vec{x}') = \mathcal{L}(-t,\vec{x})$$

From Noether's theorem applied to the $U(1)$ gauge symmetry, we know that

$$j^\mu(x) = i[\phi^*(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^*(x)]$$

(Note again that the complex conjugation induced by the T operator is required to verify the physical transformation property).

Hence,

$$C\phi(t,\vec{x})C^\dagger = e^{i\gamma}\phi^\dagger(t,\vec{x}) \Longleftrightarrow C\phi^\dagger(t,\vec{x})C^\dagger = e^{-i\gamma}\phi(t,\vec{x})$$

and it is immediately seen that $C^\dagger\mathcal{L}C = \mathcal{L}$.

Now we move to Dirac fermions. For every representation of the Dirac matrices, there are non-singular 4×4 matrices such that

$$A\gamma_\mu = \gamma_\mu^\dagger A \qquad\qquad (\psi^\dagger A \gamma_\mu \psi)^\dagger = \psi^\dagger \gamma_\mu^\dagger A \psi = \psi^\dagger A \gamma_\mu \psi$$

$$\gamma_\mu C = -C\gamma_\mu^T$$

These satisfy the useful identities

$$A^\dagger = A \qquad\qquad A\gamma_5 = -\gamma_5^\dagger A$$

$$C^T = -C \qquad\qquad \gamma_5 C = C\gamma_5^T$$

$$CA^*C^*A = 1 \qquad\qquad A\sigma_{\mu\nu} = \sigma_{\mu\nu}^\dagger A \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\sigma_{\mu\nu} C = -C\sigma_{\mu\nu}^T$$

In the commonly used representations (Dirac, Weyl, Majorana), $A = \gamma_0$. Moreover, in the Weyl representation, $\gamma_2^* = -\gamma_2$, $\gamma_2^T = \gamma_2$ $C = i\gamma_2\gamma_0$. We recall the familiar notation $\overline{\psi} = \psi^\dagger A$ and define the

<u>charge conjugate spinor:</u>

$$\psi^c = C\overline{\psi}^T = CA^T\psi^{\dagger T} = CA^T\psi^*$$

It follows that $(\psi^c)^c = \psi$ and $\overline{\psi^c} = -\psi^T C^{-1}$.

It is interesting to consider chirality eigenstates

$$\gamma^5\psi = \pm\psi \Longleftrightarrow \psi^\dagger\gamma^{5\dagger} = \pm\psi^\dagger \Longleftrightarrow \psi^\dagger A\gamma^5 = \mp\psi^\dagger A \Longleftrightarrow \overline{\psi}\gamma^5 = \mp\overline{\psi}$$

minus for left
plus for right

$$\Longleftrightarrow C\gamma^{5T}\overline{\psi}^T = \mp C\overline{\psi}^T = \mp\psi^c \Longleftrightarrow$$

$$\gamma^5\psi^c = \mp\psi^c$$

Hence, charge conjugation flips the chirality.

With $P_{L,R} = \frac{1\mp\gamma^5}{2}$ and $\psi_{L,R} = P_{L,R}\,\psi \Longrightarrow$

$$\overline{\psi_{R,L}} = \overline{\psi}\, P_{L,R} \, ; \qquad (\psi_{R,L})^c = (\psi^c)_{L,R}$$

Now, we couple to the electromagnetic field through

$$\mathcal{L} = \overline{\psi}\left[\gamma_\mu (i\partial^\mu - qA^\mu) - m\right]\psi$$

From the Dirac algebra, it follows $\gamma^0\gamma_\mu\gamma^0 = \gamma^\mu$.
We therefore postulate

$$P\,\psi(t,\vec{x})\,P^\dagger = e^{i\alpha}\gamma^0\psi(t,-\vec{x}) \Longrightarrow$$

$$P\,\overline{\psi}(t,\vec{x})\,P^\dagger = P\,\psi^\dagger(t,\vec{x})\,P^\dagger A = e^{-i\alpha}\psi^\dagger(t,-\vec{x})\,\gamma^{0\dagger} A$$

$$= e^{-i\alpha}\,\overline{\psi}(t,-\vec{x})\,\gamma^0$$

$\Longrightarrow$

$$P\,\mathcal{L}(t,\vec{x})\,P^\dagger \underset{\substack{t'=t \\ \vec{x}'=-\vec{x}}}{\overset{\uparrow}{=}} \overline{\psi}(t',\vec{x}')\,\gamma^0\left[\gamma_\mu(i\partial'^\mu - qA_\mu(t',\vec{x}')) - m\right]\gamma^0\,\psi(t',\vec{x}')$$

$$= \overline{\psi}(t,-\vec{x})\left[\gamma^\mu(i\partial_\mu - qA_\mu(t,-\vec{x})) - m\right]\psi(t,-\vec{x})$$

$$= \mathcal{L}(t,-\vec{x})$$

For time reversal, we take

$$T\,\psi(t,\vec{x})\,T^{-1} = U_T K\,\psi(t,\vec{x})\,K\,U_T^\dagger = U_T\,\psi^*(t,\vec{x})\,U_T^\dagger$$

$$= e^{i\beta}\gamma_0^*\gamma_5^* C^* A\,\psi(-t,\vec{x}) \Longrightarrow$$

$$T\,\overline{\psi}(t,\vec{x})\,T^{-1} = U_T K\,\overline{\psi}(t,\vec{x})\,K\,U_T^\dagger = U_T\,\underbrace{\overline{\psi}^*(t,\vec{x})}_{=\psi^\dagger A^*}\,U_T^\dagger$$

$$= U_T\,\psi^\dagger(t,\vec{x})\,U_T^\dagger\, A^*$$

$$= \left[U_T\,\psi^*(t,\vec{x})\,U_T^\dagger\right]^\dagger A^*$$

$$= -e^{-i\beta}\,\psi^T(-t,\vec{x})\,A\,C\underbrace{\gamma_5^T\gamma_0^T A^*}_{=-A^*\gamma_5^*\gamma_0^*}$$

$$= e^{-i\beta}\,\psi^T(-t,\vec{x})\,(C^{-1})^*\gamma_5^*\gamma_0^*$$

since $C^* A\, C A^* = 1 \Longrightarrow A C A^* = (C^{-1})^*$

Hence

$$T\mathcal{L}(t,\vec{x})T^{-1} = U_T\,\mathcal{L}^*(t,\vec{x})\,U_T^{-1}$$

$$= U_T\,\overline{\psi}^*(t,\vec{x})U_T\left(\gamma_\mu^*\left[(-i)(-\partial_\mu')-qA_\mu(t',\vec{x}')-m\right]\right)U_T\,\psi^*(t,\vec{x})U_T^\dagger$$

$$= \psi^T(t',\vec{x}')\,(C^{-1})^*\gamma_5^*\gamma_0^*\left(\gamma_\mu^*\left[i\partial_\mu'-qA_\mu(t',\vec{x}')-m\right]\right)$$
$$*\,\gamma_0^*\gamma_5^*C^*A\,\psi(-t',\vec{x}')$$

$$= \overline{\psi}(t',\vec{x}')\;\gamma^\mu\left(i\partial_\mu'-qA_\mu(t',\vec{x}')-m\right)\psi(t',\vec{x}') = \mathcal{L}(-t,\vec{x})$$

↑
commuting C^* and A past γ^μ amounts to complex conjugation and a minus sign, the γ^0 sandwich raises μ and the γ^5 sandwich yields an extra minus sign

For charge conjugation, we take

$$C\psi C^\dagger = e^{i\gamma}\psi^c \iff C\overline{\psi}C^\dagger = e^{-i\gamma}\overleftarrow{\overline{\psi^c}}$$

$$C\mathcal{L}C^\dagger = -\psi^T C^{-1}\left[\gamma_\mu(i\partial^\mu+qA^\mu)-m\right]C\,\overline{\psi}^T$$

$$= \psi^T\left[\gamma_\mu^T(i\partial^\mu+qA^\mu)+m\right]\overline{\psi}^T$$

$$= -\overline{\psi}\left[\gamma_\mu(i\overleftarrow{\partial}^\mu+qA^\mu)+m\right]\psi \quad \text{(spinors anticommute)}$$

$$= \overline{\psi}\left[\gamma_\mu(i\partial^\mu - qA^\mu) - m\right]\psi = \mathcal{L}$$

Combining charge and parity reversal for two different spinors leads to

$$(CP)\,\psi\,(CP)^\dagger = e^{i\xi_\psi}\,\gamma^0\, C\,\overline{\psi}^T$$
$$(CP)\,\chi\,(CP)^\dagger = e^{i\xi_\chi}\,\gamma^0\, C\,\overline{\chi}^T$$

$$(CP)\,\overline{\psi}\,(CP)^\dagger = e^{-i\xi_\psi}\,\overline{\psi}^* C^\dagger \gamma^{0\dagger} A$$
$$= e^{-i\xi_\psi}\,\psi^T A^* C^\dagger \gamma^{0\dagger} A$$
$$= e^{-i\xi_\psi}\,\psi^T A^* C^\dagger A\,\gamma^0$$
$$= -e^{-i\xi_\psi}\,\psi^T A^* C^* A\,\gamma^0$$
$$= -\,e^{-i\xi_\psi}\,\psi^T C^{-1}\gamma^0$$
$$(CP)\,\overline{\chi}\,(CP)^\dagger = -\,e^{-i\xi_\chi}\,\chi^T C^{-1}\gamma^0$$

The transformation properties of field bilinears follow from the sandwiching properties

$$C^T\gamma^{0T}\,\gamma^{0T}(C^{-1})^T = 1$$
$$C^T\gamma^{0T}\gamma^{5T}\gamma^{0T}(C^{-1})^T = -C^T\gamma^{5T}(C^{-1})^T = -\gamma^5$$
$$C^T\gamma^{0T}\gamma^{\mu T}\gamma^{0T}(C^{-1})^T = C^T\gamma_\mu^T\,(C^{-1})^T = -\gamma_\mu$$
$$C^T\gamma^{0T}\gamma_5^T\gamma^{\mu T}\gamma^{0T}(C^{-1})^T = -\,C^T\gamma_5^T\gamma_\mu^T\,(C^{-1})^T = \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$$

$$(CP)(\bar{\psi}\chi)(CP)^\dagger = e^{i(\xi_\chi - \xi_\psi)}\,\bar{\chi}\psi$$

$$(CP)(\bar{\psi}\gamma^5\chi)(CP)^\dagger = -\,e^{i(\xi_\chi - \xi_\psi)}\,\bar{\chi}\gamma^5\psi$$

$$(CP)(\bar{\psi}\gamma^\mu\chi)(CP)^\dagger = -\,e^{i(\xi_\chi - \xi_\psi)}\,\bar{\chi}\gamma_\mu\psi$$

$$(CP)(\bar{\psi}\gamma^\mu\gamma_5\chi)(CP)^\dagger = -\,e^{i(\xi_\chi - \xi_\psi)}\,\bar{\chi}\gamma_\mu\gamma_5\psi$$

Relative parities of particles and antiparticles

Compare transformations of the Klein-Gordon field

$$P\phi(t,\vec{x})P^\dagger = e^{i\alpha_\phi}\phi(t,-\vec{x})$$

$$P[C\phi(t,\vec{x})C^\dagger]P^\dagger = e^{-i\alpha_\phi}[C\phi(t,-\vec{x})C^\dagger]$$

with the Dirac field $\qquad \psi^c = C\bar{\psi}^T = CA^T\psi^*$

$$P\psi(t,\vec{x})P^\dagger = e^{i\alpha_\psi}\gamma^0\psi(t,-\vec{x})$$

$$P[C\psi(t,\vec{x})C^\dagger]P^\dagger = P[e^{i\gamma}C\bar{\psi}^T]P^\dagger$$

$$= e^{-i\alpha_\psi}e^{i\gamma}C[(\gamma^0\psi)^\dagger A]^T$$

$$= e^{-i\alpha_\psi}e^{i\gamma}\underbrace{CA^T\gamma^{0*}}_{=C\gamma^{0T}A^T = -\gamma^0 CA^T}\psi^*$$

$$= -e^{-i\alpha_\psi}\gamma^0[C\psi(t,-\vec{x})C^\dagger]$$

While the phases α_ϕ and α_ψ are unobservable, the relative parities can be probed. An quark-antiquark pair has an intrinsic parity of -1, such that a system of a quark antiquark pair has an intrinsic parity minus one that adds to the contribution from the orbital angular momentum L: $P = (-1)^{L+1}$

It is straightforward to work out the effect of the remaining discrete transformations on the bilinears. We give the following table for reference:

	scalar $\bar\psi\psi$	pseudoscalar $i\bar\psi\gamma^5\psi$	vector $\bar\psi\gamma^\mu\psi$	pseudovector $\bar\psi\gamma^\mu\gamma^5\psi$	antisymmetric tensor $\bar\psi\sigma^{\mu\nu}\psi$	derivative ∂^μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

Weak interactions violate C and P maximally, because

$$\bar\psi\gamma^\mu A_\mu \frac{1-\gamma^5}{2}\psi \xrightarrow{P} \bar\psi\gamma^\mu A_\mu \frac{1+\gamma^5}{2}\psi$$

$$\bar\psi\gamma^\mu A_\mu \frac{1-\gamma^5}{2}\psi \xrightarrow{C} \bar\psi\gamma^\mu A_\mu \frac{1+\gamma^5}{2}\psi$$

Note however that they conserve the combined charge-parity CP. The Yukawa couplings in the SM can in principle violate CP because they can in principle have pseudoscalar contributions. We will discuss below that this requires at least three generations of quarks, which is the case for the SM.

The combined symmetry CPT is however conserved in local Quantum Field Theory, which is known as the CPT theorem.

From above table, we notice that odd-rank tensors are odd, even rank tensors even under CPT. Since the Lagrangian is a rank 0 tensor (i.e. a scalar), we immediately notice its invariance

$$CPT\,\mathcal{L}(x)\,(CPT)^{-1} = \mathcal{L}(-x).$$

One interesting application is that for the invariant matrix element

$$i\mathcal{M}_{A\to B} = i\mathcal{M}_{CPB(CP)^{-1}\to CPA(CP)^{-1}}$$

where A and B are some multi-particle states.

Now if we take $A=B$, assume that this is a state of a single, unstable particle and take twice the imaginary part, the optical theorem tells us that this is the decay rate. Since the same applies to $CP\,A\,(CP)^{-1}$, we find that particles and anti-particles have the same life-time.

3.4 CP Violation and the CKM Matrix

While up to this point, we have not indicated the weak interaction eigenstates with a prime, in order to keep the notation compact, we will introduce the prime for this present section.

The common choice is to perform the rotation into the weak interaction basis for the down type quarks

$$\underbrace{\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix}}_{\text{weak eigenstates}} = V \underbrace{\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}}_{\text{mass eigenstates}}$$

Recall that rotations of the up-type quarks are understood to be readily absorbed within the CKM matrix V. We also recall that V is unitary and denote its components as

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Yet, there are ambiguities from field redefinitions, and rather than determining the individual entries, it is useful to think first of a parametrisation.

An $n \times n$ unitary matrix can depend on

$$2n^2 - \underset{\substack{\uparrow \\ \text{normalisations}}}{n} - \underset{\substack{\uparrow \\ \text{orthogonalities}}}{2\frac{n(n-1)}{2}} = n^2$$ real parameters.

On the other hand, an orthogonal matrix has

$n^2 - n - \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$ parameters. Hence, we have $\frac{1}{2} n(n-1)$ mixing angles and $\frac{1}{2} n(n+1)$ phases. Most phases can be removed by the redefinitions

$$u_{L\alpha} \longmapsto e^{i\vartheta^u_\alpha} u_{L\alpha}, \quad d_{L\alpha} \longmapsto e^{i\vartheta^d_\alpha} d_{L\alpha},$$

which take the effect

$$V_{\alpha\beta} \longmapsto e^{i(\vartheta^d_\beta - \vartheta^u_\alpha)} V_{\alpha\beta}$$

An overall common rephasing takes no effect, such that $2n-1$ of the phases can be removed, leaving $\frac{1}{2}n^2 - \frac{3}{2}n + 1$. Hence, for $n=3$, there are 3 angles and one physical phase. The common choice for the parametrisation is ($s_{ij} = \sin\vartheta_{ij}$, $c_{ij} = \cos\vartheta_{ij}$)

$$V=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

We will explain how to determine the elements of this matrix in the following chapters. In order to understand the various processes involved, and because of the particular hierarchical structure of the CKM matrix, we first state the results.

Often quoted are the magnitudes of the CKM matrix elements $|V_{\alpha\beta}|$ [PDG 2012 (11.26)]

$$\begin{pmatrix} 0{,}97428\pm0{,}00015 & 0{,}2253\pm0{,}0007 & 0{,}00347^{+0{,}00016}_{-0{,}00012} \\ 0{,}2252\pm0{,}0007 & 0{,}97345^{+0{,}00015}_{-0{,}00016} & 0{,}0410^{+0{,}0011}_{-0{,}0007} \\ 0{,}00862^{+0{,}00026}_{-0{,}00020} & 0{,}0403^{+0{,}0011}_{-0{,}0007} & 0{,}999152^{+0{,}000030}_{-0{,}000045} \end{pmatrix}$$

While the origin of these mixings is not known and consequently, their sizes have not been predicted, we may recognise the hierarchy

$1 \gg s_{12} \gg s_{23} \gg s_{13}$

This gives rise to the often employed phenomenological Wolfenstein parametrisation. We define

$$s_{12}=\lambda=\frac{V_{us}}{\sqrt{|V_{ud}|^2+|V_{us}|^2}} \qquad s_{23}=A\lambda^2=\lambda\left|\frac{V_{cb}}{V_{us}}\right|$$

$$s_{13}e^{i\delta}=V_{ub}^*=A\lambda^3(\rho+i\eta)=\frac{A\lambda^3(\bar\rho+i\bar\eta)\sqrt{1-A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1-A^2\lambda^4(\bar\rho+i\bar\eta)]}$$

where $\bar{\varrho}+i\bar{\eta} = -\dfrac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$

Expanding in λ, we obtain

$$V = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\varrho-i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\varrho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

We can make at this point the remark that the small mixings $\sim \lambda, \lambda^2, \lambda^3$ suppress the CP-violation originating from the remaining phase. The experimental values for the Wolfenstein parameters are [PDG 2012 (11.26)]

$\lambda = 0{,}2253 \pm 0{,}0007$ $\qquad$ $A = 0{,}808^{+0{,}022}_{-0{,}015}$

$\bar{\varrho} = 0{,}132^{+0{,}022}_{-0{,}014}$ $\qquad$ $\bar{\eta} = 0{,}341 \pm 0{,}013$

While above numbers are from global fits, the CKM elements can be determined individually. It is then interesting to test the SM with the unitarity constraints

$$\sum_i V_{ij}V_{ik}^* = \delta_{jk} \quad \text{and} \quad \sum_j V_{ij}V_{kj}^* = \delta_{ik}$$

There are six different sums that should vanish.
These relations can be geometrically interpreted as triangles in the complex plane. In the CP conserving limit, their area vanishes, and we will show now that they all have the same area.
Therefore, we consider quantities that are invariant under rephasings, which take the effect of

multiplying entire rows or columns of the CKM matrix by a phase factor.
The simplest invariants are the moduli

$$U_{\alpha i} = |V_{\alpha i}|^2$$

and the next-to-simplest ones the "quartets"

$$Q_{\alpha i \beta j} = V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^* \quad \text{with } \alpha \neq \beta,\ i \neq j$$

It turns out that higher order invariants can be expressed in terms of these two basic types.
Now consider for definiteness the orthogonality of the first two rows

$$V_{ud} V_{cd}^* + V_{us} V_{cs}^* + V_{ub} V_{cb}^* = 0,$$

multiply by $V_{us}^* V_{cs}$ and take the imaginary part $\longrightarrow$

$$V_{ud} V_{cs} V_{us}^* V_{cd}^* + |V_{us}|^2 |V_{cs}|^2 + V_{ub} V_{cs} V_{us}^* V_{cb}^* = 0$$

$\longrightarrow$

$$\mathrm{Im}\, Q_{udcs} = -\mathrm{Im}\, Q_{ubcs}$$

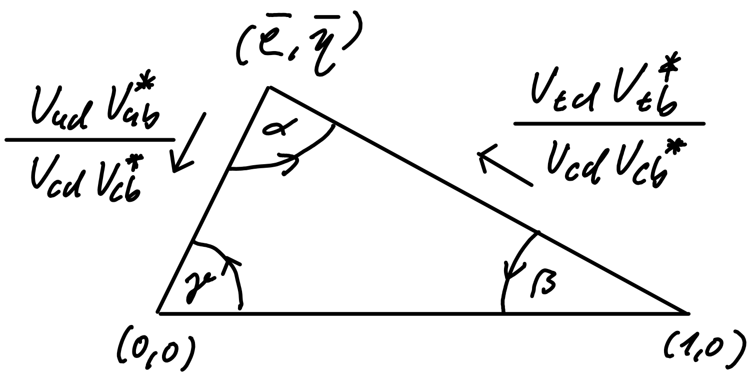
We can directly generalize this to the statement that all quartets have the same imaginary part, up to a sign. In terms of the mixing angles and the phase, the <u>Jarlskog invariant</u> is given by

$$J = \mathrm{Im}\left[V_{td}^* V_{tb} V_{ub}^* V_{ud} \right] = c_{12} c_{23} c_{13}^2 s_{12} s_{23} s_{13} \sin\delta_{13} = A^2 \lambda^6 \eta$$

From the Wolfenstein parametrisation, we see that four of the six triangles are squashed, while the remaining two take the same shape. The commonly considered combination is

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

The standard <u>unitarity triangle</u> is obtained when dividing each side by $V_{cd}V_{cb}^*$. The vertices are then at $(0,0)$, $(1,0)$ and $(\bar{\rho},\bar{\eta})$.



$$\alpha = (89{,}0^{+4{,}4}_{-4{,}2})^\circ$$

$$\gamma = (68^{+10}_{-11})^\circ$$

The lengths of the complex sides of this triangle are given by

$$R_u = \left|\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right| = \sqrt{\bar{\rho}^2+\bar{\eta}^2}, \quad R_t = \left|\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right| = \sqrt{(1-\bar{\rho})^2+\bar{\eta}^2}$$

and the angles are defined

$$\alpha = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right], \quad \beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right], \quad \gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]$$

Notice that in the standard parametrisation, $\gamma = \delta$. In terms of these lengths and angles, we can reexpress the standard unitarity relation as

$$R_u e^{i\gamma} + R_t e^{-i\beta} = 1$$

Undoing the rescaling of the triangle, the product of of its height times its base is

$$\mathrm{Im}\, V_{ud} V_{ub}^* V_{cb} V_{cd}^* = \mathrm{Im}\, Q_{udcb} = J$$

Hence, the area of all triangles equals $\frac{|J|}{2}$.

Weak basis invariants

In order to discuss the CP properties of a theory, it is most useful to identify a CP-conserving sector, i.e. QED and QCD in the SM. Furthermore, the weak gauge interactions are CP invariant as well, and CP violation is only introduced in combination with Yukawa interactions. Based on this, we can define the CP transformation properties of the individual fields and thus study the CP properties of the remaining Lagrangian. The full Lagrangian is given by

$$\mathcal{L} = \underbrace{\mathcal{L}_{CP}}_{\text{CP invariant}} + \underbrace{\mathcal{L}_{\text{remaining}}}_{\substack{\text{may contain} \\ \text{CP violation}}}$$

Now consider the most general CP transformation that leaves the gauge interactions in the SM Lagrangian invariant:

$$(CP)\, u_L'(t,\vec{x})\, (CP)^\dagger = K_L\, \gamma^0 C\, \overline{u_L'}^T(t,-\vec{x})$$

$$(CP)\, \overline{u_L'}(t,\vec{x})\, (CP)^\dagger = -\, u_L'^T(t,-\vec{x})\, C^{-1} \gamma^0\, K_L^\dagger$$

$$(CP)\, d_L'(t,\vec{x})\, (CP)^\dagger = K_L\, \gamma^0\, C\, \overline{d_L'}^T(t,-\vec{x})$$

$$(CP)\, \overline{d_L'}(t,\vec{x})\, (CP)^\dagger = -\, d_L'^T(t,-\vec{x})\, C^{-1} \gamma^0\, K_L^\dagger$$

$$(CP)\, u'_R(t,\vec{x})\,(CP)^\dagger = K_R^u\, \gamma^0\, C\, \overline{u}'^{T}_R(t,-\vec{x})$$

$$(CP)\, \overline{u}'_R(t,\vec{x})\,(CP)^\dagger = -\, u'^{T}_R(t,-\vec{x})\, C^{-1}\gamma^0\, K_R^{u\dagger}$$

$$(CP)\, d'_R(t,\vec{x})\,(CP)^\dagger = K_R^d\, \gamma^0\, C\, \overline{d}'^{T}_R(t,-\vec{x})$$

$$(CP)\, \overline{d}'_R(t,\vec{x})\,(CP)^\dagger = -\, d'^{T}_R(t,-\vec{x})\, C^{-1}\gamma^0\, K_R^{d\dagger}$$

$K_{L,R}$ are unitary matrices in flavour space, and we have omitted the phase associated with the CP conjugation of the charged bosons.

CP therefore can in general mix the generations, such that there is a family of transformations that are equally well justified. Only those quantities that are CP odd under all these transformations will give rise to physical CP-violating effects.

Hence there is physical CP invariance if and only if there exists a basis where

$$K_L^\dagger\, y_u\, K_R^u = y_u^*$$
$$K_L^\dagger\, y_d\, K_R^d = y_d^*$$

(Note that for these transformations, the up and down type left-handed quarks transform with the same matrix, in contrast to the transformations that give rise to the CKM matrix.)

Alternatively, we can express this in terms of the mass matrices in the basis of weak interaction eigenstates:

$$K_L^\dagger \, m_u \, K_R^u = m_u^* \qquad \circledast$$
$$K_L^\dagger \, m_d \, K_R^d = m_d^*$$

Since a biunitary diagonalization leads to a diagonal matrix with real eigenvalues, we can always satisfy one of these conditions, but observing them both at a time is not possible in general for three or more generations.

Now, define

$$H_u = m_u m_u^\dagger \qquad m_u^{-1\dagger} m_u^{-1}$$
$$H_d = m_d m_d^\dagger$$

Then, the conditions $\circledast$ imply

$$K_L^\dagger \, H_u \, K_L = H_u^* \qquad (**) \qquad \Longleftrightarrow \quad H_{u,d}^{*-1} = K_L^\dagger H_{u,d}^{-1} K_L$$
$$K_L^\dagger \, H_d \, K_L = H_d^*$$

and vice versa, since and

$$K_L^\dagger H_{u,d} K_L = K_L^\dagger m_{u,d} K_R^{u,d} K_R^{u,d\dagger} m_{u,d}^\dagger K_L = m_{u,d}^* m_{u,d}^T$$

can be solved by $K_R^{u,d}$ with

$$K_L^\dagger \, m_{u,d} \, K_R^{u,d} = m_{u,d}^*$$
$$\Longleftrightarrow \qquad K_R^{u,d} = m_{u,d}^{-1} \, K_L \, m_{u,d}^*$$

Now

$$K_R^{u,d\dagger} = m_{u,d}^T \, K_L^\dagger (m_{u,d}^{-1})^\dagger$$
$$\Longrightarrow \quad K_R^{u,d\dagger} K_R^{u,d} = m_{u,d}^T \underbrace{K_L^\dagger \, H_{u,d}^{-1} \, K_L}_{= H_{u,d}^{*-1} = m_u^{-1T} m^{*-1}} m_{u,d}^* = 1$$

Hence, $K_R^{u,d}$ is unitary.

Substituting $K_R^{u,d}$ into ⊛, we verify that this is a solution.

In summary, we may state that the Lagrangian is (physically) CP invariant, if and only if there is a transformation satisfying ⊛⊛.

When there are several fields with the same quantum numbers, we have the freedom of rewriting the Lagrangian using a unitary transformation that mixes these, which is called a weak basis transformation (WBT). A WBT leaves the kinetic terms and the gauge interactions unchanged.

In the quark sector of the SM, we can transform

$$\begin{aligned} Q_L &= T_L \tilde{Q}_L \\ u_R &= T_R^u \tilde{u}_R \\ d_R &= T_R^d \tilde{d}_R \end{aligned} \implies \begin{aligned} \tilde{m}_u &= T_L^+ m_u T_R^u \\ \tilde{m}_d &= T_L^+ m_d T_R^d \end{aligned} \implies \tilde{H}_{u,d} = T_L^+ H_{u,d} T_L$$

This implies that traces of arbitrary polynomials of $\tilde{H}_{u,d}$ are WB invariant. For example

$$\mathrm{tr}\, H_u^a = m_u^{(2a)} + m_c^{(2a)} + m_t^{(2a)}$$

$$\mathrm{tr}\, H_d^a = m_d^{(2a)} + m_s^{(2a)} + m_b^{(2a)}$$

are basis-independent physical quantities, just as

$$\mathrm{tr}\,(H_u^r H_d^s) = \mathrm{tr}\left(U_u m_u^{D(2r)} U_u^+ U_d m_d^{D(2s)} U_d^+ \right)$$

$$= \mathrm{tr}\,(V^+ m_u^{D(2r)} V m_d^{D(2s)})$$

$$= \sum_{i=d,s,b} \sum_{j=u,c,s} V_{ij}^{+} \, m_{u\,j}^{D(2r)} \; V_{ji} \, m_{d\,i}^{D(2s)}$$

$$= \sum_{i=d,s,b} \sum_{j=u,c,s} |V_{ji}|^2 \, m_{u\,j}^{D(2r)} \; m_{d\,i}^{D(2s)}$$

where we have recalled that
$V = U_u^+ U_d$.

$$y_{u,d} \, y_{u,d}^+ = U \, y_{u,d}^{D2} \, U_{u,d}^+ \qquad m_u^{D2} = \begin{pmatrix} m_u^2 & & \\ & m_c^2 & \\ & & m_t^2 \end{pmatrix} = U_u^+ H_u U_u$$

$$m_d^{D2} = \begin{pmatrix} m_d^2 & & \\ & m_s^2 & \\ & & m_b^2 \end{pmatrix} = U_d^+ H_d U_d$$

Now we use these considerations to derive a necessary condition for CP invariance.
Starting from

$$K_L^+ \; H_u \; K_L = H_u^*$$

$$K_L^+ \; H_d \; K_L = H_d^*$$

we use that $H_{u,d}^* = H_{u,d}^T \Longrightarrow$

$$H_u^T H_d^T - H_d^T H_u^T = K_L^+ \; H_u \; K_L \; K_L^+ \; H_d \; K_L - K_L^+ \; H_d \; K_L \; K_L^+ \; H_u \; K_L$$

$$= K_L^+ H_u H_d K_L - K_L^+ H_d H_u \, K_L \Longleftrightarrow$$

$$K_L^+ [H_u, H_d] K_L = [H_u^T, H_d^T] = -[H_u, H_d]^T$$

Multiplying this equation by itself an odd number of times yields

$$K_L^+ [H_u, H_d]^r K_L = -[H_u, H_d]^{r\,T} \quad \text{for } r \text{ odd}$$

$\Longrightarrow \ \mathrm{tr}\,[H_u, H_d]^r = 0$ for r odd

Now, for arbitrary hermitian 2×2 matrices H_1 and H_2,

$$[H_1, H_2] = [a_{1\mu}\sigma^\mu, a_{2\nu}\sigma^\nu] = [a_{1i}\sigma^i, a_{2j}\sigma^j]$$

$$= 2i\, a_{1i}\, a_{2j}\, \varepsilon^{ijk}\sigma^k$$

$$[H_1, H_2]^2 = -4\, a_{1i}\, a_{2j}\, \varepsilon^{ijk}\sigma^k\, a_{1r}\, a_{2s}\, \varepsilon^{rst}\sigma^t$$

$$= -4\left(\delta^{kt}\mathbb{1} + i\underbrace{\varepsilon^{ktu}}_{\substack{\text{antisymm.}\\ \text{in } k\leftrightarrow t}}\sigma^u\right)\underbrace{\left(a_{1i}\, a_{1r}\, a_{2j}\, a_{2s}\ \varepsilon^{ijk}\varepsilon^{rst}\right)}_{\text{symm. in } k\leftrightarrow t}$$

i.e. $[H_1, H_2]^2$ is always proportional to the identity matrix and this necessary condition for CP invariance is always satisfied for two flavours.

For an arbitrary number of generations, one finds

$$[H_u, H_d]^3 = (H_u H_d)^3 - (H_d H_u)^3 + 3 H_u H_d (H_d H_u)^2 - 3 (H_u H_d)^2 H_d H_u$$

$$\mathrm{tr}\,[H_u, H_d]^3 = 3\,\mathrm{tr}\left[H_u^2 H_d^2 H_u H_d - H_d H_u H_d^2 H_u^2\right]$$

$$= 6i\,\mathrm{Im}\,\mathrm{tr}\left[H_u^2 H_d^2 H_u H_d\right]$$

$$= 6i\,\mathrm{Im}\,\mathrm{tr}\big[U_u m_u^{D\,4} U_u^\dagger U_d m_d^{D\,4} U_d^\dagger$$
$$U_u m_u^{D\,2} U_u^\dagger U_d m_d^{D\,2} U_d^\dagger\big]$$

$$= 6i\,\mathrm{Im}\,\mathrm{tr}\left[V^\dagger m_u^{D\,4} V m_d^{D\,4} V^\dagger m_u^{D\,2} V m_d^{D\,2}\right]$$

$$= 6 i \,\mathrm{Im} \sum_{i=d,s,b} \sum_{\alpha=u,c,t} \sum_{j=d,s,b} \sum_{\beta=u,c,t} V^\dagger_{j\alpha} V_{\alpha i} V^\dagger_{i\beta} V_{\beta j} \, m^{D^4}_{u\alpha} m^{D^4}_{di} m^{D^2}_{u\beta} m^{D^2}_{dj}$$

$$= 6 i \sum_{i=d,s,b} \sum_{\alpha=u,c,t} \sum_{j=d,s,b} \sum_{\beta=u,c,t} m^{D^4}_{u\alpha} m^{D^2}_{u\beta} m^{D^4}_{di} m^{D^2}_{dj} \; \mathrm{Im}\, Q_{\alpha i \beta j}$$

where we recall that

$$Q_{\alpha i \beta j} = V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i} \quad \text{with } \alpha \neq \beta, \; i \neq j$$

and $\mathrm{Im}\, Q_{udcb} = J$

A permutation $\alpha \leftrightarrow \beta$ apparently changes the sign of the imaginary part, and all indices must be different in order to have a non-vanishing imaginary part. Hence, we can infer that for three generations

$$\mathrm{tr}\,[H_u, H_d]^3 = 6 i \,(m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)$$
$$(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)\, J$$

As a necessary condition for CP violation, it must therefore hold that $J \neq 0$ and that the quark masses are non-degenerate.

The importance of this expression is that CP violation in the SM is suppressed perturbatively because it requires Yukawa couplings at a large order and involves even in the leading contributions small factors proportional to second generation Yukawa couplings.

Fortunately however, as we will discuss, the CP violation in neutral meson systems can be non-perturbatively enhanced and be observed after all.

3.5 The Vector Bosons $W^\pm$, Z

Due to the gauge quantum numbers of the fermions, there are rather distinctive predictions for the couplings of vector bosons to matter fields. To study these, we express the couplings between the physical fields of the Electroweak sector to the fermions as

$$\mathcal{L}_f = \sum_f \overline{\psi}_f \left(i\not{\partial} - m_f - g_W \frac{m_f H}{2 M_W} \right) \psi_f$$

$$- \frac{g_W}{\sqrt{2}} \sum_f \overline{\psi}_f \left(\gamma^\mu T^+ W_\mu^+ P_L + \gamma^\mu T^- W_\mu^- P_L \right) \psi_f$$

$$- |e| \sum_f Q_f \overline{\psi}_f \gamma^\mu A_\mu \psi_f - \frac{g_W}{2\cos\vartheta_W} \sum_f \overline{\psi}_f \gamma^\mu Z_\mu (V_f - A_f \gamma^5) \psi_f$$

The fermion masses and their interactions with the Higgs boson have been expressed as

$$y_f^D \, \overline{\psi}_f \frac{v + H}{\sqrt{2}} \psi_f \underset{\uparrow}{=} \overline{\psi}_f m_f \psi + \overline{\psi}_f g_W \frac{m_f H}{2 M_W} \psi_f$$

$$v = \frac{2 M_W}{g_W}$$

For the couplings to the Z boson, we have defined vectorial and axial vectorial couplings

$$V_f = T_f^3 - 2 Q_f \sin^2\vartheta_W$$
$$A_f = T_f^3$$

Now, consider the process $Z(q) \longrightarrow f(p)\,\bar{f}(p)$. The matrix element is

$$\frac{-i g_W}{2\cos\theta_W}\,\bar{u}(p)\,\gamma^\mu (V_f - A_f\gamma^5)\, v(p')\,\varepsilon_\mu(q)$$

The sum over the <u>three</u> polarisation of the massive Z boson is $\sum_{\text{Polarisations}} \varepsilon_\mu^{(i)}(q)\,\varepsilon_\nu^{(i)*}(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}$

such that we need to evaluate the trace

$$\mathrm{tr}\left[\not{p}\,\gamma^\mu (V_f - A_f\gamma^5)\,\not{p}'\,(V_f + A\gamma^5)\,\gamma^\nu\right]\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)$$

$$= \mathrm{tr}\left[\not{p}\,\gamma^\mu (V_f^2 + A_f^2)\,\not{p}'\,\gamma^\nu\right]\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)$$

$\not{p}\gamma^\mu = -\gamma^\mu\not{p} + 2p^\mu$

$$= \mathrm{tr}\Big[\underbrace{\gamma^\mu\gamma_\mu\,\not{p}\not{p}' - 2\,\not{p}\not{p}'}_{\substack{\gamma^\mu\gamma_\mu = 4 \\ 4 * M_Z^2}} - \frac{1}{M_Z^2}\underbrace{\not{p}\not{q}\not{p}'\not{q}}_{4(2p\cdot q\, p'\cdot q - p\cdot p'\, q^2) = 0}\Big](V_f^2 + A_f^2)$$

$(p+p')^2 = 2p\cdot p' = M_Z^2$

$(q-p)^2 = 0 = M_Z^2 - 2q\cdot p$

$$= 4M_Z^2 (V_f^2 + A_f^2)$$

Now, we average over initial and sum over final polarisations:

$$\frac{1}{3}\sum_{\text{Polarisations}} |i\mathcal{M}|^2 = \frac{1}{3}\,\frac{g_W^2}{4\cos^2\theta_W}\,4M_Z^2 (V_f^2 + A_f^2) = \frac{1}{3}\,\frac{8}{\sqrt{2}}\,G_F\, M_Z^4 (V_f^2 + A_f)^2$$

Where we made use of the definition of the Fermi-constant

$$G_F = \frac{\sqrt{2}\,g_W^2}{8M_W^2} = \frac{\sqrt{2}\,g_W^2}{8M_Z^2\cos^2\theta_W} \Longleftrightarrow \frac{M_Z^2}{\cos^2\theta_W} = \frac{8G_F}{\sqrt{2}\,g_W^2}\,M_Z^4$$

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} \approx 1$$

In order to obtain the decay rate, we must divide by $2M_Z$ and multiply by the two-body phase-space factor

$$\int \frac{d^4k}{(2\pi)^4} \frac{d^4(k')}{(2\pi)^4} (2\pi)^4 \delta^4(q-k-k')\, 2\pi\, \delta(k^2)\, 2\pi\, \delta(k'^2) = \frac{1}{8\pi}$$

such that

$$\Gamma(Z \to f\bar{f}) = C \frac{G_F M_Z^3}{6\sqrt{2}\,\pi} (V_f^2 + A_f^2)$$

Here, we have introduced a colour factor, $C=3$ for quarks & $C=1$ for leptons.

Similarly, for the decays of W bosons, one obtains

$$\Gamma(W^+ \longrightarrow f\bar{f}') = C \frac{G_F M_W^3}{6\sqrt{2}\,\pi}$$

where we refer to the sum of the decays to a given quark of charge $\frac{2}{3}$ and all anti-quarks of charge $\frac{1}{3}$, e.g. $W^+ \to u\bar{d} + u\bar{s} + u\bar{b}$. For an individual mode $W \longrightarrow u_i d_j$, there is an additional factor $|V_{ij}|^2$ from the CKM matrix.

The following branching fractions therefore test the gauge interactions of matter in the Standard Model of EWSB:

	Relative Coupling	Branching Ratio
$W^+ \to e^+\nu_e, \mu^+\nu_\mu, \tau^+\nu_\tau$	1	3 * 11,1%
$W^+ \longrightarrow u\bar{d} + u\bar{s} + u\bar{b}$	3	33,3%
$W^+ \to c\bar{d} + c\bar{s} + c\bar{b}$	3	33,3%

There is no decay into top quarks because of their large mass $m_t \approx 172$ GeV.

	V_f	A_f	Rel. Coupling	Br. Ratio
$Z \to \nu_e\bar{\nu}_e, \nu_\mu\bar{\nu}_\mu, \nu_\tau\bar{\nu}_\tau$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$3 * 6{,}8\%$
$Z \to e^+e^-, \mu^+\mu^-, \tau^+\tau^-$	$-\frac{1}{2}+2\sin^2\vartheta_W$	$-\frac{1}{2}$	$\frac{1}{4}+(\frac{1}{2}-2\sin^2\vartheta_W)^2$	$3 * 3{,}4\%$
$Z \to u\bar{u}, c\bar{c}$	$\frac{1}{2}-\frac{4}{3}\sin^2\vartheta_W$	$\frac{1}{2}$	$3[\frac{1}{4}+(\frac{1}{2}-\frac{4}{3}\sin^2\vartheta_W)^2]$	$2 * 11{,}8\%$
$Z \to d\bar{d}, s\bar{s}, b\bar{b}$	$-\frac{1}{2}+\frac{2}{3}\sin^2\vartheta_W$	$-\frac{1}{2}$	$3[\frac{1}{4}+(\frac{1}{2}-\frac{2}{3}\sin^2\vartheta_W)^2]$	$3 * 15{,}2\%$

These tree-level branching ratios compare well with the experimentally observed values listed in the Particle Physics Booklet pp. 9, 10 !

For the Z boson, you also find in that table "invisible" decays. These go into neutrinos.
The theoretical tree-level ratio for the decay into one species of neutrinos over decays into one species of leptons is

$$\frac{\Gamma(Z \to \nu\bar{\nu})}{\Gamma(Z \to l^+l^-)} = \frac{2}{1+(1-4\sin^2\vartheta_W)^2} \approx 1{,}99$$

Experimentally, one finds that

$$\frac{\Gamma(Z \to \text{invisible})}{\Gamma(Z \to l^+l^-)} = 5{,}956 \pm 0{,}031,$$

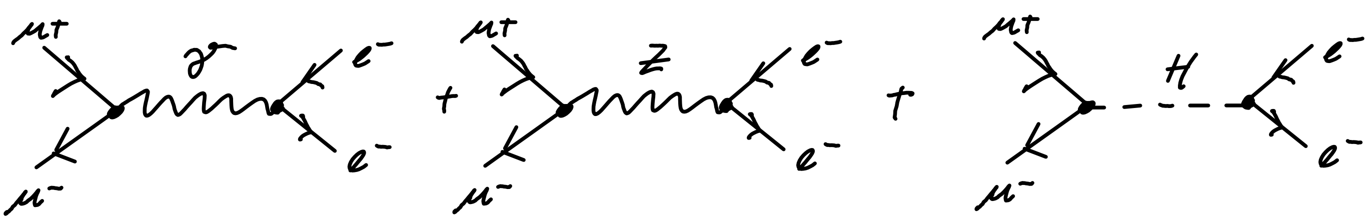
such that the number of light neutrinos coupling to Z is $2{,}991 \pm 0{,}016$.

We can also use above matrix element in order to obtain the production cross sections for the vector bosons:

$$\sigma_{f\bar{f}' \to W} = \frac{\pi}{3}\sqrt{2}\, G_F M_W^2 \,|V_{f\bar{f}'}|^2\, \delta(\hat{s} - M_W^2)$$

$$\sigma_{f\bar{f} \to Z} = \frac{\pi}{3}\sqrt{2}\, G_F M_Z^2 \,(V_f^2 + A_f^2)\, \delta(\hat{s} - M_Z^2)$$

When protons (e.g. LHC) [or protons & anti-protons (e.g. SPS, Tevatron) are collided, the full cross section can be obtained from the partonic one using the methods applied to the Drell-Yan process.
Another classic observation is the process $e^+e^- \to \mu^+\mu^-$. We discussed the production via photons in the chapter on QED. At high energies close to the Electroweak scale, one has to account for the new particles by these diagrams:



Since the Yukawa couplings of the Higgs to μ and e are very small, we can neglect these for the present purposes. Calculating the cross-section is straightforward using the methods introduced in these lectures, but yet somewhat tedious. The details can be found e.g. in the book by Greiner & Müller. The result is

$$\sigma_{e^+e^- \to \mu^+\mu^-} \approx \frac{4\pi\alpha^2}{3s}\left[1 + \frac{s^2}{16\sin^4 2\theta_W \left(s - |M_Z - \frac{i}{2}\Gamma_Z|^2\right)^2}\right]$$

Here, Γ_Z is the total decay rate of the Z-boson. It is related to the Z-boson self-energy as

$$\Gamma_Z = \frac{1}{M_Z} \operatorname{Im} \Pi(p^0 = M_Z, \vec{p} = 0)$$

and therefore occurs in the resummed propagator (cf. Chapter 2). There it regulates the divergence that would otherwise occur for $s = M_Z^2$. The Z-boson occurs as a resonance peak on top of the $\frac{1}{s}$-behaviour of the cross section. The maximum is reached for

$s = M_Z^2 - \frac{\Gamma_Z^2}{4}$ where $\sigma_{max} \approx \frac{4\pi\alpha^2}{3\Gamma_Z^2}$

For larger (smaller) values of Γ_Z, the peak becomes wider (more narrow), such that instead of a decay rate, one often speaks of a decay width.

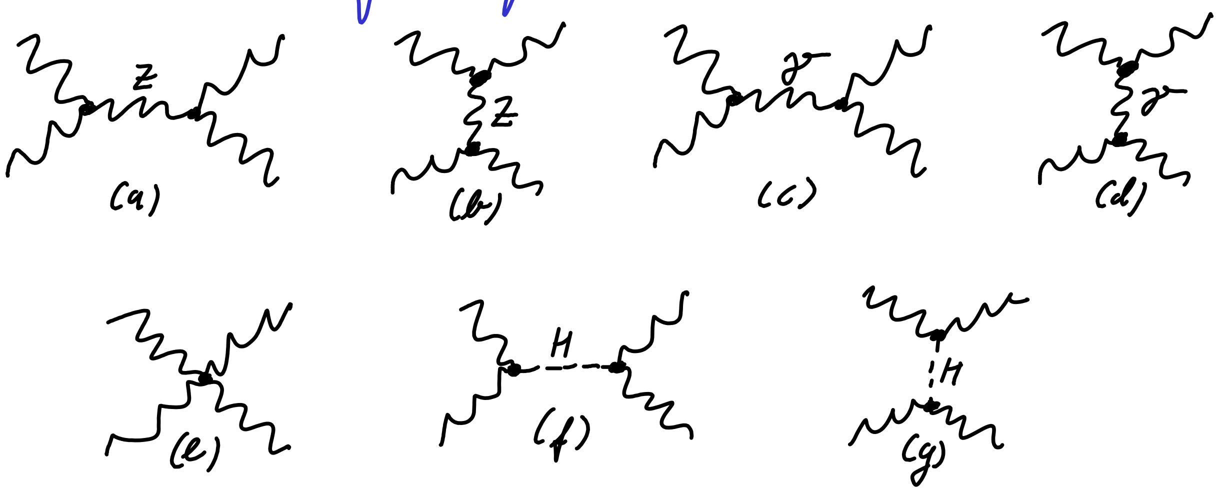
All these results provide a strong evidence that the Electric and Weak interactions unify to a single, spontaneously broken gauge theory. Yet, the charge assignments and the value of the ρ-parameter may rely on coincidences, such that there could be no additional degrees of freedom in the Electroweak sector beyond $W^\pm$ and Z. Or the Higgs boson could be too heavy to be observed at the LHC.

However, consideration of the high-energy behaviour of the scattering amplitudes of vector boson indicates that $W^\pm$ and Z are not the full story.

We therefore consider the process

$$W^+(p_+) + W^-(p_-) \to W^+(q_+) + W^-(q_-)$$

The contributing diagrams are



In the centre of mass frame, we parametrise the momenta as

$$p_\pm = (E, 0, 0, \pm p) \qquad E^2 - p^2 = M_W^2$$

$$q_\pm = (E, 0, \pm p \sin\vartheta, \pm p \cos\vartheta)$$

The interesting behaviour at high energies occurs for the longitudinal polarisations (cf. the polarisation sum above)

$$\varepsilon_L(p_\pm) = \left(\frac{p}{M_W}, 0, 0, \pm \frac{E}{M_W}\right)$$

$$\varepsilon_L(q_\pm) = \left(\frac{p}{M_W}, 0, \pm \frac{E}{M_W} \sin\vartheta, \pm \frac{E}{M_W} \cos\vartheta\right)$$

These satisfy the Lorentz gauge condition $k \cdot \varepsilon_L(k) = 0$ and are normalised such that $\varepsilon_L^2 = -1$.

The pure gauge diagrams exhibit the disastrous high-energy behaviour.

$$|i\mathcal{M}^{(a-d)}|^2 = g_W^2 \left\{ \frac{p^4}{M_W^4}\left[3 - 6\cos\vartheta - \cos^2\vartheta\right] + \frac{p^2}{M_W^2}\left[\frac{9}{2} - \frac{11}{2}\cos\vartheta - 2\cos^2\vartheta\right] \right\}$$

$$|i\mathcal{M}^{(e)}|^2 = g_W^2 \left\{ \frac{p^4}{M_W^4}\left[-3 + 6\cos\vartheta + \cos^2\vartheta\right] + \frac{p^2}{M_W^2}\left[-4 + 6\cos\vartheta + 2\cos^2\vartheta\right] \right\}$$

While the most offensive terms $\propto p^4$ cancel, still the remaining terms indicate a breakdown of perturbation theory for $p \gg M_W$.
This growth of $|i\mathcal{M}|^2$ leads to a growth of the S-matrix as well, which is why it is sometimes referred to unitarity violation in W^+W^- scattering, but one should bear in mind that this is due to the application of perturbation theory beyond its domain of validity.
But where poison grows, there also does the cure. For the Higgs exchanges, one finds:

$$|i\mathcal{M}^{(f-g)}|^2 = g_W^2 \left\{ \frac{p^2}{M_W^2}\left[-\frac{1}{2}-\frac{1}{2}\cos^2\vartheta\right] - \frac{M_H^2}{4M_W^2}\left[\frac{s}{s-M_H^2}+\frac{t}{t-M_H^2}\right]\right\}$$

Field Theory holds another spectacular cancellation in stock:

$$|i\mathcal{M}^{(a-g)}|^2 = -g_W^2 \frac{M_H^2}{4M_W^2}\left[\frac{s}{s-M_H^2}+\frac{t}{t-M_H^2}\right]$$

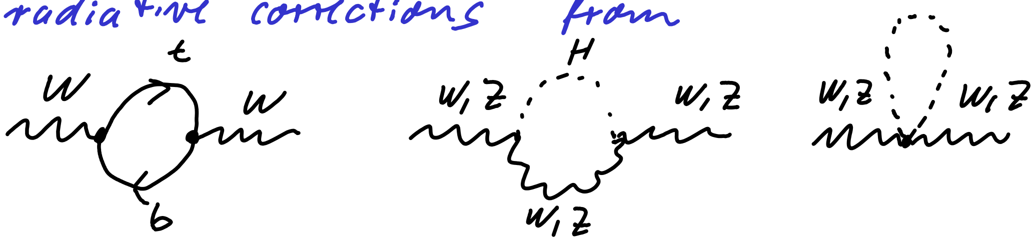
Which has a benign behaviour up to high energies.
The meaning of these results is that at high energy colliders that probe the Electroweak scale, we will either

- discover the Higgs boson H
- or discover other new degrees of freedom that ensure the perturbativity of W^+W^- scattering at high energies
- or observe a breakdown of perturbation theory in W^+W^- scatterings, the details of which will hint to another mechanism different from the Standard Model EWSB realised in Nature.

It is therefore believed (and explained to the funding bodies) that a high energy collider such as the LHC must necessarily discover the Higgs boson or uncover new laws of Nature.

3.6 The Higgs Boson

Now, where is it? A detailed analysis of W^+W^- scattering leads to the conclusion that unitarity is fixed if $M_H < 1$ TeV. In addition, there are other more powerful indications. The gauge boson masses receive radiative corrections from



Regarding the first diagram, recall that the superficial degree of divergence of the vacuum polarisation is quadratic. This divergence is cancelled by the anti-fermion in the