2. QED

2.1 Review of basics

Lagrangian of Quantum Electrodynamics

$$\boxed{\mathcal{L}_{QED}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi}$$

$$D_\mu \psi(x) = (\partial_\mu + ie A_\mu)\psi(x)$$

$$-\frac{1}{4}\underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{=: F^{\mu\nu}}(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Photon polarisation:

transversality

$$\varepsilon^\mu = (0, \vec{\varepsilon}) \quad \text{and} \quad \vec{k} \cdot \vec{\varepsilon} = 0 \quad \text{and} \quad \vec{\varepsilon} \cdot \vec{\varepsilon}^{\,*} = 1 \quad \text{(normalisation)}$$

Helicity:

Recall from our discussion of Lorentz-symmetry, that

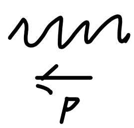
$$J_3 = J^{12} = i\left[g^{\mu 1}\delta^2{}_\nu - g^{\mu 2}\delta^1{}_\nu\right] = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

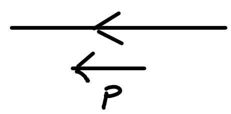
$$J_3 \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$$

Hence, for a photon travelling in 3-direction, we can choose the basis of polarisation vectors $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$ what corresponds to eigenstates of spin, or, when formulated in a direction-independent manner, of the <u>helicity</u> $h = \frac{\vec{k}}{|\vec{k}|} \cdot \vec{J} = \pm 1$. Photons with $h = +1$ are right-, with $h = -1$ are left polarised.

Feynman rules for QED

propagators:

photon:  $= -\dfrac{i g_{\mu\nu}}{p^2 + i\varepsilon}$

charged fermion:  $= \dfrac{i(\not p + m)}{p^2 + m^2 + i\varepsilon}$

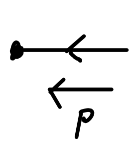
The arrow on the propagator indicates the direction from $\bar\psi$ to ψ.

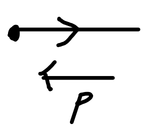
vertex:

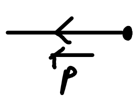
 $= -ie\gamma^\mu$, impose momentum conservation

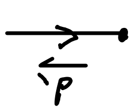
note that e is negative for the electron

external legs:

 $= u(\vec p, s)$ incoming fermion

 $= \bar v(\vec p, s)$ incoming anti-fermion

 $= \bar u(\vec p, s)$ outgoing fermion

 $= v(\vec p, s)$ outgoing anti-fermion

 $= \varepsilon^\mu$ incoming photon

 $= \varepsilon^{*\mu}$ outgoing photon

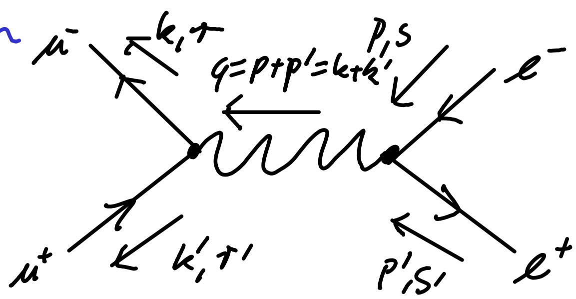
loops:

Integrate over loop momenta. Fermion loops pick up a factor of -1.

2.2 Elementary Scattering Processes and Crossing Symmetries

While we deal here with charged leptons and photons, the results worked out here essentially also applies to quark and photons and moreover quarks and gluons.
$\longrightarrow$ Ubiquity at colliders warrants detailed discussion.

We now aim to calculate the cross section for the process $e^+e^- \longrightarrow \mu^+\mu^-$. The leading contribution to the invariant matrix element is represented by the Feynman diagram



τ, τ' and s, s' stand for the spin

Application of the Feynman rules yields

$$i\mathcal{M}_{e^+e^-\to\mu^+\mu^-} = \overline{u}_\mu(k,\tau)(-ie\gamma^\mu)\, v_\mu(k',\tau')\,\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\,\overline{v}_e(p',s')(-ie\gamma^\nu)\, u_e(p,s)$$

Now suppose (as it is the case in many experimental situations) that the electron and positron beams are unpolarised and that we do not detect the spin of the produced muons. (A muon chamber, for example, would indeed only measure the momentum, not the spin of a muon). Hence, our observable is

$$\underbrace{\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}}_{\text{average over in-coming spin}} \underbrace{\sum_\tau \sum_{\tau'}}_{\text{sum over outgoing spin}} \left|i\mathcal{M}_{e^+e^-\to\mu^+\mu^-}\right|^2 =: \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

Consider the piece (the Greek indices are spinorial)

$$\sum_{\tau,\tau'} \bar{u}_{\mu\alpha}(k,\tau)\gamma^{\mu}_{\alpha\beta} v_{\mu\beta}(k',\tau')\, \bar{v}_{\mu\gamma}(k',\tau')\gamma^{\varrho}_{\gamma\delta} u_{\mu\delta}(k,\tau)$$ NB γ^0, $\gamma^0\gamma^\mu$ are hermitian!

$$= (\not{k}'-m_\mu)_{\beta\gamma}\gamma^{\varrho}_{\gamma\delta}(\not{k}+m_\mu)_{\delta\alpha}\gamma^{\mu}_{\alpha\beta} = \mathrm{tr}\left[(\not{k}'-m_\mu)\gamma^\varrho(\not{k}+m_\mu)\gamma^\mu\right]$$

We have used here the identities for the spin sums of the basis spinors quoted in the previous chapter. The same manipulation can also be applied to the electron spinors. We therefore obtain

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^2}{4q^4}\,\mathrm{tr}\left[(\not{k}'-m_\mu)\gamma^\varrho(\not{k}+m_\mu)\gamma^\mu\right]\mathrm{tr}\left[(\not{p}+m_e)\gamma_\varrho(\not{p}'-m_e)\gamma_\mu\right]$$

The evaluation of spinor traces is routinely encountered in phenomenological calculations ($\longrightarrow$ there are useful computer algebra tools for their evaluation). The present traces are manageable by the use of the following rules:

$$\mathrm{tr}\,\mathbb{1} = 4$$

$$\mathrm{tr}\,(\text{any odd \# of } \gamma\text{-matrices}) = 0$$

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\varrho\gamma^\sigma) = 4(g^{\mu\nu}g^{\varrho\sigma} - g^{\mu\varrho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\varrho})$$

A short calculation then gives

$$\mathrm{tr}\left[(\not{p}+m_e)\gamma_\varrho(\not{p}'-m_e)\gamma_\mu\right] = 4\left[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}(p\cdot p' + m_e^2)\right]$$

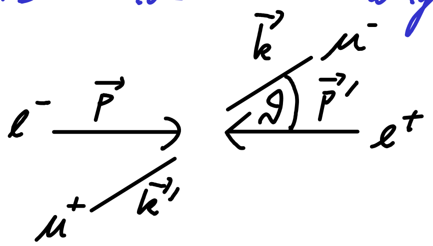
$$\mathrm{tr}\left[(\not{k}'-m_\mu)\gamma^\varrho(\not{k}+m_\mu)\gamma^\mu\right] = 4\left[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k' + m_\mu^2)\right]$$

The muon mass $m_\mu = 105{,}7$ MeV is roughly 200 times larger than the electron mass $m_e = 511{,}0$ keV, such that we may neglect it in the present calculation (corrections of order $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{128}$ at next to leading order would

be of roughly the same size).
The contraction of Minkowski indices then leads to

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2\,(p\cdot p')\right]$$

The total cross section Lorentz-transforms as an area, and differential expressions depend entirely on the chosen reference frame. Let us choose the center of mass (CMS) frame, where $\vec{p} = -\vec{p}\,'$. Consequently, also $\vec{k} = -\vec{k}\,'$, and let's denote the angle between electron and muon by ϑ:



The various Lorentz-invariants in the squared matrix element are:

$$p\cdot p' = p^0 p'^0 - \vec{p}\cdot\vec{p}\,' = 2\vec{p}^{\,2}$$

$$q^2 = (p+p')^2 = 2\underbrace{m_e^2}_{\to 0} + 2p\cdot p' = 4\vec{p}^{\,2}$$

$$p\cdot k = p'\cdot k' = p^0 k^0 - \vec{p}\cdot\vec{k} = \vec{p}^{\,2} - |\vec{p}|\sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta$$

$$p\cdot k' = p'\cdot k = p^0 k^0 + \vec{p}\cdot\vec{k} \underset{\uparrow}{=} \vec{p}^{\,2} + |\vec{p}|\sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta$$

$\omega(\vec{p}) \approx \omega(\vec{k}) \approx |\vec{p}|$

It follows

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{16|\vec{p}|^4}\Big[|\vec{p}|^2\left(|\vec{p}| - \sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta\right)^2$$

$$+ |\vec{p}|^2\left(|\vec{p}| + \sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta\right)^2 + 2\vec{p}^{\,2} m_\mu^2\Big]$$

$$= e^4\left[\left(1 + \frac{m_\mu^2}{\vec{p}^{\,2}}\right) + \left(1 - \frac{m_\mu^2}{\vec{p}^{\,2}}\right)\cos^2\vartheta\right]$$

For two final states in the center of mass frame, it is useful to integrate the differential cross section first over the moduli of the outgoing momenta, such that

$$\sigma = \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{(2\pi)^4 \delta^4(p_A + p_B - k - k')}{4\sqrt{\vec{k}^2 + m^2}\sqrt{\vec{k}'^2 + m'^2}} \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2$$

$$= \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \int d\Omega \frac{|\vec{k}|^2 d|\vec{k}|}{(2\pi)^3} \frac{(2\pi)\delta(p_A^0 + p_B^0 - \sqrt{\vec{k}^2 + m^2} - \sqrt{\vec{k}^2 + m'^2})}{4\sqrt{\vec{k}^2 + m^2}\sqrt{\vec{k}^2 + m'^2}}$$

$\uparrow$
$\vec{p}_A + \vec{p}_B = 0$
$\Rightarrow \vec{k} = -\vec{k}'$ (CMS)

$$* \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2$$

$$= \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \frac{1}{16\pi^2} \int d\Omega \frac{\vec{k}^2 \; \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2}{\frac{|\vec{k}|}{\sqrt{\vec{k}^2 + m^2}} + \frac{|\vec{k}|}{\sqrt{\vec{k}^2 + m'^2}}} \frac{1}{\sqrt{\vec{k}^2 + m^2}\sqrt{\vec{k}^2 + m'^2}}$$

$$= \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \frac{1}{16\pi^2} \int d\Omega \frac{|\vec{k}| \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2}{p_A^0 + p_B^0}$$

$\longrightarrow \sqrt{\vec{k}^2 + m^2} + \sqrt{\vec{k}^2 + m'^2} = p_A^0 + p_B^0$

We now identify $|v_A - v_B| = 2$ and $\omega_A = \omega_B = p_A^0 = p_B^0 = |\vec{p}|$, such that

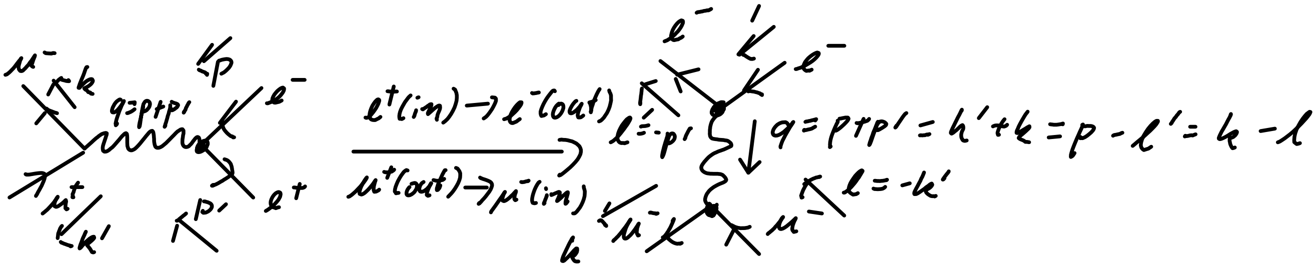
$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2} \frac{\sqrt{\vec{p}^2 - m_\mu^2}}{|\vec{p}|^3} \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

$$= \frac{1}{256\pi^2} \frac{\sqrt{\vec{p}^2 - m_\mu^2}}{|\vec{p}|^3} e^4 \left[\left(1 + \frac{m_\mu^2}{\vec{p}^2}\right) + \left(1 - \frac{m_\mu^2}{\vec{p}^2}\right)\cos^2\vartheta \right]$$

The total cross section is

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int_{-1}^{1} d\cos\vartheta \frac{d\sigma}{d\Omega} = \frac{e^4}{48\pi} \frac{\sqrt{\vec{p}^2 - m_\mu^2}}{|\vec{p}|^3} \left(1 + \frac{1}{2}\frac{m_\mu^2}{\vec{p}^2}\right)$$

Now, we consider the process $e^-\mu^- \longrightarrow e^-\mu^-$. The Feynman diagram of this process is related to the one just considered above by replacing an incoming positron by an outgoing electron and an outgoing anti-muon by an incoming muon:

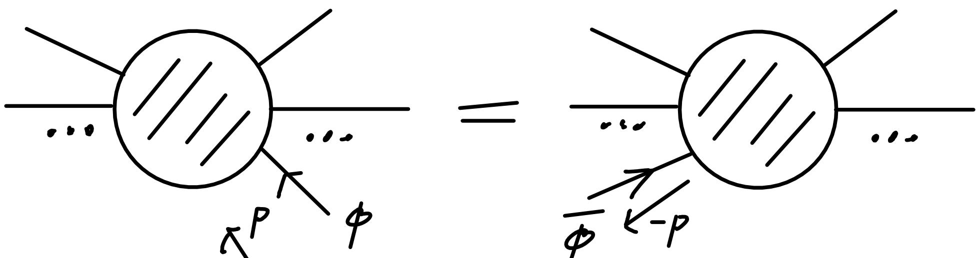


$e^+e^- \longrightarrow \mu^+\mu^-$ $\qquad$ $e^-\mu^- \longrightarrow e^-\mu^-$

We can easily convince ourselves that the spin-sums over the squared matrix elements for these processes are identical. It is possible to prove the more general

<u>crossing symmetry</u>

$$i\mathcal{M}(\phi(p) + \ldots \longrightarrow \ldots) = i\mathcal{M}(\ldots \longrightarrow \ldots \bar{\phi}(-p))$$

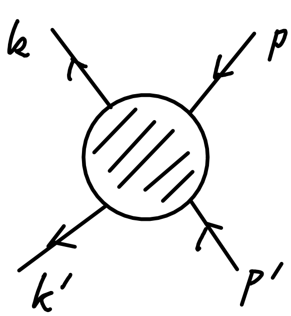


Replacing fermions by antifermions can lead to an additional sign change, because

$$\sum_{\text{spins}} u(p)\bar{u}(p) = \not{p} + m = -(\not{k} - m) = -\sum_{\text{spins}} v(k)\bar{v}(k)$$

While for $e^+e^- \longrightarrow \mu^+\mu^-$, q is the sum of the incoming or equivalently the outcoming momenta, in $e^-\mu^- \longrightarrow e^-\mu^-$, it parametrises the difference between incoming and outgoing momenta. For $2 \longleftrightarrow 2$ processes, it is useful to categorise these kinematic situations through the

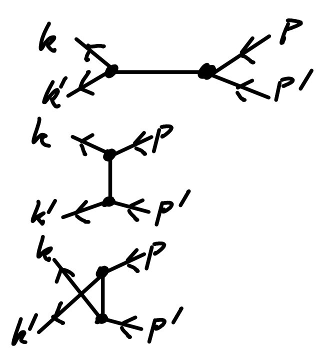
<u>Mandelstam variables:</u>



$$s = (p+p')^2 = (k+k')^2$$

$$t = (k-p)^2 = (k'-p')^2$$

$$u = (k'-p)^2 = (k-p')^2$$



One of the variables can always be eliminated by use of the relation

$$s+t+u=\sum_{i=1}^{4} m_i^2$$

As indicated by the sketched diagrams, the Mandelstam variables are the invariant momentum square of the exchanged particle.

For the process $e^+e^- \longrightarrow \mu^+\mu^-$, the Mandelstam variables are:

$$s=(p+p')^2 = 2p\cdot p' \quad \rightsquigarrow p\cdot p' = \frac{1}{2}s$$

$$t=(k-p)^2 = m_\mu^2 - 2k\cdot p \quad \rightsquigarrow p\cdot k = \frac{1}{2}(m_\mu^2 - t)$$

$$u=(k'-p)^2 = m_\mu^2 - 2k'\cdot p \quad \rightsquigarrow p\cdot k' = \frac{1}{2}(m_\mu^2 - u)$$

Knowing that

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{e^+e^-\to\mu^+\mu^-}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2\,(p\cdot p')\right]$$

$$= \frac{2e^4}{s^2}\left[t^2 - 2t\,m_\mu^2 + m_\mu^4 + u^2 - 2u\,m_\mu + m_\mu^4 + 2m_\mu^2\,(2m_\mu^2 - u - t)\right]$$

$$= \frac{2e^4}{s^2}\left[t^2+u^2-4(t+u)\,m_\mu^2 + 6\,m_\mu^4\right]$$

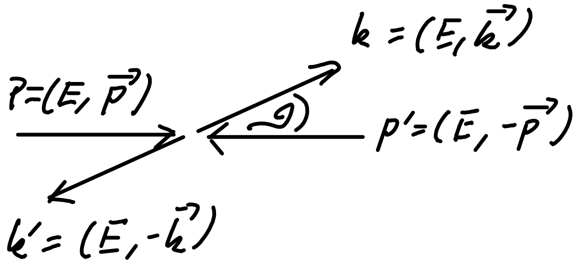
we can immediately express

$$s=(p+p')^2 \longrightarrow (p-l')^2 = t \qquad t=(k-p)^2 \longrightarrow (k-p)^2 = u$$

$$u=(k-p')^2 \longrightarrow (k+l')^2 = s$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{e^-\mu^-\to e^-\mu^-}|^2 = \frac{2e^4}{t^2}\left[s^2+u^2-4(s+u)\,m_\mu^2 + 6\,m_\mu^4\right]$$

In order to get a quick grasp of how the cross sections are related to the Mandelstam variables, we note their ϑ-dependence in case all four scattering particles are of mass m:



$|\vec{k}|=|\vec{p}|$ since all masses are the same

$$s=(p+p')^2=4E^2$$
$$t=(k-p)^2=-2|\vec{p}|^2(1-\cos\vartheta)$$
$$u=(k'-p)^2=-2|\vec{p}|^2(1+\cos\vartheta)$$

Therefore, in the ultrarelativistic limit, i.e. when we may approximate $m_e \approx m_\mu \approx 0$,

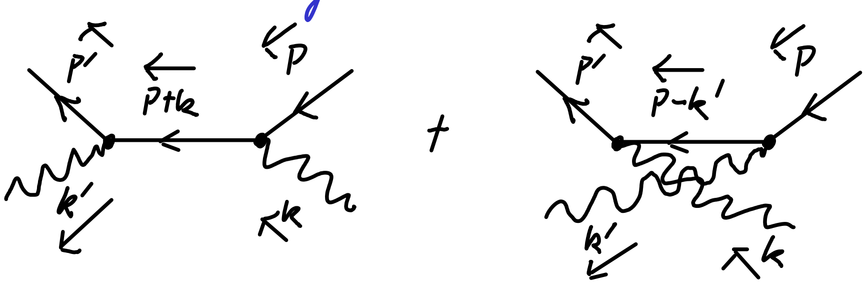
$$\frac{1}{4}\sum_{\text{spins}}\left|\mathcal{M}_{e^-\mu^-\to e^-\mu^-}\right|^2=\frac{2e^4}{4(1-\cos\vartheta)^2|\vec{p}|^4}\left[16|\vec{p}|^4+4|\vec{p}|^4(1+\cos\vartheta)^2\right]$$

$$\frac{d\sigma_{e^-\mu^-\to e^-\mu^-}}{d\Omega}=\frac{1}{8\vec{p}^2}\frac{1}{16\pi^2}\frac{|\vec{p}|\,\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2}{2|\vec{p}|}$$

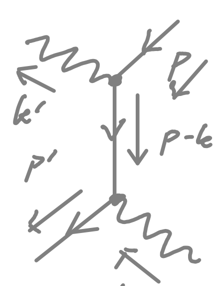
$$=\frac{e^4}{128\pi^2}\frac{1}{\vec{p}^2}\frac{4+(1+\cos\vartheta)^2}{(1-\cos\vartheta)^2}$$

Next, we study Compton Scattering, $e^-\gamma\to e^-\gamma$. The new calculational features that we encounter are the interference of two diagrams and the polarisation sum over external photons.

The two diagrams are:



The second diagram is of course t-channel



Let m denote the electron mass.

The Feynman rules yield the invariant Matrix element:

$$i\mathcal{M} = \bar{u}(p',s')(-ie\gamma^\mu)\varepsilon^*_\mu(k')\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\varepsilon}(-ie\gamma^\nu)\varepsilon_\nu(k)\,u(p,s)$$

$$+\bar{u}(p',s')(-ie\gamma^\nu)\varepsilon_\nu(k)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2+i\varepsilon}(-i\gamma^\mu)\varepsilon^*_\mu(k')\,u(p,s)$$

$$= -ie^2\varepsilon^*_\mu(k')\varepsilon_\nu(k)\bar{u}(p',s')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2+i\varepsilon}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2+i\varepsilon}\right]u(p,s)$$

Make use of

$$\left.\begin{aligned}(p+k)^2-m^2 &= 2p\cdot k\\ (p-k')^2-m^2 &= -2p\cdot k'\end{aligned}\right\}\ \text{because } p^2=m^2 \text{ and } k^2=0$$

$$(\not{p}+m)\gamma^\nu u(p) \underset{\not{p}\gamma^\nu+\gamma^\nu\not{p}=2p^\nu}{=} (2p^\nu-\gamma^\nu\not{p}+\gamma^\nu m)u(p,s) \underset{\text{Dirac Eq.}}{=} 2p^\nu u(p,s)$$

and $\varepsilon \to 0$

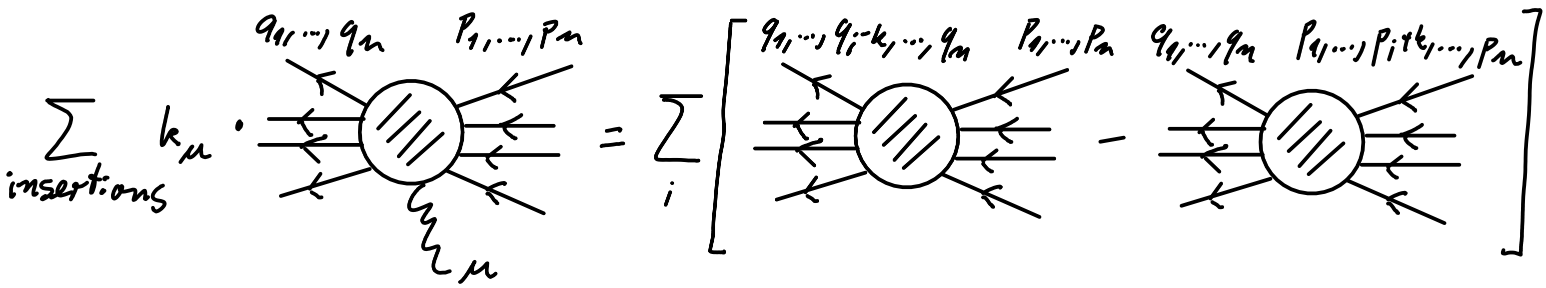
$\Longrightarrow$

$$i\mathcal{M} = -ie^2\varepsilon^*_\mu(k')\varepsilon_\nu(k)\bar{u}(p',s')\left[\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2p\cdot k}+\frac{\gamma^\nu\not{k}'\gamma^\mu-2\gamma^\nu p^\mu}{2p\cdot k'}\right]u(p,s)$$

When squaring this amplitude, a simplification occurs again when averaging over polarisations. To argue how the polarisation sum is performed, we use the <u>Ward-identity</u>, which is a special case of the <u>Ward-Takahashi identity</u> (that we give without proof):

$$k_\mu \, i \underbrace{\mathcal{G}^\mu(\dots, k)}_{\substack{\text{sum of all} \\ \text{Green functions} \\ \text{with photon inserted}}} = e \sum \Big[i\mathcal{G}(q_1, \dots, q_i - k; p_1, \dots) - i\mathcal{G}(q_1, \dots; p_1, \dots, p_i + k, \dots) \Big]$$

Diagrammatically:



Suppose now all external propagators on the left-hand side are on shell. Then, one external fermion line in each of the diagrams on the right-hand side is off shell. They therefore do not contain the "correct" poles for the S matrix and have to sum to zero as contributions to a matrix element. Define $i\mathcal{M}^\mu$ as the invariant matrix element without the photon polarisation factor ε_μ. It follows the <u>Ward identity</u>

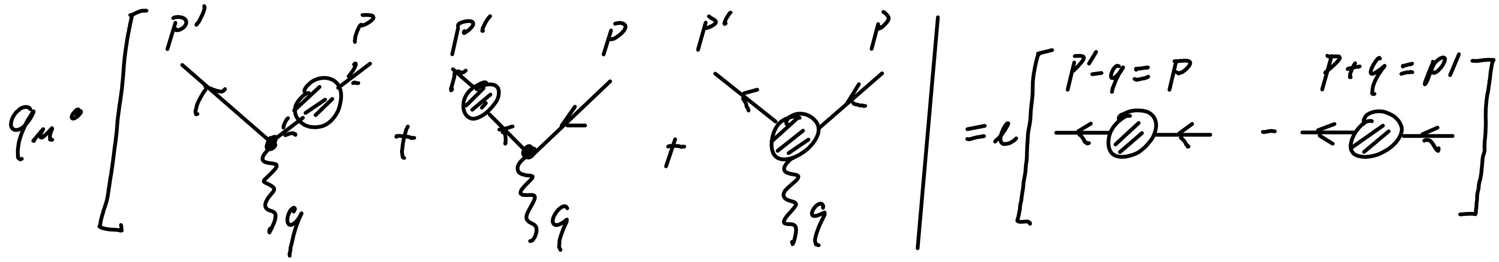
$$k_\mu \, i\mathcal{M}^\mu = 0$$

A simple argument that also leads to this important identity is that the photon attaches to a vertex $-ie\bar{\psi}\gamma^\mu\psi = -ie\, j^\mu$ (where j^μ is here understood as an operator). Current conservation implies that $\partial_\mu \langle \dots j^\mu(x) \dots \rangle = 0$, such that, when writing

$$i\mathcal{M}^\mu(k) = \int d^4x\, e^{ikx} \langle \dots j^\mu(x) \dots \rangle ,$$

we immediately obtain the Ward identity.

As an aside we also recapitulate the important consequence for renormalization:



$$iS(p) = \frac{i}{\not p - m - \Sigma} = \frac{iZ_2}{\not p - m}\Bigg|_{\text{on shell}}$$

ignore higher orders

$$iS(p')\, q_\mu (-ie)\Gamma^\mu\, iS(p) = e\left(iS(p) - iS(p+q)\right)$$

$\longrightarrow$

$$-i\, \underbrace{q_\mu \Gamma^\mu}_{= q_\mu Z^{-1}\gamma^\mu} = (iS(p+q))^{-1} - (iS(p))^{-1} = \frac{\not q}{iZ_2}$$

$$\Longrightarrow Z_1 = Z_2$$

Now back to the photon polarisation sum. Without loss of generality, take $k^\mu = (k, 0, 0, k)$, such that the two transverse polarisations are

$\varepsilon_1^\mu = (0, 1, 0, 0)$ and $\varepsilon_2^\mu = (0, 0, 1, 0)$.

Then

$$\sum_{\substack{\text{photon} \\ \text{pol.}}} |\varepsilon^*_\mu\, i\mathcal{M}^\mu(k)|^2 = \sum_{i=1,2} \varepsilon^*_{i\mu} \varepsilon_{i\nu} \mathcal{M}^\mu(k) \mathcal{M}^{*\nu}(k) = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2$$

$$= |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2 + \underbrace{|\mathcal{M}^3(k)|^2 - |\mathcal{M}^0(k)|^2}_{=0 \text{ by Ward identity}} = -g_{\mu\nu}\mathcal{M}^\mu(k)\,\mathcal{M}^{*\nu}(k)$$

Hence, the rule for performing a photon polarisation sum is

$$\sum_{\substack{\text{photon}\\ \text{pol.}}} \varepsilon^*_\mu \varepsilon_\nu \longmapsto -g_{\mu\nu}$$

It is now straightforward, but get tedious, to obtain cross section for Compton scattering. The spin/polarisation sums lead to

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = \frac{e^4}{4}\, \underset{\uparrow \text{ photon } k}{g_{\mu\varrho}}\, \underset{\uparrow \text{ photon } k'}{g_{\nu\sigma}}\, \mathrm{tr}\Big\{ (\not p' + m)\left[\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not k' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right]$$
$$* (\not p + m)\left[\frac{\gamma^\sigma \not k \gamma^\varrho + 2\gamma^\varrho p^\sigma}{2p\cdot k} + \frac{\gamma^\varrho \not k' \gamma^\sigma - 2\gamma^\sigma p^\varrho}{2p\cdot k'}\right]\Big\}$$

(Note the transposition of the Dirac matrices in the second term in square brackets).

Now, write

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = \frac{e^4}{4}\left[\frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathrm{III}}{(2p\cdot k')(2p\cdot k)} + \frac{\mathrm{IV}}{(2p\cdot k')^2}\right]$$

The numerators are traces of Dirac matrices. As a homework problem, evaluate these and express them in terms of the Mandelstam variables. For this purpose, you can use the rules for evaluating Dirac traces, but we recommend to use a computer algebra package, such as FORM.

The answers are

$$\mathrm{I} = 16\left[2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right]$$

$$\mathrm{II} = \mathrm{III} = -8\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right]$$

$$\mathrm{IV} = 16\left[2m^4 + m^2(u-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right]$$

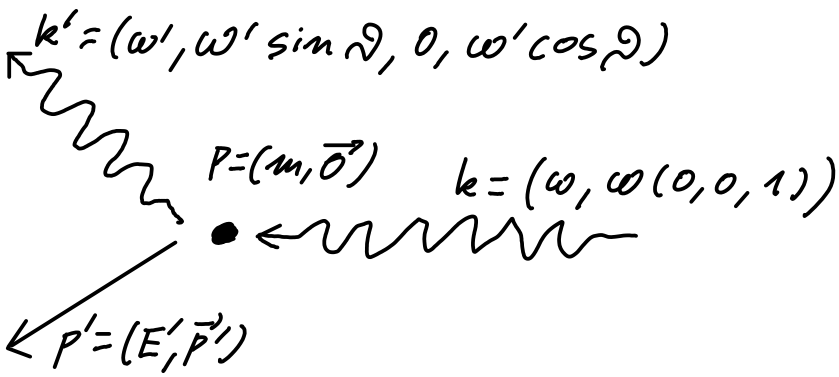
Notice that in order to obtain these, it is sufficient to compute e.g. I & II and then to take $k \leftrightarrow -k'$, what results in $s \leftrightarrow u$.

The final answer is

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

This formula applies to the scattering of photons on effectively free electrons, i.e. when the electron is bound, its binding energy is much less than the photon energy. We recall that the opposite case, the scattering on bound electrons, is called Rayleigh scattering.

We take the electron to be at rest and parametrise the kinematic situation as follows:



We aim to express the differential cross section as a function of ω and ϑ. Compton's formula relates the final photon wavelength to these parameters:

$$m^2 = p'^2 = (p+k-k')^2 = p^2 + 2p\cdot(k-k') - 2k\cdot k'$$

$$= m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\vartheta) \implies \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1-\cos\vartheta}{m}$$

For the photon energy, it follows

$$\omega' = \left(\frac{1}{\omega} + \frac{1-\cos\vartheta}{m} \right)^{-1} = \left(\frac{m + \omega(1-\cos\vartheta)}{\omega m} \right)^{-1} = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\vartheta)}$$

Now, integrate the final phase-space except for $d\Omega$:

$$\int \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega'} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} (2\pi)^4 \delta^4(k'+p'-k-p)$$

$$= \int_0^\infty \frac{\omega'^2 d\omega\, d\Omega}{(2\pi)^3} \frac{1}{4\omega' E'} 2\pi\, \delta\left(\omega' + \sqrt{\omega^2 + \omega'^2 - 2\omega\omega'\cos\vartheta + m^2} - \omega - m\right)$$

$\vec{p}\,' = \vec{k} - \vec{k}'$

$p'^{02} = (\vec{k} - \vec{k}')^2 + m^2 = E'^2$

$$= 2\pi \int_{-1}^{1} d\cos\vartheta \frac{1}{(2\pi)^2} \frac{\omega'}{4E'} \left| 1 + \frac{\omega' - \omega\cos\vartheta}{E'} \right|^{-1}$$

$$= \frac{1}{8\pi} \int_{-1}^{1} d\cos\vartheta \frac{\omega'}{\underbrace{E' + \omega'}_{=\omega+m} - \omega\cos\vartheta} = \frac{1}{8\pi} \int_{-1}^{1} d\cos\vartheta \frac{\omega'}{m + \omega(1-\cos\vartheta)}$$

$$= \frac{1}{8\pi} \int_{-1}^{1} d\cos\vartheta \frac{\omega'^2}{\omega m}$$

Substituting the matrix element and this into the cross-section yields

$$\frac{d\sigma}{d\cos\vartheta} = \frac{1}{4m\omega} \frac{1}{8\pi} \frac{\omega'^2}{\omega m} \frac{1}{4} \sum_{\text{spins}} |i\mathcal{M}|^2$$

Now, $p\cdot k = m\omega$ and $p \cdot k' = m\omega'$. We can substitute

$$\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2$$

$$= \frac{m\omega'}{m\omega} + \frac{m\omega}{m\omega'} + 2m^2\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right) + m^4\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right)^2$$

$$= \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2(1-\cos\vartheta) + (1-\cos\vartheta)^2$$

$$= \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 1 + \cos^2\vartheta = \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\vartheta$$

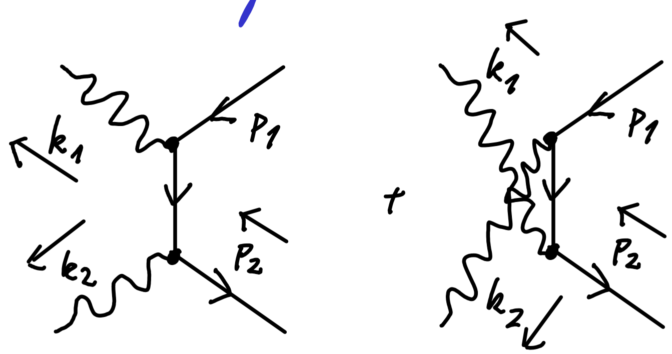
such that we obtain the Klein-Nishina formula

$$\frac{d\sigma}{d\cos\vartheta} = \underbrace{\frac{e^4}{16\pi}}_{=\pi\alpha^2} \frac{1}{m^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\vartheta\right]$$

The non-relativistic limit is when $\omega \to 0$ such that $\omega \to \omega'$. We obtain then the Thomson cross section

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} d\cos\vartheta\,(1+\cos^2\vartheta) = \frac{8\pi\alpha^2}{3m^2}$$

As another example for the use of crossing symmetry, we consider pair annihilation, $e^+e^- \to \gamma\gamma$. The leading-order diagrams are



The amplitude for this process can be obtained from the one for Compton scattering by making the replacements

$$p \mapsto p_1 \qquad k' \mapsto k_1 \qquad p' \mapsto -p_2 \qquad k \mapsto -k_2$$

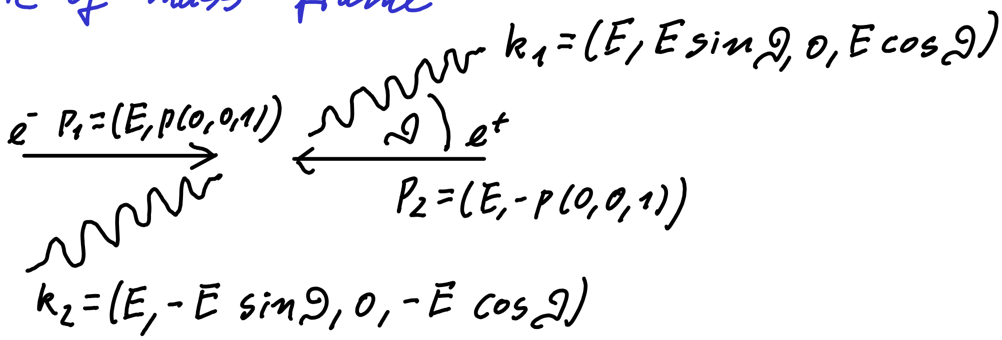
such that

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = 2e^4\left[\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} + 2m^2\left(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\right) - m^4\left(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\right)^2\right]$$

Note the additional minus sign due to the replacement of a fermion by an antifermion.

Recall that pair creation is an important interaction of high energetic ($\omega > 2m$) photons with matter. In above diagrams, one of the photons is then an off-shell photon from the field of an electron or a nucleus.

Pair annihilation is routinely observed in e^+e^- colliders. Therefore, it is useful to parametrise the kinematics in the centre-of-mass frame



$$p_1 \cdot k_1 = E^2 - Ep\cos\vartheta \qquad p_1 \cdot k_2 = E^2 + Ep\cos\vartheta$$

$$\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} = 2\frac{E^4 + E^2p^2\cos^2\vartheta}{E^4 - E^2p^2\cos^2\vartheta} = 2\frac{E^2 + p^2\cos^2\vartheta}{E^2 - p^2 + p^2\sin^2\vartheta} = 2\frac{E^2 + p^2\cos^2\vartheta}{m^2 + p^2\sin^2\vartheta}$$

$$\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2} = \frac{2E^2}{E^4 - E^2p^2\cos^2\vartheta} = \frac{2}{m^2 + p^2\sin^2\vartheta}$$

The phase-space integral is

$$\int \frac{d^3k_1}{(2\pi)^3}\frac{1}{2E}\int\frac{d^3k_2}{(2\pi)^3}\frac{1}{2E}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$$

$$= \frac{1}{4E^2}\int_0^\infty \frac{k_1^2\, dk_1\, d\Omega}{(2\pi)^3}\, 2\pi\, \delta(p_1+p_2-2E) = \frac{1}{8\pi}\int_{-1}^{1} d\cos\vartheta$$

$$v_1 - v_2 = \frac{2p}{E}$$

Substituting these pieces into the cross section yields

$$\frac{d\sigma}{d\cos\vartheta} = \frac{e^4}{32\pi pE}\left[\frac{E^4 + E^2p^2\cos^2\vartheta}{E^4 - E^2p^2\cos^2\vartheta} + \frac{2m^2}{m^2 + p^2\sin^2\vartheta} - \frac{2m^4}{(m^2 + p^2\sin^2\vartheta)^2}\right]$$

For $E \gg m$, this gives

$$\frac{d\sigma}{d\cos\vartheta} = \frac{e^4}{32\pi E^2} \frac{1+\cos^2\vartheta}{\sin^2\vartheta}$$

Finally, notice that since the two photons are identical, in order to obtain the total cross section, $d\cos\vartheta$ should only be integrated from 0 to 1.

2.3 Rutherford & Mott Scattering, Bremsstrahlung

The cross section for the process $e^-\mu^- \longrightarrow e^-\mu^-$

$$\frac{d\sigma_{e^-\mu^-\rightarrow e^-\mu^-}}{d\Omega} = \frac{e^4}{128\pi^2} \frac{1}{\vec{p}^{\,2}} \frac{4+(1+\cos\vartheta)^2}{(1-\cos\vartheta)^2} \quad \text{(ultrarelativistic limit)}$$

diverges when $\vartheta \longrightarrow 0$. Clearly, the interpretation is that all electrons get deflected by the Coulomb potential of the muon (or vice versa). If the distance (or the impact parameter $\vec{b}$) is large, the deflection angle ϑ is however very small. In the analysis of collider data, these small ϑ events correspond to a small transfer of transverse momentum and are discarded. Only events with large ϑ correspond to central collisions and therefore probe the microscopic structure of the colliding particles.
Nonetheless, small ϑ processes are of great importance in Nuclear-, Particle and Astrophysics, because these describe the energy loss of charged particles in matter (e.g. the atmosphere, the rock above an underground detector or scintillation detectors that collect the light caused by atomic recombination when ionising radiation

traverses). The charged particles lose their energy by a large number of small ϑ scattering events with atomic electrons, therefore causing the atoms to ionise. An important question is therefore, how far muons get within matter, or what scintillation signal they may cause within a detector.

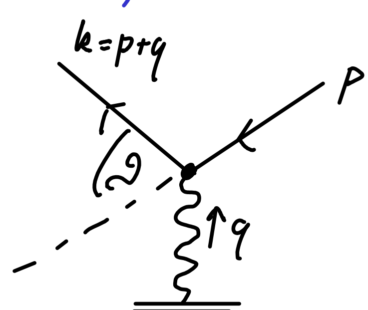
We therefore expand the above scattering cross-section for small ϑ:

$$(1+\cos\vartheta)^2 = 1 + 2\sqrt{1-\sin^2\vartheta} + 1-\sin^2\vartheta \approx 1 + 2\left(1-\frac{1}{2}\sin^2\vartheta\right) + 1-\sin^2\vartheta$$
$$= 4 - 2\sin^2\vartheta \approx 4 - 8\sin^2\frac{\vartheta}{2}$$

$$(1-\cos\vartheta)^2 = 1 - 2\sqrt{1-\sin^2\vartheta} + 1-\sin^2\vartheta$$
$$\approx 1 - 2\left(1-\frac{1}{2}\sin^2\vartheta - \frac{1}{8}\sin^4\vartheta\right) + 1 - \sin^2\vartheta = \frac{1}{4}\sin^4\vartheta$$
$$\approx 4\sin^4\frac{\vartheta}{2}$$

$$\longrightarrow \frac{d\sigma_{e^-\mu^-\to e^-\mu^-}}{d\Omega} \underset{\substack{\uparrow \\ \vartheta \ll 1}}{\approx} \frac{e^4}{128\pi^2}\frac{1}{\vec{p}^{\,2}}\frac{8\left(1-\sin^2\frac{\vartheta}{2}\right)}{4\sin^4\frac{\vartheta}{2}} \underset{\substack{\uparrow \\ \alpha = \frac{e^2}{4\pi}}}{=} \frac{\alpha^2}{4\vec{p}^{\,2}}\frac{1-\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}}$$

If our above reasoning is correct, this simplified formula for small ϑ accounts for the Coulomb potential but not for the spin-interactions (interactions between magnetic dipole fields) of the scattering particles. To check this, we calculate the following process:



The photon is not an external particle here, but part of the Coulomb field of one of the scattering particles.

The Coulomb potential of a point charge Ze is

$$A_0(\vec{x}) = -\frac{Ze}{4\pi|\vec{x}|} \longrightarrow$$

$$A_0(\vec{q}) = -\frac{Ze}{4\pi}\int \frac{d^3x}{|\vec{x}|} e^{-i\vec{q}\cdot\vec{x}} = -\frac{Ze}{4\pi} 2\pi \int_{-1}^{1} d\cos\vartheta \int_0^\infty |\vec{x}|\, d|\vec{x}|\, e^{-i|\vec{q}||\vec{x}|\cos}$$

$$= \frac{Ze}{2}\int_0^\infty d|\vec{x}| \left[\frac{e^{i|\vec{q}||\vec{x}|}}{i|\vec{q}|} - \frac{e^{-i|\vec{q}||\vec{x}|}}{i|\vec{q}|}\right]$$

$$= -\frac{Ze}{|\vec{q}|^2}\left[\cos\left(|\vec{q}||\vec{x}|\right)\right]_0^\infty = \frac{Ze}{\vec{q}^{\,2}} \quad \& \quad \vec{A}(\vec{q}) = \vec{0}$$

(The boundary term for $|\vec{x}| \to \infty$ can be neglected by replacing $e^{-i\vec{q}\cdot\vec{x}} \mapsto e^{i\vec{q}\cdot\vec{x} - \varepsilon|\vec{x}|}$.) The four-momentum is $q \approx (0, \vec{q})$, what corresponds to a negligible energy transfer (elastic scattering). The invariant matrix element is

$$i\mathcal{M} = -\bar{u}(p+q, r)\, ie^2 Z \gamma^0 u(p,s) \frac{1}{\vec{q}^{\,2}}$$

$$\gamma^\mu \gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2} g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} g_{\mu\nu} \mathbb{1}_4 2g^{\mu\nu} = 4\,\mathbb{1}_4$$

$$\not{p}\gamma_\mu = g^{\varrho\nu} p_\varrho \gamma_\nu \gamma_\mu = -\gamma_\mu\gamma_\nu p_\varrho g^{\varrho\nu} + \{\gamma_\mu, \gamma_\nu\} p_\varrho g^{\varrho\nu}$$

$$= -\gamma_\mu\gamma_\nu p_\varrho g^{\varrho\nu} + 2g_{\mu\nu}\mathbb{1}_4 p_\varrho g^{\varrho\nu} = -\gamma_\mu \not{p} + 2p_\mu$$

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{2} e^4 Z^2 \,\mathrm{tr}\left[(\not{p} + \not{q} + m)\gamma^0(\not{p} + m)\gamma_0\right] \frac{1}{|\vec{q}|^4}$$

$$= \frac{1}{2} e^4 Z^2 \,\mathrm{tr}\left[-(\not{p}+\not{q})\not{p} + 2p^0(p^0+q^0) + m^2\right]\frac{1}{|\vec{q}|^4}$$

$$= 2e^2 Z^4\left[2p^0(p^0+q^0) - p\cdot(p+q) + m^2\right]\frac{1}{|\vec{q}|^4}$$

$$\vec{q}^{\,2} = \left[(\vec{p}+\vec{q}) - \vec{p}\right]^2 = \underbrace{(\vec{p}+\vec{q})^2}_{=\vec{p}^{\,2}\text{, energy conservation}} + \vec{p}^{\,2} - 2|\vec{p}||\vec{p}+\vec{q}|\cos\vartheta = 2\vec{p}^{\,2}(1-\cos\vartheta)$$

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$$

$$\longrightarrow \vec{q}^{\,2} = 4\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}$$

$$p\cdot(p+q) = \vec{p}^{\,2} + m^2 - \vec{p}^{\,2}\cos\vartheta = m^2 + 2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2} \Rightarrow \vec{p}\cdot\vec{q} = -2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4Z^2\left(2\vec{p}^{\,2}+2m^2-m^2-2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}+m^2\right)\frac{1}{|\vec{q}|^4}$$

$$\underset{\substack{\uparrow\\ \frac{|\vec{p}|}{|p^0|}=\beta}}{=} e^4Z^2\left(\vec{p}^{\,2}+m^2\right)\left(1-\beta^2\sin^2\frac{\vartheta}{2}\right)\frac{1}{4|\vec{p}|^4\sin^4\frac{\vartheta}{2}}$$

Now dropping the Lorentz-factors for one of the incoming particles in the cross-section formula:

$$\sigma = \frac{1}{v\,2\sqrt{\vec{p}^{\,2}+m^2}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{\vec{k}^{\,2}+m^2}}\,2\pi\,\delta(k^0-p^0)\;\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{1}{4v\left(\vec{p}^{\,2}+m^2\right)}\frac{1}{(2\pi)^2}\int_{-1}^{1}d\cos\vartheta\int|\vec{k}|^2\,dk\,2\pi\,\delta\left(\sqrt{\vec{k}^{\,2}+m^2}-\sqrt{\vec{p}^{\,2}+m^2}\right)\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{1}{8\pi}\int_{-1}^{1}d\cos\vartheta\;\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{\vec{p}^{\,2}+m^2}{32\pi\,|\vec{p}|^4}\,e^4Z^2\int_{-1}^{1}d\cos\vartheta\,\frac{1-\beta^2\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}}$$

$$= \frac{\pi}{\beta^2|\vec{p}|^2}\,\frac{\alpha^2}{2}\,Z^2\int_{-1}^{1}d\cos\vartheta\,\frac{1-\beta^2\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}}$$

We obtain the Mott cross-section:

$$\frac{d\sigma}{d\cos\vartheta} = 2\pi\left(\frac{\alpha Z}{2\beta|\vec{p}|}\right)^2\frac{1-\beta^2\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}} = 2\pi\,\frac{d\sigma}{d\Omega},$$

which is, by the way, the relativistic generalisation of the Rutherford cross-section, that describes, e.g. the scattering of α-particles of gold nuclei, which led to the discovery of the nucleus in the celebrated Geiger-Marsden experiment at Manchester.

Furthermore, we note the agreement with electron-muon scattering at small angles.
In order to calculate the energy loss in a material, we replace ϑ by the momentum transfer $|\vec{q}|$,

$$\frac{d\sigma}{d\cos\vartheta} = 2\pi \left(\frac{\alpha Z}{2\beta|\vec{p}|}\right)^2 \frac{16|\vec{p}|^4}{|\vec{q}|^4} \left(1 - \beta^2 \frac{\vec{q}^{\,2}}{4\vec{p}^{\,2}}\right)$$

$$\frac{d\vec{q}^{\,2}}{d\cos\vartheta} = 4\vec{p}^{\,2} \frac{d\,\frac{1}{2}(1-\cos\vartheta)}{d\cos\vartheta} = -2\vec{p}^{\,2}$$

$$\longrightarrow \frac{d\sigma}{d\vec{q}^{\,2}} = 2 \frac{d\sigma}{d\cos\vartheta} \frac{d\cos\vartheta}{d\vec{q}^{\,2}} = \left(\frac{\alpha Z}{\beta}\right)^2 \frac{8\pi}{|\vec{q}|^4} \left(1-\beta^2 \frac{\vec{q}^{\,2}}{4\vec{p}^{\,2}}\right)$$

$\vec{p}\cdot\vec{q}$ can have either sign

Now in each scattering, an electron in the material picks up the energy

$$\Delta E = \frac{1}{2}\frac{\vec{q}^{\,2}}{m_e} \Rightarrow d\vec{q}^{\,2} = 2m_e\, d\Delta E$$

This is also the energy, that the traversing particle loses.

$$\frac{d\sigma}{2m_e\, d\Delta E} = 8\pi \left(\frac{Z\alpha}{\beta\, 2m_e\Delta E}\right)^2 \left(1-\beta^2 \frac{2m_e\Delta E}{4(m_e\gamma\beta)^2}\right)$$

$$\frac{d\sigma}{d\Delta E} = 16\pi m_e \left(\frac{Z\alpha}{\beta\, 2m_e\Delta E}\right)^2 \left(1 - \frac{\Delta E}{2m_e\gamma^2}\right)$$

$$= \frac{4\pi}{m_e}\left(\frac{Z\alpha}{\beta\Delta E}\right)^2 \left[1 - \frac{\Delta E}{2m_e}(1-\beta^2)\right]$$

Now the energy loss per unit distance is

$$-\frac{dE}{dx} = \varrho_{el} \int_{\Delta E_{min}}^{\Delta E_{max}} \Delta E \frac{d\sigma}{d\Delta E}\, d\Delta E$$

where ϱ_{el} is the number density of electrons

$$-\frac{dE}{dx} = e_{el}\left[\frac{4\pi}{m_e}\left(\frac{Z\alpha}{\beta}\right)^2\left(\log \Delta E - \frac{\Delta E}{2m_e}(1-\beta^2)\right]_{\Delta E_{min}}^{\Delta E_{max}}$$

$$= e_{el}\,\frac{4\pi}{m_e}\left(\frac{Z\alpha}{\beta}\right)^2\left(\log\frac{\Delta E_{max}}{\Delta E_{min}} - \frac{\Delta E_{max}-\Delta E_{min}}{2m_e}(1-\beta)^2\right)$$

The maximum energy transfer is twice the electron energy in the center of mass system minus the rest energy

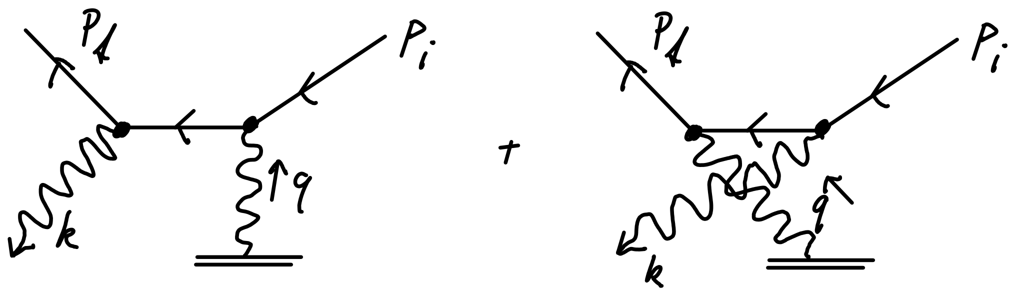
$$2m_e\gamma^2 - 2m_e = 2m_e\frac{1-1+\beta^2}{1-\beta^2} = 2m_e\frac{\beta^2}{1-\beta^2} = 2m_e\gamma^2\beta^2$$

The lower bound is given by the mean excitation potential $I \simeq 10\,eV\,Z$, where Z is the atomic number (not to be confused with the Z that occurs in the remaining formulae and that denotes the charge of the ionising particle). We finally obtain the <u>Bethe formula</u>:

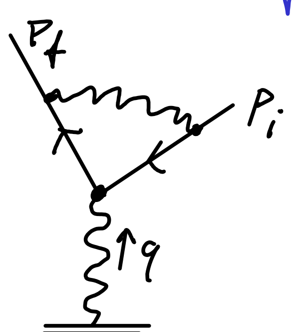
$$-\frac{dE}{dx} = e_{el}\,\frac{4\pi}{m_e}\left(\frac{Z\alpha^2}{\beta}\right)^2\left[\log\frac{2m_e\beta^2}{I(1-\beta^2)} - \beta^2\right]$$

Note that for low velocities, the stopping power sharply increases. A beam of a certain energy therefore loses most of its energy after traveling a certain distance into the material. This makes ionising particle beams interesting for medical reasons.

The Bethe formula applies to particles that are much heavier than electrons, which have a more complicated stopping behaviour. In matter, besides scattering with atomic electrons, they get deflected in the field of nuclei. Due to the large charge of nuclei of large atomic number, electrons can get strongly accelerated, what leads to the emission of Bremsstrahlung typically in the form of X-rays. The diagrams associated with that process are



These diagrams yield well defined answers for the cross section of producing photons above a certain momentum $|\vec{k}|$. However, even for large $|q^2|$, the total cross section (integrating over the full photon phase space d^4k) is divergent. The answer (worked out in the book by Itzykson and Zuber and in the 2012 version of these notes) is that inclusion of



leads to a finite total cross section. This is a consequence of the Bloch-Nordsieck theorem that in turn follows from the more general Kinoshita-Lee-Nauenberg theorem.

Form factors / magnetic moment / EDM, running coupling: check w. Martin's notes.