1. Warmup

We revisit some basic cornerstones of relativistic QFT, in particular those of special relevance for this course and to fix notation & conventions. For more details & proofs, please cf. the lectures & books on QFT.

1.1 Relativistic Fields

Contravariant vector: $x^\mu = (x^0, \vec{x})$ $\quad (\mu = 0, 1, 2, 3)$

Covariant vector: $x_\mu = (x^0, -\vec{x})$

Metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv g^{\mu\nu}$$

A Lorentz transformation Λ leaves $g_{\mu\nu}$ invariant:

$$g_{\mu\nu} \longmapsto \Lambda^\rho{}_\mu \, g_{\rho\sigma} \, \Lambda^\sigma{}_\nu = [\Lambda^T g \Lambda]_{\mu\nu} = g_{\mu\nu}$$

$\Longrightarrow$ The Λ form the group $SO(3,1)$

Raising & lowering of indices:

$$x_\mu = g_{\mu\nu} x^\nu$$
$$x^\mu = g^{\mu\nu} x_\nu$$

Boost in 1-direction

$$\Lambda = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

φ: rapidity $\qquad \beta = \frac{v}{c}$

$$\sinh\varphi = \beta\gamma \qquad \cosh\varphi = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

Rotation around 3-axis: $$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Contravariant vectors transform as:

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

Covariant vectors as

$$x_\mu \longmapsto x'_\mu = \Lambda_\mu{}^\nu x_\nu = x_\nu (\Lambda^{-1})^\nu{}_\mu$$

exercise: show

Contractions such as $v_\mu w^\mu = \sum_{\mu=0}^{4} v_\mu w^\mu = v \cdot w$ are Lorentz invariant.

Note: Can build tensors with multiple indices, e.g. $F^{\mu\nu}$ (field strength), $T^{\mu\nu}$ (stress-energy)

Poincaré Group:

Lorentz transformations & space-time translations

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$\longrightarrow$ a 10 parameter group (3 boosts, 3 rotations, 4 space-time translations).

Lie-Algebra of the Poincaré Group

$$\Lambda^\mu{}_\nu = e^{-\frac{i}{2}\omega_{\rho\sigma}\left[J^{\rho\sigma}\right]^\mu{}_\nu}$$

$$a^\mu = e^{-i\varepsilon_\rho P^\rho} x^\mu$$

Generators:

$$P^\mu = i\partial^\mu$$

$$[J^{\rho\sigma}]^\mu{}_\nu = i\left[g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu\right]$$

Rearrangement

$$\vec{P} = \begin{pmatrix} P^1 \\ P^2 \\ P^3 \end{pmatrix}, \quad H = P^0, \quad \vec{J} = \begin{pmatrix} J^{23} \\ J^{31} \\ J^{12} \end{pmatrix}, \quad \vec{K} = \begin{pmatrix} J^{10} \\ J^{20} \\ J^{30} \end{pmatrix}$$

Parameters translation: $\varepsilon^\mu = a^\mu$

boost ψ in i-direction: $\omega_{i0} = -\omega_{0i} = \psi$

rotation φ around k-axis: $\omega_{ij} = -\omega_{ji} = \varepsilon_{ijk}\varphi$

Lie-Algebra of the Lorentz group:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\sigma\mu}J^{\rho\nu} + g^{\sigma\nu}J^{\rho\mu}\right)}$$

In above rearrangement

Lie-Algebra of the Poincaré-Group

$$[J^i, J^j] = i\,\varepsilon_{ijk}J^k$$

$$[J^i, K^j] = i\,\varepsilon_{ijk}K^k$$

$$[K^i, K^j] = -i\,\varepsilon_{ijk}J^k$$

$$[P^i, J^j] = i\,\varepsilon_{ijk}P^k$$

$$[P^i, K^j] = -iH\delta_{ij}$$

$$[J^i, H] = [P^i, H] = [H, H] = 0$$

$$[K^i, H] = iP^i$$

A different basis

$$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}), \qquad \vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$$

$\longrightarrow$

$$[A^i, A^j] = i\varepsilon^{ijk} A^k$$
$$[B^i, B^j] = i\varepsilon^{ijk} B^k$$
$$[A^i, B^j] = 0$$

Eigenvalues:

$$A(A+1)\mathbb{1} = \vec{A}^2$$
$$B(B+1)\mathbb{1} = \vec{B}^2$$

$\longrightarrow$ Representations (B, A)

$$\underset{SU(2)_L}{B}, \quad \underset{SU(2)_R}{A}$$

Parity

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\vec{J} \longrightarrow P\vec{J}P = \vec{J}$$
$$\vec{K} \longrightarrow P\vec{K}P = -\vec{K}$$

such that

$$\vec{A} \longrightarrow P\vec{A}P = \vec{B}$$
$$\vec{B} \longrightarrow P\vec{B}P = \vec{A}$$

Outlook: QED & QCD are parity conserving but not the weak interactions. Key characteristic of the Standard Model.

Chiral Spinors

Consider right handed $(0, \frac{1}{2})$

$$\longrightarrow \quad \vec{A} = \frac{\vec{\sigma}}{2}, \quad \vec{B} = 0$$

$$J^{i0} = K^i = -i(\vec{A} - \vec{B}) = -\frac{i}{2}\vec{\sigma}$$

$$J^{ij} = \varepsilon^{ijk} J^k = \varepsilon^{ijk}(A^k + B^k) = \frac{1}{2}\varepsilon^{ijk}\sigma^k$$

$\longrightarrow$ Define

$$\sigma^\mu = (\mathbb{1}, \sigma^i), \quad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$$

$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$$

$$\longrightarrow \quad J^{\mu\nu} = \bar{\sigma}^{\mu\nu}, \quad D_{(0,\frac{1}{2})}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}$$

Note This representation acts on complex two-component vectors called left-handed spinors:

$$\xi \longmapsto \xi' = D_{(0,\frac{1}{2})}(\Lambda)\,\xi$$

$(\frac{1}{2}, 0)$ representation: left-handed spinors

$$\vec{A} = 0, \quad \vec{B} = \frac{\vec{\sigma}}{2}$$

$\longrightarrow$

$$J^{i0} = i\frac{\sigma^i}{2}$$

$$J^{ij} = \frac{1}{2}\varepsilon_{ijk}\sigma^k \quad \Rightarrow \quad J^{\mu\nu} = \sigma^{\mu\nu}$$

$\longrightarrow$

$$D_{(\frac{1}{2},0)}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}$$

Note:

$$D^\dagger_{(\frac{1}{2},0)} = D^{-1}_{(0,\frac{1}{2})}$$

In analogy with the Lorentz indices, it is useful to introduce upper and lower spinorial indices that may take the values 1,2. Undotted indices are left-handed, dotted ones are right-handed. On the Pauli-matrices, these appear as follows:

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}, \quad \bar{\sigma}^{\mu\nu\,\dot{\alpha}}{}_{\dot{\beta}}, \quad \sigma^{\mu}{}_{\alpha\dot{\beta}}, \quad \bar{\sigma}^{\mu\dot{\alpha}\beta}$$

The indices are contracted according to ${}^{\alpha}{}_{\alpha}$ and ${}_{\dot{\alpha}}{}^{\dot{\alpha}}$.
We define $\varepsilon^{12} = -\varepsilon^{21} = \varepsilon_{21} = -\varepsilon_{12} = 1$ where the indices may be either dotted or undotted. This tensor is invariant:

$$\varepsilon^{\alpha\gamma} D_{(\frac{1}{2},0)\alpha}{}^{\delta} D_{(\frac{1}{2},0)\gamma}{}^{\beta} = \varepsilon^{\delta\beta}$$

and accordingly for lower and dotted indices.

Just as for $g_{\mu\nu}$ for Minkowski indices, we can therefore employ ε for raising & lowering spinor indices.
Since left- and right handed spinors are related by complex conjugation, we adopt the convention of considering all spinors as left-handed. I.e. ξ_α transforms with $D_{(\frac{1}{2},0)}$ and $(\xi^\dagger)_{\dot{\beta}}\, \varepsilon^{\dot{\beta}\dot{\alpha}}$ with $D_{(0,\frac{1}{2})}$.

Without going into further details, we notice that we can construct all representations of the Lorentz-group from Tensor products, for example

$$\underbrace{\left(\tfrac{1}{2},0\right) \otimes \left(0,\tfrac{1}{2}\right)}_{\text{spinors, } 2\times 2 \text{ dof.}} = \underbrace{\left(\tfrac{1}{2},\tfrac{1}{2}\right)}_{\text{vector } V^\mu,\ 4 \text{ d.o.f.}} \longrightarrow \text{cf. } \sigma^\mu{}_{\alpha\dot\beta} \ \& \ \bar\sigma^{\mu\dot\alpha\beta}$$

or

$$\left(0,\tfrac{1}{2}\right) \otimes \left(0,\tfrac{1}{2}\right) = \underbrace{(0,0)}_{\text{scalar}} \oplus \underbrace{(0,1)}_{F^{\mu\nu}} \longrightarrow \text{cf. } \sigma^{\mu\nu}{}_\alpha{}^\beta \ \& \ \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}$$

$\oplus \to$ direct sum/product

etc.

Since QED & QCD are chirally invariant we construct the Dirac representation as the direct sum of a left and a right-handed spinor: $\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)$
The representation matrices take the block-diagonal form:

$$D_{\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)} = \begin{pmatrix} D_{\left(\frac{1}{2},0\right)} & 0 \\ 0 & D_{\left(0,\frac{1}{2}\right)} \end{pmatrix}$$

It is useful to introduce the Dirac-matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

and to define

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}$$

such that

$$D_{\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$$

Above representation of the Dirac-matrices is called the chiral (or Weyl) representation. Other representations can be obtained by similarity transformations
$\gamma^\mu \longmapsto S\gamma^\mu S^{-1}$.

The Dirac-matrices satisfy a so-called Clifford-Algebra

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\mathbb{1}\,g^{\mu\nu}.$$

Basic examples for fields of different spin

Klein-Gordon Equation

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow (\partial^2 + m^2)\phi = 0$$

Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi^A} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)} = 0$$

Dirac Equation

Define the Dirac spinor $\psi = \begin{pmatrix} \xi_\alpha \\ \chi^{+\dot\alpha} \end{pmatrix}$

and $\overline{\psi} = \psi^+\gamma^0 = (\xi^+_{\dot\beta} \quad \chi^\beta)\begin{pmatrix} 0 & \delta^{\dot\beta}{}_{\dot\alpha} \\ \delta_\beta{}^\alpha & 0 \end{pmatrix} = (\chi^\alpha \quad \xi^+_{\dot\alpha})$

Lorentz invariants up to 2nd order in $\psi, \overline{\psi}$:

$$\overline{\psi}\psi = \chi^\alpha\xi_\alpha + \xi^+_{\dot\alpha}\chi^{+\dot\alpha}$$

$$\overline{\psi}\gamma^\mu\partial_\mu\psi = \xi^+_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\xi_\alpha + \chi^\beta\sigma^\mu{}_{\beta\dot\beta}\partial_\mu\chi^{+\dot\beta}$$

Use the notation: $\not{V} = V_\mu\gamma^\mu = V^\mu\gamma_\mu$

$$\mathcal{L} = \overline{\psi}(i\not\partial - m)\psi$$

$$\Rightarrow (i\not\partial - m)\psi = 0$$

Majorana Equation

$$\mathcal{L} = \xi^{\dagger}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} i \partial_\mu \xi_\alpha - \frac{1}{2} m \xi^\alpha \xi_\alpha - \frac{1}{2} m \xi^{\dagger}_{\dot{\alpha}} \xi^{\dagger\dot{\alpha}}$$

$$\Rightarrow \quad i \partial_\mu \bar{\sigma}^\mu \xi - m (\underbrace{\varepsilon \xi}_{\varepsilon^{\alpha\beta}\xi_\beta})^{\dagger} = 0$$

Notice that the same physical field can be described with Dirac-spinors when imposing the condition $\xi = \chi$.

Proca-Equation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\rho A_\rho$$

$$\frac{\partial}{\partial A^\sigma} F^{\mu\nu} F_{\mu\nu} = 0$$

$$\frac{\partial}{\partial(\partial^\rho A^\sigma)} F^{\mu\nu} F_{\mu\nu} = 2 \left(F_{\rho\sigma} - F_{\sigma\rho} \right) = 4 F_{\rho\sigma}$$

$$\Rightarrow \quad \partial^\rho F_{\rho\sigma} - m^2 A_\sigma = 0$$

$$\partial^2 A_\sigma - \partial_\sigma \partial^\rho A_\rho - m^2 A_\sigma = 0$$

Notice that for $m=0$, this equation as well as the Lagrangian are gauge invariant, in the sense that the transformation

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu \alpha(x)$$

has no effect.

1.2 Reaction rates

Recall that reaction rates can be derived from n-point time ordered Green functions (as shown via the LSZ reduction formula).

Operator formulation (based on canonical quantization)

Example: 2-point function

interacting vacuum

interaction picture operators

$$iG(x,y) = \langle \Omega | T\, \phi_H(x) \phi_H(y) | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T\, \phi(x)\phi(y)\, e^{-i\int_{-T}^{T} d\tau\, H_I(\tau)} | 0 \rangle}{\langle 0 | T\, e^{-i\int_{-T}^{T} d\tau\, H_I(\tau)} | 0 \rangle}$$

free vacuum

Heisenberg field operators

Can be generalized straightforwardly to n points & more than one field.

Expand numerator using Wick's theorem $\longrightarrow$ Feynman rules (see QFT)

Functional formulation (based on path integral quantization)

Useful object: the generating functional

$$Z[J] = \lim_{T \to \infty} \langle \Omega | T\, e^{-i\int_{-T}^{T} d\tau \left(H(\tau) - \int d^3x\, J(\vec{x},\tau)\phi(\vec{x},\tau) \right)} | \Omega \rangle$$

Path integral representation:

$$Z[J] = N \int D\phi\, e^{i\int d^4x \{ \mathcal{L}(x) + J(x)\phi(x) \}}$$

The normalisation $\mathcal{N}$ is chosen such that $Z[J(x)\equiv 0]=1$.

Time-ordered n-point Green functions are computed as

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J(x)=0}$$

$$= \mathcal{N}\int \mathcal{D}\phi\; \phi(x_1)\cdots\phi(x_n)\, e^{i\int d^4x\,\{\mathcal{L}(x)+J(x)\phi(x)\}}\bigg|_{J(x)=0}$$

For definiteness, take $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

$$= \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4$$

Completing the square, one obtains after a straightforward computation ($\to$ QFT)

$$\frac{Z[J]}{\mathcal{N}} = \int \mathcal{D}\phi'\, e^{i\int d^4x\left[\frac{1}{2}\phi'(-\partial^2-m^2)\phi' - \frac{\lambda}{4!}\phi^4\right] - \frac{1}{2}\int d^4x\, d^4y\, J(x)\, i\Delta^F(x,y)\, J(y)}$$

$\uparrow$ no prime here!

With

$$[i\phi(y)]^n\, e^{i\int d^4x\,(\mathcal{L}_0 + J(x)\phi(x))} = \frac{\delta^n}{\delta J^n(y)}\, e^{i\int d^4x\,(\mathcal{L}_0+J(x)\phi(x))}$$

it follows:

$$\frac{Z[J]}{\mathcal{N}} = e^{-i\int d^4x\,\frac{\lambda}{4!}\frac{\delta^4}{\delta J^4(x)}}\int \mathcal{D}\phi\, e^{i\int d^4x\left[\frac{1}{2}\phi(-\partial^2-m^2)\phi + J(x)\phi(x)\right]}$$

$$= e^{-i\int d^4x\,\frac{\lambda}{4!}\frac{\delta^4}{\delta J^4(x)}}\underbrace{\int \mathcal{D}\phi\, e^{i\int d^4x\,\frac{1}{2}\phi(-\partial^2-m^2)\phi - \frac{1}{2}\int d^4x\, d^4y\, J(x)\, i\Delta^F(x,y)\, J(y)}}_{=:\,\mathcal{N}^{-1}Z_0[J]}$$

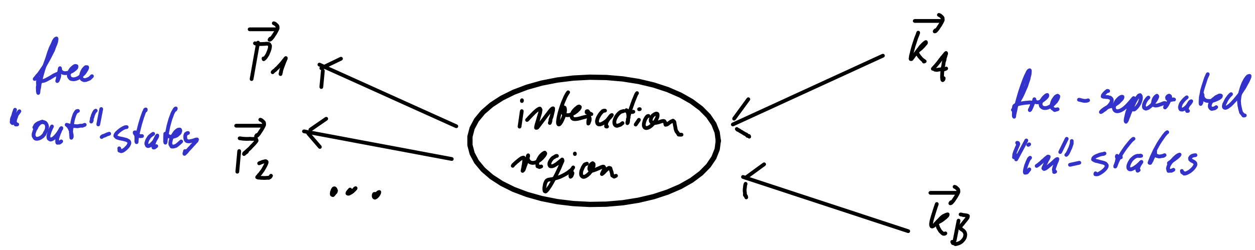
$$i\Delta^F(x_1,x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \iff i\Delta^F(x_1,x_2) = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\varepsilon}$$

$$\Longrightarrow (-\partial^2_{x_1}-m^2)\, i\Delta^F(x_1,x_2) = i\delta^4(x_1-x_2)$$

The remaining path integral is Gaußian and can be evaluated in terms of a functional determinant. Often (as for computations of S-matrix elements) this is not necessary because it amounts to a normalization factor.
Analogous expressions exist for spin $\frac{1}{2}$ and spin 1 fields.

1.3 Scattering Processes

Reactions of highly energetic particles in colliders, in astrophysical or cosmological contexts can often be described as scatterings



The probability of a scattering event is described by a unitary operator, the S-matrix:

$$_{out}\langle \vec{p}_1 \vec{p}_2 \cdots | \vec{k}_A \vec{k}_B \rangle_{in} = \langle \underbrace{\vec{p}_1 \vec{p}_2 \cdots}_{f} | S | \underbrace{\vec{k}_A \vec{k}_B}_{i} \rangle$$

i.e. the probability that the initial states i scatter to the final states f is given by

$$P_{fi} = S^{+}_{if} S_{fi}$$

The unitarity guarantees probability conservation in the form

$$\sum_f S^\dagger_{if} S_{fk} = \delta_{ik}$$

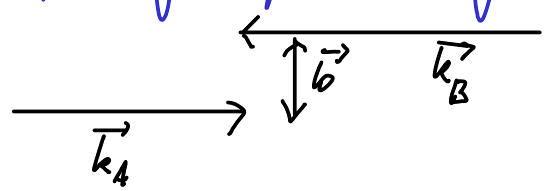
In order to connect this to observables such as cross sections and decay rates, it is useful to define the <u>T-matrix</u>

$$S = \mathbb{1} + i\, T$$

and the <u>invariant matrix-element</u>

$$\langle \vec{p}_1 \vec{p}_2 \cdots | i T | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \, \delta^{(4)}\Big(k_A + k_B - \sum_i p_i\Big)\, i \mathcal{M}(\vec{k}_A, \vec{k}_B \longrightarrow \vec{p}_1 \vec{p}_2 \ldots)$$

Notice that we also encounter a momentum-conserving δ-function when computing n-point functions.
It is useful (and it reflects the situation in collider experiments) to work in the <u>collinear frame</u>, where $\vec{k}_A \parallel \vec{k}_B$.
Nonetheless, the collisions are in general not head-on, but they are <u>spatially</u> separated by the <u>impact parameter</u> $\vec{b}$:



An intuitive definition of the <u>cross section</u> is

$$\sigma = \int d^2 b \;\; \mathcal{P}(\vec{b})$$

, i.e. it is the effective area either of the particles "sees" in the particular interaction.

(d^2b:) impact parameter

($\mathcal{P}(\vec{b})$:) probability of a scattering of particles at impact parameter $\vec{b}$

This can be shown to be equivalent to the more common definition of σ as the ratio of the transition rate over the flux. Next, we need to find an expression for $P(\vec{b})$. For that purpose, we note that the in states of the particles A and B are not momentum eigenstates, but rather wave-packets centered around certain momenta $\vec{p}$.

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{k})}} \Psi(\vec{k}) |\vec{k}\rangle$$

$\hookrightarrow$ function, that describes the momentum-distribution of the wave-packet, centered around $\vec{p}$

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')$$

$$\sqrt{2\omega(\vec{p})}\, a^\dagger(\vec{p})|0\rangle = |\vec{p}\rangle$$

The proper normalisation follows as

$$\langle\phi|\phi\rangle = \int \frac{d^3k\, d^3k'}{(2\pi)^6} \frac{\Psi^*(\vec{k}')\Psi(\vec{k})}{2\sqrt{\omega(\vec{k})\omega(\vec{k}')}} \langle 0|\sqrt{2\omega(\vec{k}')}\, a(\vec{k}')\sqrt{2\omega(\vec{k})}\, a^\dagger(\vec{k})|0\rangle$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^6} \Psi^*(\vec{k}')\Psi(\vec{k})(2\pi)^3\delta^3(\vec{k}-\vec{k}') = \int \frac{d^3k}{(2\pi)^3} |\Psi(\vec{k})|^2 \overset{!}{=} 1$$

Check particle number

$$\int \frac{d^3q}{(2\pi)^3} \langle\phi|a^\dagger(\vec{q})a(\vec{q})|\phi\rangle = \int \frac{d^3q\, d^3k\, d^3k'}{(2\pi)^9} (2\pi)^3\delta^3(\vec{k}'-\vec{q})(2\pi)^3\delta^3(\vec{k}-\vec{q}) * \Psi^*(\vec{k}')\Psi(\vec{k})$$

$$= \int \frac{d^3q}{(2\pi)^3} |\Psi(\vec{k})|^2 = 1$$

The probability that we are after is:

$$P = \left| {}_{out}\langle \vec{p}_1 \vec{p}_2 \dots | \phi_A \phi_B \rangle_{in} \right|^2$$

Making use of translation invariance, we write

$$|\phi_A \phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{\Psi_A(\vec{k}_A)\Psi_B(\vec{k}_B)\, e^{-i\vec{b}\cdot\vec{k}_B}}{2\sqrt{\omega(\vec{k}_A)\,\omega(\vec{k}_B)}} |\vec{k}_A \vec{k}_B\rangle_{in}$$

i.e. the function Ψ_B in the transversal direction ($\parallel \vec{b}$) is the Fourier transform of the spatial distribution of the particle B orthogonal to the collision axis (which we take to be the z-axis, for definiteness.

Putting everything together, we obtain for the differential cross-section

$$d\sigma = \prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\,\omega(\vec{p}_f)}\int d^2b\left(\prod_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\varphi_i(\vec{k}_i)}{\sqrt{2\omega(\vec{k}_i)}}\int\frac{d^3k_i'}{(2\pi)^3}\frac{\varphi_i^*(\vec{k}_i')}{\sqrt{2\omega(\vec{k}_i')}}\right)$$

$$* \, e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}\left({}_{out}\langle\{p_f\}|\vec{k}_A\vec{k}_B\rangle_{in}\right)\left({}_{out}\langle\{p_f\}|\vec{k}_A'\vec{k}_B'\rangle_{in}\right)^*$$

$$= \prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\,\omega(\vec{p}_f)}\left(\prod_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\varphi_i(\vec{k}_i)}{\sqrt{2\omega(\vec{k}_i)}}\int\frac{d^3k_i'}{(2\pi)^3}\frac{\varphi_i^*(\vec{k}_i')}{\sqrt{2\omega(\vec{k}_i')}}\right)$$

$$* \,(2\pi)^2\delta^2(\vec{k}_B^{\perp}-\vec{k}_B'^{\perp})\,(2\pi)^4\delta^4\Big(\sum_f p_f-k_A-k_B\Big)\,(2\pi)^4\delta^4\Big(\sum_f p_f-k_A'-k_B'\Big)$$

$$* \; i\mathcal{M}(\vec{k}_A\vec{k}_B\to\vec{p}_1\vec{p}_2\ldots)\,(-i)\mathcal{M}^*(\vec{k}_A'\vec{k}_B'\to\vec{p}_1\vec{p}_2\ldots)$$

$$\underset{\substack{\vec{k}_B'^{\perp}=\vec{k}_B^{\perp}\\ \vec{k}_A'^{\perp}=\sum_f\vec{p}_f^{\perp}-\vec{k}_B'^{\perp}}}{=} \prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\,\omega(\vec{p}_f)}\left(\prod_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\varphi_i(\vec{k}_i)}{\sqrt{2\omega(\vec{k}_i)}}\int\frac{dk_i'^z}{2\pi}\frac{\varphi_i^*(\vec{k}_i')}{\sqrt{2\omega(\vec{k}_i')}}\right)$$

$$* \,(2\pi)^4\delta^4\Big(\sum_f p_f-k_A'-k_B'\Big)\,2\pi\,\delta\Big(\sum_f\omega_f-\omega_A'-\omega_B'\Big)$$

$$* \,2\pi\,\delta\Big(\sum_f p_f^z-k_A^z-k_B^z\Big)$$

$$* \; i\mathcal{M}(\vec{k}_A\vec{k}_B\to\vec{p}_1\vec{p}_2\ldots)\,(-i)\mathcal{M}^*(\vec{k}_A'\vec{k}_B'\to\vec{p}_1\vec{p}_2\ldots)$$

Now,

$$\int\frac{dk_A'^z\,dk_B'^z}{(2\pi)^2}\,2\pi\,\delta\Big(\sum_f\omega_f-\omega_A'-\omega_B'\Big)\,2\pi\,\delta\Big(\sum_f p_f^z-k_A'^z-k_B'^z\Big)$$

$$\underset{k_B^z=\sum_f p_f^z-k_A^z}{=}\int\frac{dk_A'^z}{2\pi}\,2\pi\,\delta\Big(\sqrt{\vec{k}_A'^2+m_A^2}+\sqrt{\vec{k}_B'^2+m_B^2}-\sum_f\omega_f\Big)=\frac{1}{\left|\frac{k_A'^z}{\omega_A}-\frac{k_B'^z}{\omega_B}\right|}=\frac{1}{|v_A-v_B|}$$

Furthermore, recall that we assume that Ψ_A and Ψ_B are narrowly centered around $\vec{p}_A$ and $\vec{p}_B$. It is therefore good enough to evaluate $\mathcal{M}$ and v_i for these momenta. We obtain the result

$$d\sigma = \frac{(2\pi)^4 \delta^4(\sum_f p_f - k_A - k_B)}{2\omega_A 2\omega_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2\omega_f} \right) |\mathcal{M}(\vec{k}_A \vec{k}_B \to \vec{p}_1 \vec{p}_2 \ldots)|^2$$

A similar derivation can be performed for the case of the <u>decay rate</u> Γ of a single decaying particle

$$d\Gamma = \frac{1}{2\omega_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2\omega_f} \right) |\mathcal{M}(\vec{p}_A \to \vec{p}_1 \vec{p}_2 \ldots)|^2 (2\pi)^4 \delta^4(p_A - \sum_f p_f)$$

It remains to link the S matrix element, and hence also the T matrix

$$(2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_i p_i)\, i\mathcal{M}(\vec{k}_A, \vec{k}_B \to \vec{p}_1 \vec{p}_2 \ldots)$$

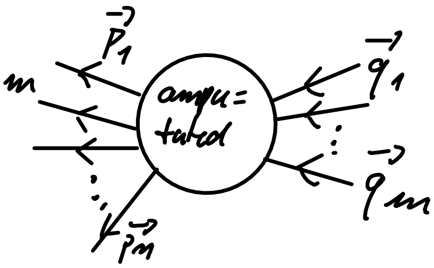
to n-point Green functions, i.e. Feynman diagrams. The desired relation is the <u>LSZ reduction formula</u>

$$\prod_{i=1}^{n} \int d^4x_i\, e^{i p_i \cdot x_i} \prod_{j=1}^{m} d^4 y_j\, e^{-i q_j \cdot y_j} \langle \Omega | T[\phi(x_1) \cdots \phi(x_n) \phi(y_1) \cdots \phi(y_m)] | \Omega \rangle$$

$$= \left(\prod_{i=1}^{n} \frac{\sqrt{Z}\, i}{p_i^2 - m^2 + i\varepsilon} \right) \left(\frac{\sqrt{Z}\, i}{q_j^2 - m^2 + i\varepsilon} \right) \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{q}_1 \cdots \vec{q}_m \rangle$$

We now recall that in the perturbative expressions for the n-point functions, the external legs are represented by propagators. These are now "amputated" by the factors $\frac{1}{\sqrt{Z}}(q_i^2 - m^2 + i\varepsilon)$, where $\sqrt{Z}$ is the field strength renormalization.

$\longrightarrow$

$$\langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{q}_1 \cdots \vec{q}_m \rangle = (\sqrt{Z})^{n+m}$$ 

This cartoon version of the LSZ formula applies to all particles (in particular, to spin $\frac{1}{2}$-fermions and to gauge bosons as well).

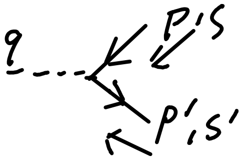
1.4 Example: Production of a Resonance

All known particles heavier than the proton (and many of the lighter ones) are unstable. While it is possible to produce these in scattering processes involving many particles, the "golden" channel for a discovery is their production in a $2 \longrightarrow 1$ process, where the invariant mass square of the incoming particle pair equals the mass of the heavier particle. We will see shortly why this situation is referred to as a "resonance".

Standard Model interactions encompass interferences as well as some tedious numerator algebra. So instead, consider the toy model

$$\mathcal{L} = \overline{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \sum_i y_i \overline{\psi}_i \phi \psi_i$$

The field ϕ may represent a vector boson or a Higgs boson, i.e. it serves as a model for an unstable particle. Now consider the cross section for $\overline{\psi}_i \psi_i \longrightarrow \phi$



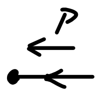
$$\left.\begin{aligned} p &= (p^0, \vec{p}) \\ p' &= (p^0, -\vec{p}) \end{aligned}\right\} \text{CMS} \qquad p^2 = p'^2 = 0$$

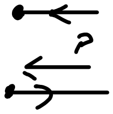
$$i\mathcal{M} = -i y_i \, \bar{v}(p',s') \, u(p,s) \qquad |i\mathcal{M}|^2 = |y_i|^2 \, \bar{v}_\alpha u_\alpha \bar{u}_\beta v_\beta = [u\bar{u}]_{\alpha\beta} [v\bar{v}]_{\beta\alpha}$$

$$\frac{1}{4}\sum_{s,s'} |i\mathcal{M}|^2 = \frac{1}{4} y_i^2 \underbrace{\mathrm{tr}[\not p \not p']}_{= 4 \not p \not p' = 4p^{0\,2} + 4\vec{p}^{\,2} = 4\left(\frac{m^2}{4} + \frac{m^2}{4}\right) = 2m^2} = 2 y_i^2 \, p^{0\,2} \leftarrow p^0 = \frac{m}{2}$$

$$\sigma = \frac{1}{8p^{0\,2}} \int \frac{d^4q}{(2\pi)^4} \, 2\pi \, \delta(q^2 - m^2) \, (2\pi)^4 \delta^4(p + p' - q) \, 2 y_i^2 \, p^{0\,2}$$

We have made use here of the following Feynman rules (cf. Peskim & Schroeder on Yukawa theory):

 (incoming fermion) $\qquad u(p,s)$

 (incoming antifermion) $\qquad \bar{v}(p,s)$

Further, we have averaged over the incoming polarizations, as we will review when considering the QED processes.

$$\sum_s u(p,s)\bar{u}(p,s) = \not p + m_f \qquad \sum_s v(p,s)\bar{v}(p,s) = \not p - m_f$$

In the cross section, we find one δ-function too many in order to give rise to a well defined & finite cross section.

On the other hand, we know that since ϕ has a finite life-time only, there should be no sharp on-shell δ-function due to the uncertainty relation. Hence, we expect that within the correct expression, we should replace the on-shell δ-function by some finite-width representation, when the width is given by the life time of ϕ. This is what we derive in the following.

The decay rate of ϕ is given by

$$\Gamma_\phi(\vec q) = \frac{m^2}{\omega_\phi} \sum_i g_i^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{4|\vec p||\vec p'|} (2\pi)^4 \delta^4(q-p-p')$$

Note that this expression is valid as well for $\vec q \neq 0$, and we have $q^0 = \omega_\phi = \sqrt{\vec q^{\,2} + m^2}$, $p^0 = |\vec p|$, $p'^0 = |\vec p'|$.

Now, we can evaluate

$$\Gamma_\phi(\vec q) = \frac{m^2}{\omega_\phi} \sum_i g_i^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{4p^0 p'^0} 2\pi\, \delta(q^0 - p^0 - p^{0\prime})$$

$$\vec p' = \vec q - \vec p \iff p'^0 = \sqrt{\vec q^{\,2} + \vec p^{\,2} - 2|\vec q||\vec p|\cos\vartheta}$$

$$q^0 - |\vec p| - \sqrt{\vec q^{\,2} + \vec p^{\,2} - 2|\vec q||\vec p|\cos\vartheta} = 0$$

$$q^{0\,2} - 2q^0|\vec p| + \vec p^{\,2} = \vec q^{\,2} + \vec p^{\,2} - 2|\vec q||\vec p|\cos\vartheta$$

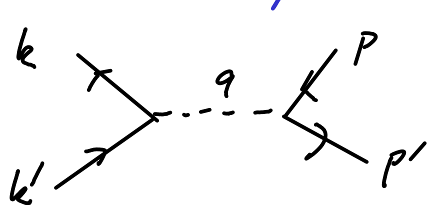
$$\cos\vartheta = -\frac{q^2 - 2q^0|\vec p|}{2|\vec q||\vec p|}$$

$$\cos\vartheta < 1 \Rightarrow -q^2 < -2|\vec p|(q^0 - |\vec q|) \iff |\vec p| < \frac{1}{2}(q^0 + |\vec q|)$$

$$\cos\vartheta > -1 \Rightarrow -q^2 > -2|\vec p|(q^0 + |\vec q|) \iff |\vec p| > \frac{1}{2}(q^0 - |\vec q|)$$

$$\Gamma_\phi(\vec q) = \frac{m^2}{q^0} \sum_i g_i^2 \int_{\frac{1}{2}(q^0 - |\vec q|)}^{\frac{1}{2}(q^0 + |\vec q|)} \cancel{\vec p^{\,2}}\, d|\vec p|\, \frac{1}{2\pi} \frac{1}{4\cancel{p^0}\cancel{p'^0}} \frac{\cancel{p'^0}}{|\vec q|\cancel{|\vec p|}} = \frac{m^2 \sum_i g_i^2}{8\pi q^0}$$

Consider next the squared matrix element for $\overline{\psi_i}\, \psi_i \longrightarrow \overline{\psi_j}\, \psi_j$, i.e. the diagram



From the scalar propagator, $|i\mathcal{M}|^2$ would contain a factor

$$\frac{1}{q^2-m^2+i\varepsilon}\,\frac{1}{q^2-m^2-i\varepsilon} = \frac{1}{[q^2-m^2]^2+\varepsilon^2}$$

$$\left(\text{Note: } \frac{\varepsilon}{[q^2-m^2]^2+\varepsilon^2} = \pi\,\delta(q^2-m^2)\right)$$

This is ill defined for $q^2=m^2$, but the problem is fixed because being unstable, ϕ has a self-energy with an imaginary part. We assume that the the real part of the self-energy is cancelled by appropriate counterterms for $q^2=m^2$.

The imaginary part can be obtained by evaluating the self-energy introducing Feynman parameters and performing a Wick rotation. In the Feynman parameter integrals, there will be logarithms with negative argument and therefore, there will result an imaginary part. In the present context, it is however more instructive to go directly for the imaginary part without performing a Wick rotation.

$$i\Pi(q) = \sum_i g^2 \,\mathrm{tr} \int \frac{d^4p}{(2\pi)^4}\,\frac{i\not{p}}{p^2+i\varepsilon}\,\frac{i(\not{q}-\not{p})}{(q-p)^2+i\varepsilon}$$

$$\rightsquigarrow \mathrm{Im}\,\Pi(q) = 2\sum_i g^2 \int \frac{d^4p}{(2\pi)^4}\, p\cdot(q-p)\left[\frac{1}{p^2+i\varepsilon}\,\frac{1}{(q-p)^2+i\varepsilon} - \frac{1}{p^2-i\varepsilon}\,\frac{1}{(q-p)^2-i\varepsilon}\right]$$

$$\frac{2}{x\pm i\varepsilon} = \frac{1}{x\pm i\varepsilon} + \frac{1}{x\mp i\varepsilon} + \frac{1}{x\pm i\varepsilon} - \frac{1}{x\mp i\varepsilon} = 2\,PV\frac{1}{x} \mp 2\pi i\,\delta(x)$$

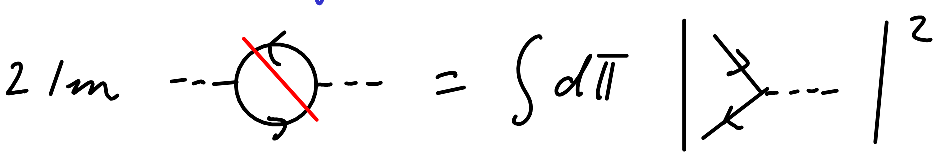
$$\mathrm{Im}\,\Pi(q) = 2\sum_i g_i^2 \int \frac{d^4p}{(2\pi)^4}\, p\cdot(q-p) \quad 2\pi\,\delta(p^2)\, 2\pi\,\delta((q-p)^2)$$

$$= q^2 \sum_i g_i^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{4p^0p'^0} (2\pi)^4 \delta^4(q-p-p') = \omega_\phi \Gamma_\phi$$

The replacement of the propagators by the δ-functions can be justified when considering two subsequent integrations using the residue theorem. A simpler and more powerful method derives from the so called Closed-Time-Path formalism, which we however not discuss in these lectures.

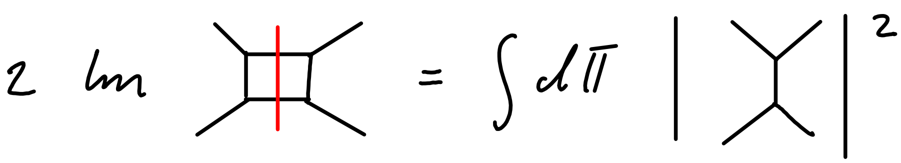
Now recall that the decay rate was $\frac{1}{2\omega_\phi}$ times the the phase space integral $(d\Pi)$ over the $1 \to 2$ diagram. The present calculation can be generalised to the optical theorem.

For our present example, it can be stated diagrammatically that

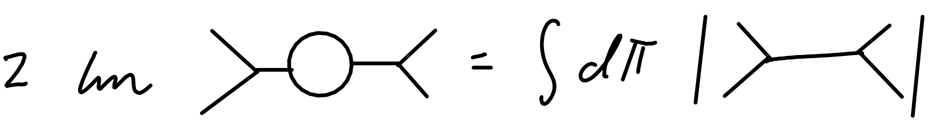


Where the red line indicates an on-shell cut.

To give another example, it can also be applied to



tt interference



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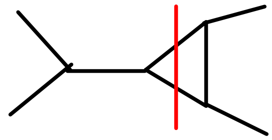
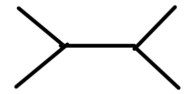
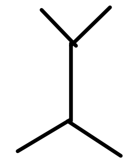
$2\,\mathrm{Im}$  $= \int d\Pi$  $*$  * st interference

as well as to other possibilities.

Now back to the process $\psi_i \overline{\psi_i} \to \psi_j \overline{\psi_j}$

$$i\mathcal{M}_{\psi_i\overline{\psi_i}\to\psi_j\overline{\psi_j}} = -\,y_i\,\overline{v}(p',s')\,u(p,s)\,\frac{i}{(p+p')^2-m^2+i\,\mathrm{Im}[\Pi(p+p')]}\,y_j\,\overline{u}(k,r)\,v(k',r')$$

$$\frac{1}{4}\sum_{\substack{s,s',\\ r,r'\\ j}} \left|i\mathcal{M}_{\psi_i\overline{\psi_i}\to\psi_j\overline{\psi_j}}\right|^2 = y_i^2\sum_j y_j^2\,\frac{(p+p')^4}{[(p+p')^2-m^2]^2+[\mathrm{Im}\,\Pi(p+p')]^2}$$

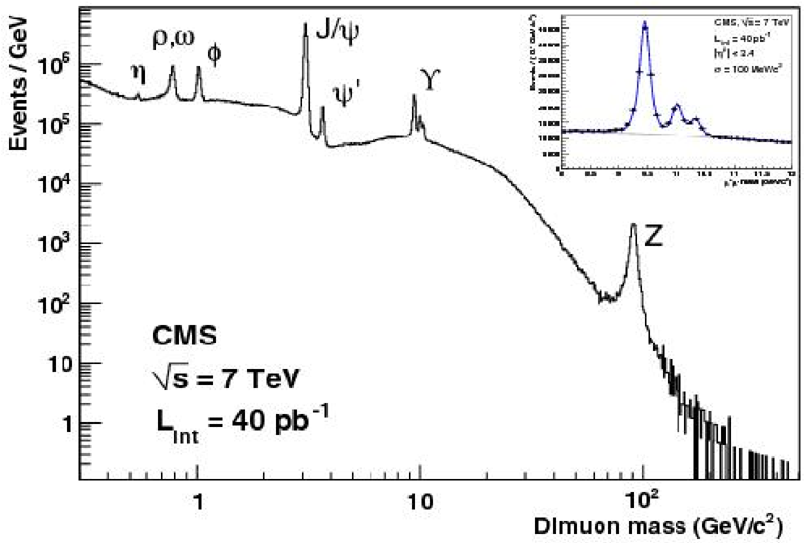
$$p\cdot p' = \frac{1}{2}(p+p')^2$$

$$\mathrm{Im}\,\Pi(q) = q^2\sum_i y_i^2\int\frac{d^3p}{(2\pi)^3}\,\frac{d^3p'}{(2\pi)^3}\,\frac{1}{4p^0p'^0}\,(2\pi)^4\,\delta^4(q-p-p')$$

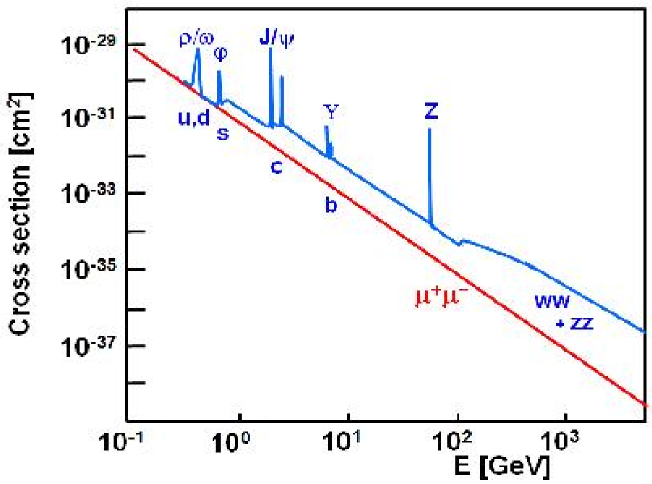
$$\sigma = \frac{y_i^2\sum_j y_j^2}{2p^0\,2p'^0\,2}\int\frac{d^3k}{(2\pi)^3}\,\frac{d^3k'}{(2\pi)^3}\,\frac{(p+p')^4}{2k^0\,2k'^0}\,\frac{(2\pi)^4\delta^4(p+p'-k-k')}{[(p+p')^2-m^2]^2+[\mathrm{Im}\,\Pi(p+p')]^2}$$

$$= \frac{y_i^2}{8p^0p'^0}\,(p+p')^2\,\frac{\mathrm{Im}\,\Pi(p+p')}{[(p+p')^2-m^2]^2+[\mathrm{Im}\,\Pi(p+p')]^2}$$

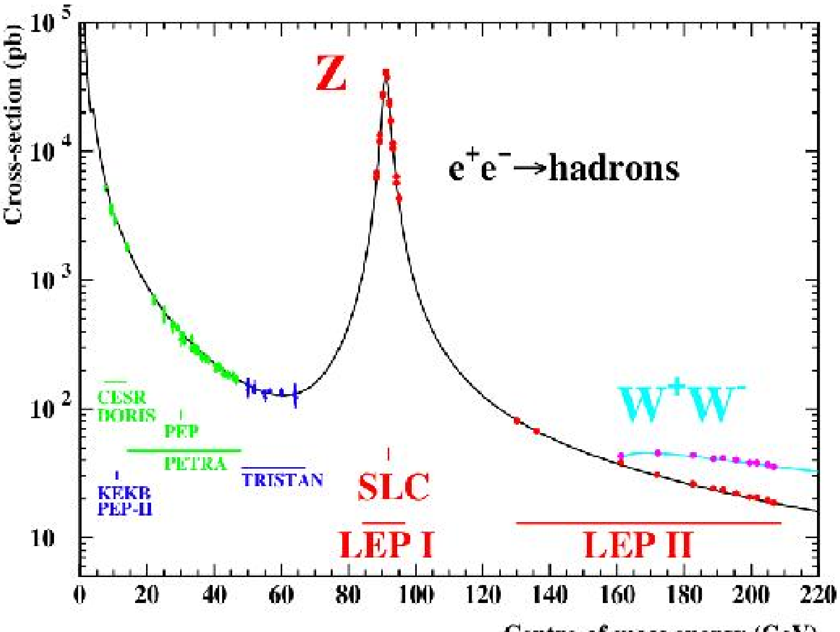
When we recall the representation $\delta(x) = \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$ we observe that indeed this formula generalises the standard cross section to an expression suitable for unstable particles by a replacement of the on-shell δ-function by a finite-width distribution associated with the decay rate of the particle. The behaviour of the cross section near "resonances" is thus quantitatively explained.



Final state
Event count at LHC



Spectrum seen at lepton colliders



Detail of spectrum

The included figures exhibit many of the Standard Model resonances (i.e. particles). These may correspond to fundamental (e.g. the Z boson) or composite particles.

Apparently, most of the visible resonances (except for the η) have spin 1. The spin 1 mesons have a longer lifetime because we will see that they decay mostly into three gluons, compared to the η, η_c, η_b that can decay into two gluons. They are therefore much broader (32 MeV for η_c compared to 93 keV for J/ψ) such that their peaks are not visible. Mesons that consist of quarks and antiquarks of different flavours decay either through weak interactions (heavy mesons) or the chiral anomaly (neutral pion) such that they do not appear as resonances in e^+e^- annihilation.