3. Quantum Chromodynamics - QCD

3.1 QCD Lagrangian

We have seen the emergence of QED from the local invariance with respect to $U(1)$-phase rotation. This can be generalised to to other continuous groups and indeed for $SU(2)$ and $SU(3)$, this is realised within the Standard Model. While $SU(2)$ is spontaneously broken at low energies (we can distinguish electrons & neutrinos, up and down quarks), $SU(3)$ remains as an unbroken symmetry. The associated gauge theory is called QCD. Since $U(1)$ is an Abelian group while the other continuous groups (other than the trivial one and those isomorphic to $U(1)$) are non-Abelian, one refers to the associated gauge theories as non-Abelian gauge theories. The crucial new feature is that the gauge sector is non-linear (gauge bosons interact with each other at tree-level). Moreover, the β-function may be negative, which means that the theory becomes <u>asymptotically free</u> at high energies.

Let us assume that the generators of a Lie group are given by t^a. The group is characterised by the <u>Lie algebra</u>

$$[t^a, t^b] = i f^{abc} t^c$$

where f^{abc} are the <u>structure constants.</u> One may choose a basis for t^a such that f^{abc} is totally antisymmetric. Recall that we have encountered a Lie-algebra and its structure constants already for the Lorentz group, which is not gauged, however. For $SU(2)$, $t^i = \frac{\sigma^i}{2}$ and $f^{abc} = \varepsilon^{abc}$.

Now, we consider the transformation

$$\psi(x) \longmapsto e^{i\alpha^a(x)t^a}\psi(x)$$

In order to construct a covariant derivative, define a matrix-valued function with $U(x,x)=1$ and that transforms as

$$U(x,y) \longmapsto e^{i\alpha^a(x)t^a} U(x,y) e^{-i\alpha^a(y)t^a}$$

This operator undoes the effect of different gauge transformation parameters at different space-time points. We use it to define the covariant derivative as

$$n^\mu D_\mu \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)\right]$$

Now consider a small gauge transformation α:

$$U(x+\varepsilon n, x) = \mathbb{1} + ig\,\varepsilon n^\mu t^a A^a_\mu(x) + \mathcal{O}(\varepsilon^2)$$

$$\longmapsto \mathbb{1} + ig\,\varepsilon n^\mu t^a\left[A^a_\mu(x) + \Delta A^a_\mu(x)\right] + \mathcal{O}(\varepsilon^2)$$

$$\overset{!}{=} e^{i\alpha^a(x+\varepsilon n)t^a}\left[\mathbb{1} + ig\,\varepsilon n^\mu t^b A^b_\mu(x) + \mathcal{O}(\varepsilon^2)\right] e^{-i\alpha^c(x)t^c}$$

$$= \left[\mathbb{1} + i\alpha^a(x)t^a + i\varepsilon \frac{\partial \alpha^a(x)}{\partial x^\mu} n^\mu t^a \mathcal{O}(\varepsilon^2, \alpha^2)\right]$$

$$* \left[\mathbb{1} + ig\,\varepsilon n^\mu t^b A^b_\mu(x) + \mathcal{O}(\varepsilon^2)\right] * \left[\mathbb{1} - i\alpha^c(x)\, t^c + \mathcal{O}(\alpha^2)\right]$$

$$\Longrightarrow ig\,\Delta A^a_\mu(x) t^a = i\frac{\partial \alpha^a(x)}{\partial x^\mu} t^a - g\,\alpha^a(x)\left[t^a, t^b\right] A^b_\mu(x)$$

(note that we sum over double indices in each term)

$$\Longrightarrow A^a_\mu(x)\, t^a \longmapsto \left[A^a_\mu(x) + \Delta A^a_\mu(x)\right] t^a$$

$$= A^a_\mu(x) t^a + \frac{1}{g}\frac{\partial \alpha^a(x)}{\partial x^\mu} t^a - f^{abc}\alpha^a(x) A^b_\mu(x)\, t^c$$

$$\Longrightarrow \quad A_\mu^a(x) \longmapsto A_\mu^a(x) + \frac{1}{g}\frac{\partial \alpha^a(x)}{\partial x^\mu} + f^{abc} A_\mu^b(x)\, \alpha^c(x)$$

When α is finite, we can more formally write:

$$ig\,\varepsilon n^\mu t^a \left[A_\mu^a(x) + \Delta A_\mu^a(x)\right]$$

$$= e^{i\alpha^a(x) t^a}\left[ig\,\varepsilon n^\mu t^b A_\mu^b(x) + \mathcal{O}(\varepsilon^2)\right] e^{-i\alpha^c(x) t^c}$$

$$- e^{i\alpha(x) t^a}\, \varepsilon n^\mu \frac{\partial}{\partial x^\mu} e^{-i\alpha^c(x) t^c}$$

$$\Longrightarrow \quad A_\mu^a t^a \longmapsto e^{i\alpha^a(x) t^a}\left(A_\mu^b t^b + \frac{i}{g}\frac{\partial}{\partial x^\mu}\right) e^{-i\alpha^c(x) t^c}$$

With these results, the covariant derivative is given by

$$D_\mu = \partial_\mu - ig A_\mu^a t^a$$

In analogy with QED, we define the field strength tensor as

$$F_{\mu\nu}^a\, t^a = \frac{i}{g}\left[D_\mu, D_\nu\right] = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) t^a + g f^{abc} A_\mu^b A_\nu^c\, t^a$$

Note that this quantity is not gauge-invariant, as it transforms (for infinitesimal α)

$$F_{\mu\nu}^a t^a \longmapsto F_{\mu\nu}^a t^a + \frac{1}{g}\cancel{\partial_\mu \frac{\partial \alpha^a(x)}{\partial x^\nu}}\, t^a - \frac{1}{g}\cancel{\partial_\nu \frac{\partial \alpha^a(x)}{\partial x^\mu}}\, t^a$$

$$+ \partial_\mu f^{abc} A_\nu^b\, \alpha^c(x)\, t^a - \partial_\nu f^{abc} A_\mu^b(x)\, \alpha^c(x)\, t^a$$

$$+ g f^{abc}\left[\underline{\frac{1}{g}\frac{\partial \alpha^b}{\partial x^\mu}} + \underline{f^{bde} A_\mu^d(x)\, \alpha^e(x)}\right] A_\nu^c\, t^a$$

$$+ g f^{abc} A_\mu^b \left[\underline{\frac{1}{g}\frac{\partial \alpha^c}{\partial x^\nu}} + \underline{f^{cfg} A_\nu^f(x)\, \alpha^g(x)}\right] t^a$$

$$f^{abc} f^{bde} A_\mu^d \alpha^e A_\nu^c + f^{abc} f^{cfg} A_\mu^b \alpha^g A_\nu^f$$

$$= f^{aeg} f^{efb} A_\mu^f \alpha^b A_\nu^g + f^{afc} f^{cgb} A_\mu^f \alpha^b A_\nu^g$$

$$= - f^{abc} \alpha^b f^{cfg} A_\mu^f A_\nu^g$$

Where we have made use of the Jacobi-identity:

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

$$-f^{aed} f^{bcd} + f^{bed} f^{acd} - f^{ced} f^{abd} = 0$$

$$\Longrightarrow -f^{age} f^{bfe} - f^{afc} f^{bgc} = + f^{abc} f^{gfc}$$

$$\Longrightarrow F^a_{\mu\nu} \longmapsto F^a_{\mu\nu} - f^{abc} \alpha^b F^c_{\mu\nu}$$

For totally anti-symmetric structure constants however, we easily see that $F^a_{\mu\nu} F^{a\,\mu\nu}$ is gauge invariant.

The fundamental representation of $SU(3)$ is given by unitary 3×3 matrices with unit determinant. These can be generated by exponentiating i times Hermitian, traceless 3×3 matrices. A basis of these matrices is eight dimensional (in general, for $SU(N)$, N^2-1 dimensional). We also choose the convention that $\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$. The standard choice for QCD is $t^a = \lambda^a$, with λ^a being the <u>Gell-Mann matrices</u>

$$\lambda^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The gauge particles of QCD, that are quanta of the fields A^a_μ $(a=1,\dots,8)$, are called <u>gluons</u>.

Strongly interacting fields (i.e. fields that interact with gluons) can live in any non-trivial representation of $SU(3)$. A representation is any set of operators that satisfy the same multiplication table as the original group.

Each representation can be defined by eight representation matrices that satisfy the Lie Algebra. The quarks of QCD are Dirac fermions that live in the fundamental representation, which is obviously three-dimensional. We summarise our consideration within the

QCD Lagrangian

$$\mathcal{L} = \overline{\psi}\,(i\not{D})\,\psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - m\,\overline{\psi}\psi$$

There are several flavours of quarks $f = \{u, d, c, s, t, b\}$, In the above Lagrangian, we have suppressed the flavour indices. We will make the flavour indices explicit whenever this is notationally convenient.

Before stating the Feynman rules, we should make two remarks. First, unlike for QED, the term $\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$ is not quadratic in A^a_μ, i.e. it contains interactions of three and four gauge bosons. It is an essential feature of non-Abelian gauge theories, that the pure (gauge fields only) gauge sector is self-interacting already at tree-level. Second, which is related to the first remark, the gauge coupling g is the same for all strongly interacting particles. This is because due to the non-linear term in the gauge transformation of A^a_μ, different values of g for different fields cannot be compensated by redefinitions of $\alpha(x)$. A natural and rather elegant explanation for the fractional integer values for the electric charges of elementary particles ($q(e^-) = -1$, $q(u) = \frac{1}{3}$, $q(d) = -\frac{2}{3}$) would be if the $U(1)$ symmetry of QED was emerging from a higher non-Abelian gauge group.

3.2 Feynman Rules for QCD

Our Feynman rules for QED are based on the so called Gupta-Bleuler method: After gauge fixing, canonical quantisation is performed and a propagator obtained. The unphysical, longitudinally polarised, negative norm states are then weeded out by hand. This works, because provided all incoming states within a Feynman graph are physical, the same applies to the outgoing ones.

For QCD, it turns out that this is no longer true when simply adapting the Feynman rules of QED: negative norm states may be produced. Within the functional formalism, the gauge fixing of the quantum theory can be performed more systematically by means of the Faddeev-Popov method. It turns out that the effects from the unphysical polarisations are cancelled by unphysical, fermionic but spin-0 "ghost fields". Neither the unphysical polarisations nor the ghosts will appear as external particles. While ghosts in principle are present in QED as well, they decouple due to the Abelian nature of the theory. In QCD, they couple to gluons and may appear in loops due to the self-interacting nature of the gauge sector.

We encourage to study the Faddeev-Popov method. However, as it is covered in the lectures on Quantum Field Theory and as the focus of the present lectures should be less theoretical, we omit their derivation and directly present the gauge-fixed in fundamental representation

in adjoint representation

Lagrangian of QCD:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) + \overline{\psi}(i\not{D} - m)\psi + \overline{c}\,(-\partial^\mu D_\mu)\,c$$

The anti-commuting scalar field c lives in the adjoint representation (the same that the gluons live in). Hence, there are eight ghosts c^a, $a = 1, \dots, 8$.

As defined above, the covariant derivative is valid for any representation. For the adjoint representation, it is possible to express the representation matrices in terms of the structure constants:

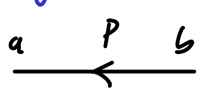
$$[t^b_{adj}]_{ac} = i f^{abc}$$

such that

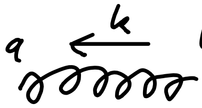
$$D_\mu = \partial_\mu \mathbb{1}_{8\times 8} - ig A^b_\mu [t^b_{adj}] \implies D^{ab}_\mu = \mathbb{1}\,\delta^{ab} + g A^c_\mu f^{acb}$$

We can now state the

Feynman Rules for QCD:

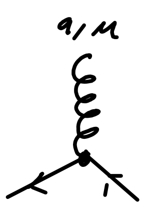


$= \dfrac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}\,\delta^{ab}$ quark propagator

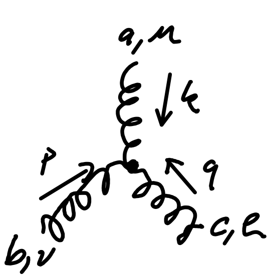


$= \dfrac{-i}{k^2 + i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\dfrac{k_\mu k_\nu}{k^2}\right)\delta^{ab}$ gluon propagator

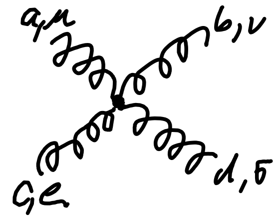
$= \dfrac{i}{k^2 + i\varepsilon}\,\delta^{ab}$ ghost propagator



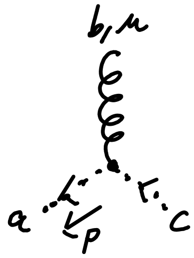
$= i g^\mu \lambda^a$



$= g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$



$$= -ig^2 \left[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right) \right]$$

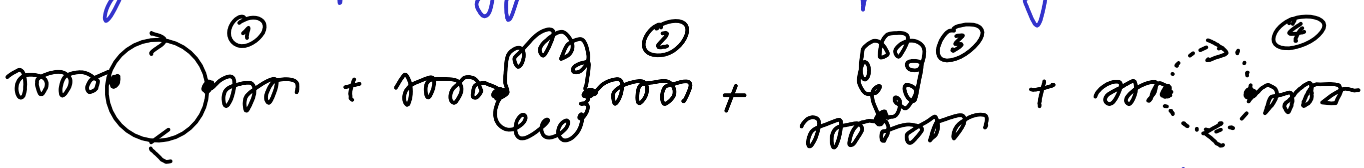


$$= -g f^{abc} p^\mu$$

3.3 The β-Function of QCD – Asymptotic Freedom

We now determine the one-loop corrections to QCD-gauge boson self-energy, quark self-energy & the vertex correction. The goal is to determine how the coupling strength varies with the energy scale of the interaction.

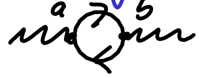
The gluon self-energy receives the following contributions:



Before we start, we remark that the Ward-Takahashi identities generalise in the non-Abelian case to the so-called Slavov-Taylor identities. We do not discuss these intricate identities here, but note without proof that the Ward identity

$$k_\mu \, i\Pi^{ab\mu\nu}(k) = 0$$

still holds. In the following, we will see this explicitly.

For the first diagram, , we remember an old friend. It is the same diagram as in QED, except for a few proportionality factors, that we now explain.

The SU(3) colour algebra gives

$$[\lambda^a]_{ef}\,\delta_{fg}\,[\lambda^b]_{gh}\,\delta_{he} = \mathrm{tr}[\lambda^a\lambda^b]$$

from the fermion propagators

from the vertices

Traces over representation matrices t^a (here in the concrete form λ^a) are frequently encountered within non-Abelian gauge theories. It is therefore useful to treat these in a bit more generality. For a given representation R, the quadratic Casimir operator is defined as

$$t^a_R t^a_R = C_2(R)\,\mathbb{1}_{d(R)\times d(R)}$$

where $d(R)$ is the dimension of the representation (i.e. the dimension of the vector space that the t^a_R are acting on. For $R=F$ the fundamental representation of $SU(N)$, $d(F)=N$, for the adjoint representation $R=G$, $d(G)=N^2-1$).

It is also customary to identify the representations by their dimension, such that $F\equiv N$ and $G\equiv N^2-1$ for $SU(N)$.

In the trace of the self-energy diagram, there appears

$$\mathrm{tr}[t^a_R t^b_R] = C(R)\,\delta^{ab}$$

(The generators are chosen by convention in such a way that this trace is proportional to δ^{ab}, and it may be proved that indeed, it is possible to do so).

Recall that $a,b=1,\dots,d(G)$. Therefore, there is the relation

$$d(R)\,C_2(R) = d(G)\,C(R)$$

For $SU(N)$,

$$C(N)=\frac{1}{2},\ C_2(N)=\frac{N^2-1}{2N},\ C(N^2-1)=C_2(N^2-1)=N,\ f^{acd}f^{bcd}=C_2(G)\,\delta^{ab}.$$

Finally, we account for the fact that n_f flavours of quarks may propagate within the loop, which gives a simple factor. The vacuum polarisation diagram therefore generalises to

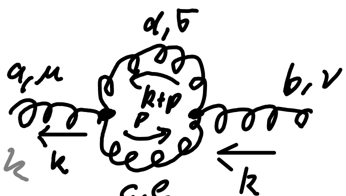
$$i\Pi^{(1)ab}_{\mu\nu}(k) = -\mathrm{tr}[\lambda^a\lambda^b](g_{\mu\nu}k^2 - k_\mu k_\nu)\frac{ig^2 n_f}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\Delta_\varepsilon - \log\frac{m^2 - x(1-x)k^2}{\mu^2}\right]$$

$$\Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi$$

We are interested here in the scale-dependence of the strong coupling constant. For this purpose, we only need to keep the divergent contributions and the dependence on the scale μ, such that

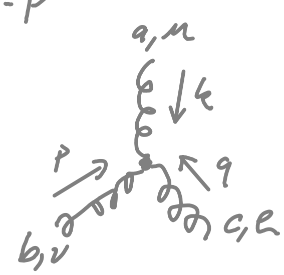
$$i\Pi^{(1)ab}_{\mu\nu}(k) = i(g_{\mu\nu}k^2 - k_\mu k_\nu)\delta^{ab}\left[-\frac{g^2}{(4\pi)^2}\frac{4}{3}n_f C(R)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \dots\right)\right]$$

where # stands for a dimensionful constant that is irrelevant for the determination of the β-function.
For the remaining contributions to the gluon self-energy, we quote the expressions for the Feynman diagrams and the final result for the μ-dependence.



$$= i\Pi^{(2)ab}_{\mu\nu}(k) = \frac{1}{2}\mu^\varepsilon g^2\int\frac{d^dp}{(2\pi)^d}\frac{-i}{p^2+i\varepsilon}\frac{-i}{(p+k)^2+i\varepsilon} f^{acd} f^{bcd} N^{\mu\nu}$$

$q \to -k$
$p \to -p$



$$= g f^{abc}\left[g^{\mu\nu}(k-p)^e + g^{\nu e}(p-q)^\mu + g^{e\mu}(q-k)^\nu\right]$$

$$N^{\mu\nu} = \left[-g^{\mu e}(k-p)^\sigma - g^{e\sigma}(2p+k)^\mu + g^{\sigma\mu}(2k+p)^\nu\right]$$

$$*\left[\delta^\nu{}_e(k-p)_\sigma + g_{e\sigma}(2p+k)^\nu - \delta_\sigma{}^\nu(2k+p)_e\right]$$

$k \to -k$ $\quad \mu \to \mu$ $\quad a \to a$
$p \to -p$ $\quad \nu \to e$ $\quad b \to c$
$q \to k+p$ $\quad e \to \sigma$ $\quad c \to d$

$k \to k$ $\quad \mu \to \nu$ $\quad a \to b$
$p \to p$ $\quad \nu \to e$ $\quad b \to c$
$q \to -k-p$ $\quad e \to \sigma$ $\quad c \to d$

Introducing a Feynman parameter

$$\frac{1}{p^2+i\varepsilon}\,\frac{1}{(p+k)^2+i\varepsilon} = \int_0^1 dx\, \frac{1}{\left[(1-x)p^2 + x\,(p+k)^2 + i\varepsilon\right]^2}$$

and performing a Wick rotation, one obtains

$$i\Pi^{(2)\,ab}_{\mu\nu}(k) = i\,\frac{g^2}{(4\pi)^{\frac{d}{2}}}\,\mu^{\varepsilon}\, C_2(G)\,\delta^{ab}\int_0^1 dx\,\frac{1}{\left[-x(1-x)k^2+i\varepsilon\right]^{2-\frac{d}{2}}}$$

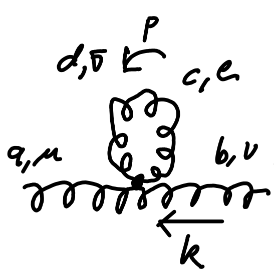
$$* \Big(\Gamma\left(1-\tfrac{d}{2}\right) g^{\mu\nu} k^2\left[\tfrac{3}{2}(d-1)\,x(1-x)\right]$$

$$+\Gamma\left(2-\tfrac{d}{2}\right) g^{\mu\nu} k^2\left[\tfrac{1}{2}(2-x)^2 + \tfrac{1}{2}(1+x)^2\right]$$

$$-\Gamma\left(2-\tfrac{d}{2}\right) k^{\mu}k^{\nu}\left[\left(1-\tfrac{d}{2}\right)(1-2x)^2 + (1+x)(2-x)\right]\Big)$$

Notice that the first term is singular for $d \to 2$, due to a quadratic divergence. However, after cancellation with additional loops, we will find below that just like QED, QCD is free of quadratic divergences.

The next diagram is



$$= i\Pi^{(3)\,ab}_{\mu\nu} = \frac{1}{2}\,\mu^{\varepsilon}\int \frac{d^dp}{(2\pi)^d}\,\frac{-i\,g_{e\sigma}}{p^2+i\varepsilon}\,\delta^{cd}\,(-i g^2)$$

$$* \left[f^{abe} f^{cde}\left(g^{\mu e} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu e}\right) + f^{ace} f^{bde}\left(g^{\mu\nu} g^{e\sigma} - g^{\mu\sigma} g^{\nu e}\right) + f^{ade} f^{bce}\left(g^{\mu\nu} g^{e\sigma} - g^{\mu e} g^{\nu\sigma}\right)\right]$$

The first term vanishes due to the antisymmetry of f^{cde}. The last two simplify using the identity $f^{acd} f^{bcd} = C_2(G)\,\delta^{ab}$, such that

$$i\Pi^{(3)\,ab}_{\mu\nu} = -\mu^{\varepsilon} g^2 C_2(G)\,\delta^{ab}\int \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+i\varepsilon}\, g^{\mu\nu}(d-1)$$

Recall that

$$I_{r,m} = \int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^r}{[l^2 - \Delta + i\varepsilon]^m}$$

$$= i(-1)^{r-m} \frac{1}{(4\pi)^{2-\frac{\varepsilon}{2}}} \left(\frac{1}{\Delta}\right)^{-m+r+2-\frac{\varepsilon}{2}} \frac{\Gamma(r+2-\frac{\varepsilon}{2})\Gamma(m-r-2+\frac{\varepsilon}{2})}{\Gamma(2-\frac{\varepsilon}{2})\Gamma(m)}$$

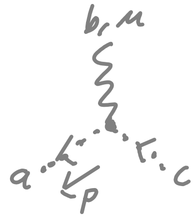
Above integral is therefore $\quad x^{\varepsilon} = e^{\varepsilon \log x} \approx 1+\varepsilon \log x$

$$(d-1) I_{0,1} = -i \frac{1}{(4\pi)^d} (0)^{-1+\frac{d}{2}} \Gamma(1-\frac{d}{2})$$

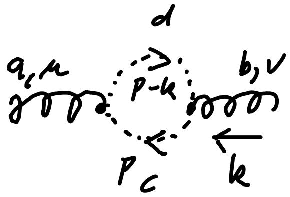
This vanishes for $d \to 4$, but is singular for $d=2$. We therefore need to account for this diagram in order to demonstrate the cancellation of the quadratic divergences. In order to make this diagram to match the form of the previous one in terms of the Feynman parameter, we multiply the integrand by $1 = \frac{(p+k)^2}{(p+k)^2}$. This way, one finds

$$i\Pi^{(3)\,ab}_{\mu\nu}(k) = \frac{i g^2 \mu^{\varepsilon}}{(4\pi)^{\frac{d}{2}}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{[-x(1-x)k^2 + i\varepsilon]^{2-\frac{d}{2}}}$$

$$* \left(-\Gamma(1-\tfrac{d}{2})\, g^{\mu\nu} k^2 \left[\tfrac{1}{2} d(d-1)\, x(1-x)\right] - \Gamma(2-\tfrac{d}{2})\, g^{\mu\nu} k^2 \left[(d-1)(1-x)^2\right]\right)$$

The ghost loop is 

$= -g f^{abc} p^{\mu}$



$$= i\Pi^{(4)}_{\mu\nu}(k) = -\mu^{\varepsilon} g^2 \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 + i\varepsilon} \frac{i}{(p-k)^2 + i\varepsilon}$$

from the ghost loop

$$* f^{dac} (p-k)^{\mu} f^{cbd} p^{\nu}$$

$$= \frac{i\mu^{\varepsilon} g^2}{(4\pi)^{\frac{d}{2}}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{\left[-x(1-x)k^2 + i\varepsilon\right]^{2-\frac{d}{2}}}$$

$$* \left(-\Gamma\left(1-\frac{d}{2}\right) g^{\mu\nu} k^2 \left[\frac{1}{2} x(1-x)\right] + \Gamma\left(2-\frac{d}{2}\right) k^{\mu} k^{\nu} \left[x(1-x)\right] \right)$$

The coefficient of the quadratic divergence $\Gamma(1-\frac{d}{2}) g^{\mu\nu} k^2 x(1-x)$ from the diagrams ②, ③, ④ is

$$\underbrace{\frac{3}{2}(d-1)}_{②} \underbrace{-\frac{1}{2} d(d-1)}_{③} \underbrace{-\frac{1}{2}}_{④} = -\frac{1}{2} d^2 + 2d - 2 = \left(1-\frac{d}{2}\right)(d-2)$$

This coefficient renders the terms $\propto \Gamma(1-\frac{d}{2})$ as vanishing when $d \to 2$, thus there are no quadratic divergences in QCD.

Yet, these terms are divergent for $d \to 4$. In order to bring them to the form of the purely logarithmic divergences, write

$$\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) = \Gamma\left(2-\frac{d}{2}\right)$$

The coefficient of the term $\Gamma(2-\frac{d}{2}) g^{\mu\nu} k^2$ is thus

$$\underbrace{(d-2)\, x(1-x)}_{②, ③, ④} + \underbrace{\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2}_{②} \underbrace{- (d-1)(1-x)^2}_{③}$$

$$= (-2d+4)x^2 + (3d-5)x + \frac{7}{2} - d$$

The integrand is symmetric in $x \longleftrightarrow (1-x)$. Therefore, under the integral, we may replace

$$x \longrightarrow \frac{1}{2} x + \frac{1}{2}(1-x) = \frac{1}{2},$$

such that the coefficient becomes

$$4\left(1-\frac{d}{2}\right)x^2 + 2dx + \frac{d}{2} - 4x - \frac{1}{2} + \frac{7}{2} - d$$

$$= 4\left(1-\frac{d}{2}\right)x^2 - 4\left(1-\frac{d}{2}\right)x + \left(1-\frac{d}{2}\right) + 2$$

$$= \left(1-\frac{d}{2}\right)(1-2x)^2 + 2$$

Now, we collect the coefficients of $\Gamma\left(2-\frac{d}{2}\right)k^\mu k^\nu$:

$$\underbrace{-\left(1-\frac{d}{2}\right)(1-2x)^2 - (1+x)(2-x)}_{2} + \underbrace{x(1-x)}_{4} = -\left(1-\frac{d}{2}\right)(1-2x)^2 - 2$$

Putting these results together, one obtains

$\xrightarrow{}$ pure gauge

$$i\Pi^{PG\,ab}_{\mu\nu}(k) = i\Pi^{②ab}_{\mu\nu}(k) + i\Pi^{③ab}_{\mu\nu}(k) + i\Pi^{④ab}_{\mu\nu}(k)$$

$$= \frac{ig^2\mu^\varepsilon}{(4\pi)^{\frac{d}{2}}} C_2(G)\delta^{ab}\int_0^1 dx \frac{\Gamma\left(2-\frac{d}{2}\right)}{\left[-x(1-x)k^2+i\varepsilon\right]^{2-\frac{d}{2}}}\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) * \left[\left(1-\frac{d}{2}\right)(1-2x)^2+2\right]$$

This is manifestly transversal, as required by the Ward identity. In view of our goal to obtain the β-function, we extract the divergence and the dependence on the renormalisation scale μ as:

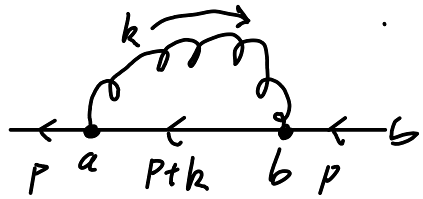
$$i\Pi^{PG\,ab}_{\mu\nu}(k) = i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\delta^{ab}\left[\frac{-g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)C_2(G)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \ldots\right)\right]$$

By imposing renormalisation conditions, we require that the corrections to the gluon self-energy are cancelled by the counterterm

$$\delta Z_3 = \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f\, C(R)\right)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#}\right)$$

In QED, the vacuum polarisation is sufficient to compute the running coupling, as the contributions from the fermion self-energy and the vertex correction neutralise due to the Ward-Takahashi identity. The latter does not hold for QCD, such that we now calculate these two corrections as well.

We know the fermionic wave function correction already. It remains to determine the colour factor



$$-i\Sigma(p) = -\mu^{\varepsilon} g^2 \int \frac{d^dk}{(2\pi)^d}\, t^a \frac{\delta^{ab}}{k^2-\lambda^2+i\varepsilon}\, t^b \gamma^\mu \frac{\not p + \not k + m}{(p+k)^2 - m^2 + i\varepsilon} \gamma^\mu$$

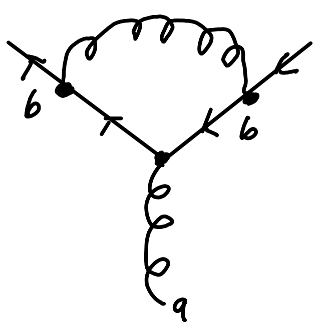
Compared to the QED case, there is an additional factor of a $d(R)\times d(R)$ matrix, $t^a_R t^a_R = C_2(R)\, \mathbb{1}_{d(R)\times d(R)}$. In the following, we suppress the explicit notation of the matrix structure. For a massless quark, we obtain

$$-i\Sigma(p) = i \not p \frac{g^2}{(4\pi)^2} C_2(R) \left(\Delta_\varepsilon + \log \frac{\mu^2}{\#} + \dots \right)$$

The corresponding counterterm is

$$\delta Z_2 = -\frac{g^2}{(4\pi)^2} C_2(R) \left(\Delta_\varepsilon + \log \frac{\mu^2}{\#} + \dots \right)$$

For the vertex correction, there are two contributions. The first one is again familiar from QED, up to a colour factor:



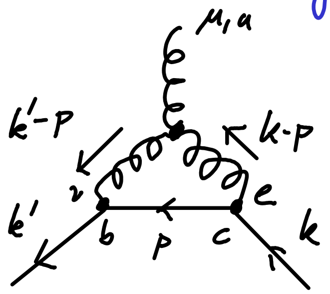
The new feature is here the product of representation matrices

$$t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b] = C_2(R) t^a + i t^b f^{abc} t^c$$
$$= C_2(R) t^a + \frac{i}{2} f^{abc} [t^b, t^c] = C_2(R) t^a - \frac{1}{2} f^{abc} f^{bcd} t^d$$
$$= \left[C_2(R) - \frac{1}{2} C_2(G) \right] t^a$$

Hence, this diagram evaluates to

$$i \frac{g^3}{(4\pi)^2} \left[C_2(R) - \frac{1}{2} C_2(G) \right] t^a \gamma^\mu \left(\Delta_\varepsilon + \log \frac{\mu^2}{\#} + \dots \right)$$

The second diagram is



$$= \int \frac{d^4p}{(2\pi)^4} (ig\gamma_\nu t^b) \frac{i\not{p}}{p^2+i\varepsilon} (ig\gamma_e t^c) \frac{-i}{(k'-p)^2+i\varepsilon} \frac{-i}{(k-p)^2+i\varepsilon}$$
$$* g f^{abc}\left[g^{\mu\nu}(2k'-k-p)^e + g^{\nu e}(-k'+k+2p)^\mu + g^{e\mu}(2k-k'-p^\nu)\right]$$

The colour factor is

$$f^{abc}t^bt^c = \frac{1}{2} f^{abc}[t^b,t^c] = \frac{i}{2} f^{abc} f^{bcd} t^d = \frac{i}{2} C_2(G) t^a$$

The result for this diagram is

$$\frac{ig^3}{(4\pi)^2} \frac{3}{2} C_2(G) t^a \gamma^\mu \left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \ldots\right)$$

The divergences in the vertex correction are then cancelled by the counterterm

$$\delta Z_1 = -\frac{g^2}{(4\pi)^2}\left[C_2(R) + C_2(G)\right]\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \ldots\right)$$

As for QED, we now take the point of view that $\delta Z_{1,2,3}$ are parameters in a fundamental Lagrangian, while g and μ may be chosen arbitrarily, with the constraint that they reproduce the observed coupling g_{eff}, where

$$g_{eff} = g\left[1 - \frac{g^2}{(4\pi)^2}\left[C_2(R)+C_2(G)\right]\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \ldots\right) - \delta Z_1\right]^{-1}$$
$$* \left[1 - \frac{g^2}{(4\pi)^2} C_2(R)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#} + \ldots\right) - \delta Z_2\right]$$
$$* \left[1 + \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(R)\right)\left(\Delta_\varepsilon + \log\frac{\mu^2}{\#}\right) - \delta Z_3\right]^{\frac{1}{2}}$$

When g_{eff} is independent of μ, $g(\mu)$ must have the property

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} = \frac{g^3}{(4\pi)^2}\left[-2\left(C_2(R)+C_2(G)\right) + 2C_2(R) - \frac{5}{3}C_2(G) + \frac{4}{3}n_f C(R)\right]$$

$$= -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(R)\right]$$

Recall that in the Chapter on QED, we noted that provided we may expand

$\beta(g) = b\,g^n + \dots$, it follows that

$$g^{n-1}(\mu) = \frac{g^{n-1}(\mu_0)}{1 - b(n-1)g^{n-1}(\mu_0)\log\frac{\mu}{\mu_0}}$$

For the present case, $SU(N)$ gauge theory with n_f fundamental Dirac fermions,

$$C_2(G = N^2-1) = N, \quad C(R=N) = \frac{1}{2},$$

$$b = -\frac{1}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2}{3}n_f\right], \quad n = 3$$

such that we obtain the <u>running coupling of QCD</u>

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{1}{2}\alpha(\mu_0)\left[\frac{11}{3}N - \frac{2}{3}n_f\right]\log\frac{\mu}{\mu_0}}$$

A striking feature is that provided $N > \frac{2}{11}n_f$, the coupling decreases with larger μ (recall that $\alpha(\mu)$ may be interpreted as the measured coupling at momentum transfer $-q^2 \sim \mu^2$). That is, the theory is <u>asymptotically free</u> and perturbatively well-defined up to arbitrarily high energies.

Asymptotic freedom would therefore be an attractive possibility of UV-completion of field theory.

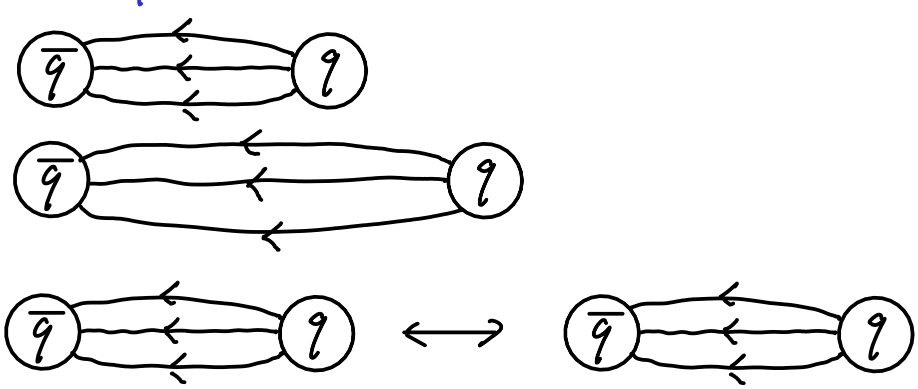
Unfortunately however, gravitational interactions have a rather different behaviour under loop corrections than

QED and QCD and exhibit in particular no asymptotic freedom.
In order to give a quantitative meaning to above relation, we have to fix $\alpha(\mu_0)$ at a certain scale μ_0. Usually, the reference scale is chosen to be the rather well-known Z_0-boson mass
$M_Z = (91,1876 \pm 0,0021)$ GeV (PDG booklet 2012, p. 9).
(In the following, we will refer to this reference in short by "PDG"). The observed value is
$\alpha(M_Z) = 0,1184 \pm 0,0007$
The running of the coupling is well-confirmed experimentally, cf. PDG, Fig. 9.4.
Another interesting number to keep in mind is the value of $\Lambda_{QCD} = \mu$, for which the Landau pole (vanishing denominator) of the running coupling occurs. This celebrated quantity roughly takes the value

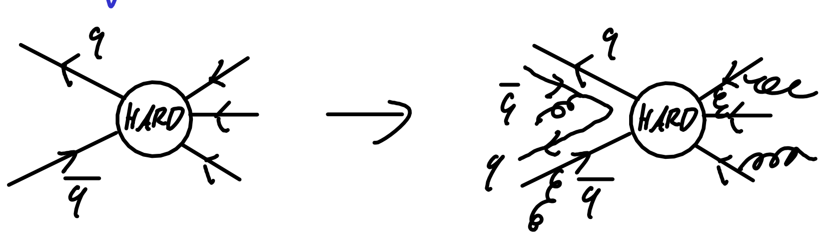
$$\Lambda_{QCD} = (250 \pm 100)\ \text{MeV}$$

From above formula with $n_f = 4$ or 5 (accounting for the quarks u, d, c, s and partly b, while t is way above the Z mass), we would obtain a larger value, around 10 GeV. In order to obtain Λ_{QCD} theoretically, one indeed needs to include next-to-next-to leading order (NNLO) corrections, as one extends to regions where perturbation theory converges slowly and eventually breaks down. In summary, our formula for the running coupling should only be applied when $\mu \gg \Lambda_{QCD}$

and be loop-improved otherwise.
Now, what happens below Λ_{QCD}? Way above this energy scale, our analytic calculation exhibits asymptotic freedom, below, there is <u>confinement</u>.
This is the phenomenon that there are no isolated particles that carry colour charge (quarks and gluons).
So far, there is no analytic proof, but overwhelming experimental evidence as well as theoretical evidence from lattice simulations of QCD.
Typical pictures of confinement include that upon separation of two quarks, a flux tube of constant force forms.
At some point, the formation of a second quark anti-quark pair becomes more favourable:



In high energy scatterings, colour-non singlet states interact at the elementary level, but at large distances, they are accompanied by additional particles, such that there are only colour-singlets. A simple process of this kind may be pictured as



Factorisation states that the amplitudes can be obtained by convoluting the hard amplitudes with functions that encompass the soft, non-perturbative properties that should be obtained experimentally or on the lattice. The proof is technically involved and could be the content of a seperate lecture course. Nonetheless, the concept is somewhat intuitive and we will use it in the following discussion on deep inelastic scattering.

Before getting into that, we should state that at low energy, the strongly interacting particles are bound states, that are colour singlets and only participate in strong interactions through their constituents. These particles are called hadrons. Mesons are quark-antiquark pairs, whereas baryons consist (after subtracting antiquarks from quarks) of three valence-quark. The two most common specimen are the nucleons, the proton (uud) and the neutron (udd), with total spin $\frac{1}{2}$. Besides up and down, there are the quark flavours

	charge Q_i	mass
u "up"	$\frac{2}{3}$	1,7 – 3,1 MeV
d "down"	$-\frac{1}{3}$	4,1 – 5,7 MeV
c "charm"	$\frac{2}{3}$	1,3 GeV
s "strange"	$-\frac{1}{3}$	100 MeV
t "top"	$\frac{2}{3}$	173 GeV
b "bottom"	$-\frac{1}{3}$	4,2 GeV

In addition to the valence quarks, there are pairs (quark & antiquarks) of sea quarks in the nucleon. These may be from each flavour, but in practice, only c and s are of importance next to u and d. (Recall that $m_p = 938,3$ MeV, $m_n = 939,6$ MeV.) Of importance for the structure of hadrons are of course also the gluons. Unlike from QED, where the wave-functions of e.g. atoms or positronium can be calculated, the non-perturbative nature of QCD at low energies has so far prevented us from achieving an analytic understanding of the structure of hadrons. Fortunately, in order to make predictions for high-energy interactions, only a coarse knowledge in terms of structure functions is needed. In the following Section, we are concerned with the properties of the proton in high-energy interactions. A more complete characterisation of hadrons will be given once we have obtained a more detailed understanding of flavour in the light of Electroweak unification, i.e. in the next semester.

The values of the lighter quark masses are determined from the observation of their bound states. In the next semester, we discuss how the u, d, s masses are related to the masses of the π and K mesons due to chiral symmetry breaking, while the c, d masses follow from more detailed considerations of the potential of their mesonic (quark-antiquark pairs) bound states.

Nonetheless, it is interesting to demonstrate the reality of quarks from pair production in e^+e^- collisions. Recall that for the total cross section for muon production, we obtained:

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{\pi\alpha^2}{3}\,\frac{\sqrt{\vec{p}^{\,2}-m_\mu^2}}{|\vec{p}|^3}\left(1+\frac{m_\mu^2}{\vec{p}^{\,2}}\right) \underset{m_\mu \ll |\vec{p}|}{\approx} \frac{4\pi\alpha^2}{3E_{CMS}}$$

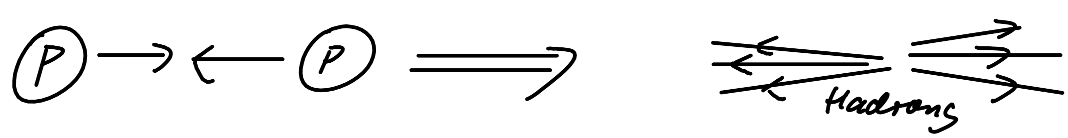
We can directly infer that the cross section for hadron (particles containing strongly interacting constituents) is given by

$$\sigma_{e^+e^- \to \text{hadrons}} = \overset{\substack{\text{number of} \\ \text{colours}}}{3} \Big(\sum_i \overset{\text{quark charge}}{Q_i^2} \Big) \sigma_{e^+e^- \to \mu^+\mu^-}$$

Of course, not all quarks are produced at low energies, but the heavier flavours only kick in when E_{CMS} exceeds twice their mass. This behaviour is neatly demonstrated in Figure 5.3 of Peskin & Schroeder. The height of the first step is $3 * \left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right) = 2$, of the second step $2 + 3 * \left(\frac{2}{3}\right)^2 = 3\,\frac{1}{3}$ and of the third step $3\,\frac{1}{3} + 3 * \left(\frac{1}{3}\right)^2 = 3\,\frac{2}{3}$.

3.4 Deep Inelastic Scattering

An excellent account of how experimental results led to the discovery of quarks and QCD is given in Chapters 14 and 17 of Peskin and Schroeder, which we recommend to read. Proton-proton collisions lead to a large number of hadrons along the collinear axis

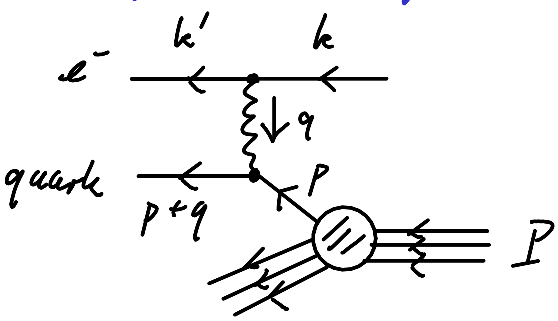


and only few events with large transversal momentum. This has suggested the picture that the proton consists of a large number of constituents that are weakly bound together in a jelly-like structure.
However, this behaviour is precisely what is predicted

by the running coupling. For the collinear scatterings, the momentum exchange q^2 is small, even though $|\vec{q}|$ may be large, whereas transversal scatterings require a large $|q^2|$. The latter are suppressed due to the running coupling, which was not known at the time the first experiments of this type were performed.

On the other hand, it was observed that there is a large hard scattering rate of electrons from the proton, what is inconsistent with the notion of a large number of smaller constituents. Nonetheless, in these hard scatterings, the proton does typically not remain intact, but typically is shattered into hadrons. This is why these are referred to as <u>deeply inelastic scattering</u> (DIS). Together with the theory prediction of asymptotic freedom, one may consistently conclude that the proton consists of a small number of partons, i.e. the valence quarks u, u, d, gluons and additional sea quarks.

The Feynman diagram for DIS is



When we cut the quark propagator p, assume that all particle masses are small compared to $|q^2|$ and $p^2 \sim \Lambda_{QCD}^2 << |q|^2$, we can apply Guttenberg's method to infer the matrix element for electron-quark collision from electron-muon scattering:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4 Q_i^2}{\hat{t}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{4}\right)$$

Here, Q_i is the electric charge of the struck parton and the hats indicate that the Mandelstam variables parametrise the parton-electron collision, i.e. p is treated as an incoming momentum in a $2 \longleftrightarrow 2$ process.

In the CMS, $\hat{s} = (p+k)^2 = 4E^2$, where E is the energy of the electron or quark, which are equal in the CMS.

Using the result from Page 10 of Chapter 2 of these lectures, we obtain ($\hat{s}+\hat{t}+\hat{u}=0$)

$$\frac{d\sigma}{d\cos\vartheta_{CMS}} = \frac{1}{2\hat{s}}\,\frac{1}{16\pi}\,\frac{8e^4 Q_i^2}{\hat{t}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{4}\right) = \frac{\pi\alpha^2 Q_i^2}{\hat{s}}\left(\frac{\hat{s}^2+(\hat{s}+\hat{t})^2}{\hat{t}^2}\right)$$

Using Page 13, Chapter 2, we may relate

$$\hat{t} = -\frac{1}{2}\hat{s}(1-\cos\vartheta_{CMS}) \iff \cos\vartheta_{CMS} = 2\frac{\hat{t}}{\hat{s}}+1$$

such that

$$\frac{d\sigma}{d\hat{t}} = \frac{d\sigma}{d\cos\vartheta_{CMS}}\,\frac{d\cos\vartheta_{CMS}}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{\hat{s}^2}\left(\frac{\hat{s}^2+(\hat{s}+\hat{t})^2}{\hat{t}^2}\right)$$

which is no longer dependent on the reference frame

Next, we relate the kinematic variables to observable parameters (in particular, the parton momentum p is not directly observed).

Since $\hat{t}$ is negative, it is conventionally written as the positive quantity

$$Q^2 = -\hat{t} = -q^2$$

Now, since we assume that the transversal energy scale of the partons $\sim \Lambda_{QCD}$ is much smaller than Q, we can characterise each parton by the fraction ξ of the proton's longitudinal energy-momentum,

$$p = \xi P$$

Then

$$\hat{s} = (p+k)^2 = 2p\cdot k = 2\xi P\cdot k = \xi s$$

On the other hand (massless partons)

$$0 \simeq (p+q)^2 = 2p\cdot q + q^2 = 2\xi P\cdot q - Q^2$$

When we define

$$x = \frac{Q^2}{2P\cdot q}$$

it follows that $\xi = x$ when above approximations are valid. Since k and P are known from the experimental setup and $q = k - k'$, we can determine the longitudinal parton fraction by a measurement of k', which is clearly easier and more precise than an attempt of reconstructing it from the produced hadrons.

In terms of these new variables, we write

$$\frac{(\hat{s}+\hat{t})^2}{\hat{s}^2} = \frac{(xs - Q^2)^2}{x^2 s^2} = \left(1 - \frac{Q^2}{xs}\right)^2 = \frac{\hat{u}^2}{\hat{s}^2}$$

Now, we define the <u>parton distribution functions</u> (PDFs) $f_i(\xi)$ such that the probability of finding the parton i within the interval $[\xi, \xi + d\xi]$ is given by $f_i(\xi)d\xi$.

We therefore obtain the cross section formula

$$\frac{d^2\sigma}{dx\, dQ^2} = \sum_i f_i(x)\, Q_i^2\, 2\pi\alpha^2\, \frac{1+\left(1-\frac{Q^2}{xs}\right)^2}{Q^4}$$

Removing the last factor defines a quantity that is independent of Q, a behaviour known as <u>Bjorken scaling</u>, which is excellently verified experimentally, cf. Figure 14.2 of Peskin & Schroeder.

We can express the differential dQ^2 by another dimensionless variable,

$$y = \frac{P\cdot q}{P\cdot k} = \frac{2P\cdot q}{s} = \frac{p\cdot(k-k')}{p\cdot k} = \frac{\hat{s}+\hat{u}}{\hat{s}} \iff -(1-y) = \frac{\hat{u}}{\hat{s}}$$

In the rest frame of the proton, $y = \frac{q^0}{k^0}$, i.e. the fraction of the electron energy that is transferred to the hadrons.

Notice that $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Now, $Q^2 = xys$, such that $\frac{dy}{dQ^2} = \frac{1}{xs}$ and

$$\frac{d^2\sigma}{dx\, dy} = \left(\sum_i x f_i(x)\, Q_i^2\right) \frac{2\pi\alpha^2 s}{Q^4} \left[1+(1-y)^2\right]$$

The dependence on the factor in square brackets is called the <u>Callan-Gross relation</u>.

From the Bjorken scaling, we can extract $\sum_i x f_i(x)\, Q_i^2$, but not the individual parton distribution functions.

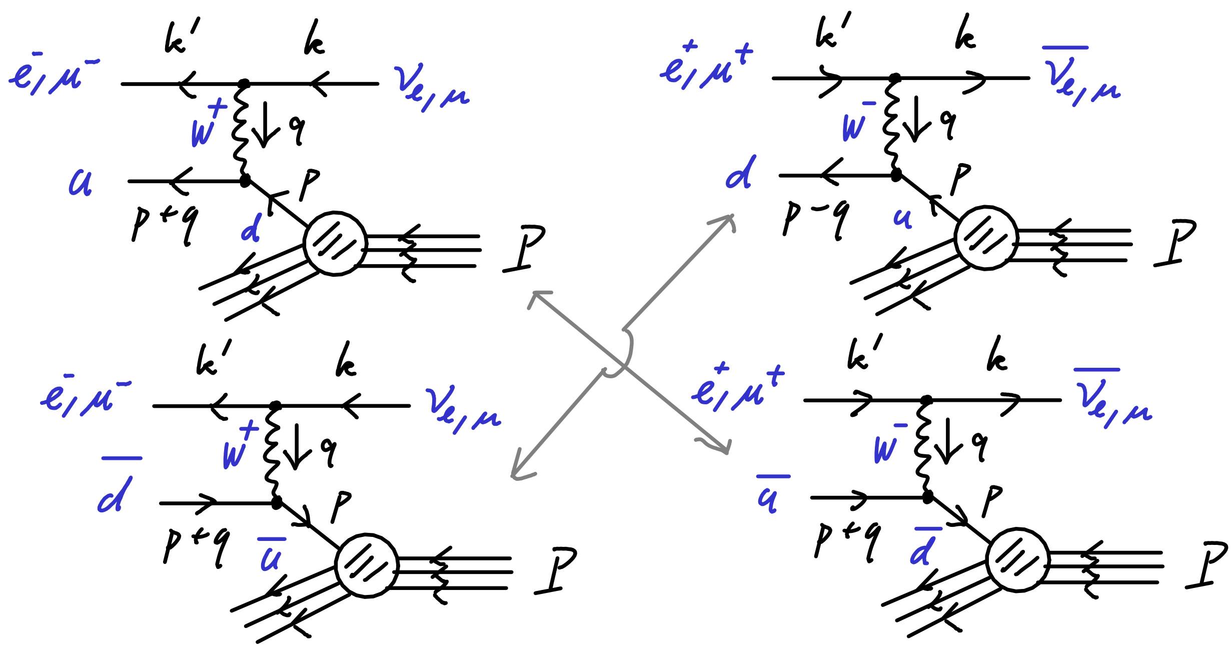
Fortunately, the remaining information may be obtained using neutrinos as a probe.

The interaction with quarks is given by

$$\Delta\mathcal{L} = \frac{g^2}{2}\frac{1}{m_W^2}\left[\bar{l}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\nu\right]\left[\bar{u}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)d\right] + h.c.$$

where l is either e^-, μ^- and ν either ν_e, ν_μ and g is the weak coupling constant.

The diagrams for DIS are now given by



Therefore, we may use neutrinos to probe the d-quark and $\bar{u}$ contents and anti-neutrinos to probe the u-quarks and the $\bar{d}$s. We have replaced the propagator of the W boson by $\frac{1}{m_W^2}$, where $m_W = 80{,}4$ GeV. Hence, our calculation is valid when $\Lambda_{QCD} \ll Q \ll m_W$. Besides, we have indicated which cross sections are identical due to crossing symmetry.

For the first (or fourth) diagram, we need the matrix element

$$i\mathcal{M}_{\nu d \to l^- u} = \frac{1}{2}\frac{g^2}{m_W^2}\bar{u}(k',s')\gamma^\mu \frac{1-\gamma^5}{2} u(k,s)\, \bar{u}(p+q,r')\gamma_\mu \frac{1-\gamma^5}{2} u(p,r)$$

$$\longrightarrow$$

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^4}{8\, m_W^4} \operatorname{tr}\left[\not{k}'\gamma^\mu \frac{1-\gamma^5}{2} \not{k}\gamma^\nu \frac{1-\gamma^5}{2}\right]$$

$$* \operatorname{tr}\left[(\not{p}+\not{q})\gamma_\mu \frac{1-\gamma^5}{2} \not{p}\gamma_\nu \frac{1-\gamma^5}{2}\right]$$

$$= \frac{g^4}{16\, m_W^4}\; 4\hat{s}^2$$

We have evaluated the traces using FORM (nuDIS.frm). Moreover, the initial neutrinos are always left-polarised, such that we only average over the polarisation of the quark. We obtain

$$\frac{d\sigma_{\nu d\to l^- u}}{d\cos\vartheta_{cms}} = \frac{1}{2\hat{s}}\,\frac{1}{8\pi}\,\frac{g^4}{4m_W^2}\,\hat{s}^2 = \frac{\hat{s}}{2}\,\frac{d\sigma_{\nu d\to l^- u}}{d\hat{t}}$$

$$\frac{d\sigma_{\nu d\to l^- u}}{d\hat{t}} = \frac{\pi g^4}{2(4\pi)^2\hat{s}^2}\,\frac{\hat{s}^2}{m_W^4} = \frac{G_F^2}{\pi} = \frac{d\sigma_{\bar\nu\bar d\to l^+\bar u}}{d\hat{t}}$$

where $G_F = \frac{\sqrt{2}\,g^2}{8m_W^2}$ is the Fermi constant.

Now, for the second (third) diagram, we obtain

$$i\mathcal{M}_{\nu\bar u\to l^-\bar d} = \frac{1}{2}\,\frac{g}{m_W^2}\,\bar{u}(k',s')\gamma^\mu\,\frac{1-\gamma^5}{2}\,u(k,s)\,\bar{v}(p,r)\,\gamma_\mu\,\frac{1-\gamma^5}{2}\,v(p+q,r')$$

$$\longrightarrow$$

$$\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 = \frac{g^4}{8m_W^4}\,\mathrm{tr}\left[\not{k}'\gamma^\mu\,\frac{1-\gamma^5}{2}\,\not{k}\,\gamma^\nu\,\frac{1-\gamma^5}{2}\right]$$

$$* \,\mathrm{tr}\left[\not{p}\,\gamma_\mu\,\frac{1-\gamma^5}{2}\,(\not{p}+\not{q})\gamma_\nu\,\frac{1-\gamma^5}{2}\right]$$

$$= \frac{g^4}{16m_W^4}\,4\hat{u}^2$$

$-(1-y) = \frac{\hat{u}}{\hat{s}}$

such that

$$\frac{d\sigma_{\nu\bar u\to l^-\bar d}}{d\hat{t}} = \frac{\pi g^4}{2(4\pi)^2\hat{s}^2}\,\frac{\hat{u}^2}{m_W^4} = \frac{G_F^2}{\pi}(1-y)^2 = \frac{d\sigma_{\bar\nu u\to l^+ d}}{d\hat{t}}$$

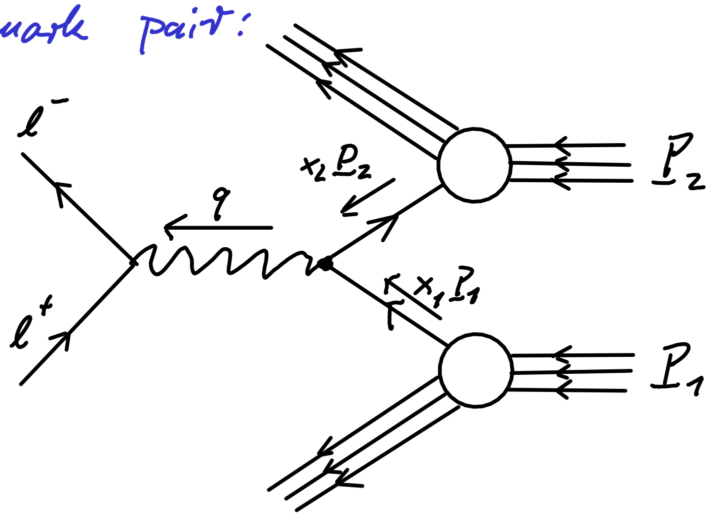
Now, recall that $\frac{dy}{dQ^2} = \frac{1}{xs} = -\frac{dy}{d\hat{t}}$, such that

$$\frac{d^2\sigma_{\nu p\to\mu^- X}}{dx\,dy} = \frac{G_F^2 s}{\pi}\left[x f_d(x) + x f_{\bar u}(x)(1-y)^2\right]$$

$$\frac{d^2\sigma_{\bar\nu p\to\mu^+ X}}{dx\,dy} = \frac{G_F^2 s}{\pi}\left[x f_u(x)\cdot(1-y)^2 + x f_{\bar d}(x)\right]$$

Deep inelastic neutrino scatterings are therefore a powerful mean of extracting the individual quark distribution functions.

An important application of PDFs is the Drell-Yan process, where a lepton pair is produced from the annihilation of a quark-antiquark pair:



Again, we can infer from the formula for $e^+e^- \to \mu^+\mu^-$ that

$$\sigma_{q_f \bar{q}_f \to l_+ l_-} = \frac{1}{3} Q_f^2 \frac{4\pi\alpha^2}{3\hat{s}}$$

where

$$\hat{s} = M^2 = q^2$$

$$P_1 = (E, 0, 0, E) \qquad P_2 = (E, 0, 0, -E) \quad (\text{CMS})$$

$$q = x_1 P_1 + x_2 P_2 = ((x_1 + x_2)E, 0, 0, (x_1 - x_2)E)$$

$$\to q^2 = M^2 = 4 x_1 x_2 E^2 = x_1 x_2 s \Rightarrow x_{1,2} = \frac{M^2}{s x_{2,1}}$$

We can also express q as

$$q = M \begin{pmatrix} \cosh\psi \\ 0 \\ 0 \\ \sinh\psi \end{pmatrix}$$ where ψ is the rapidity of the virtual photon.

It follows that

$$\cosh\psi = \frac{q^0}{M} = \frac{x_1 + x_2}{2\sqrt{x_1 x_2}} = \frac{1}{2}\left(\sqrt{\frac{x_1}{x_2}} + \sqrt{\frac{x_2}{x_1}}\right) \Rightarrow e^{\psi} = \sqrt{\frac{x_1}{x_2}}$$

$$\Longrightarrow \quad x_1 = \frac{M}{\sqrt{s}} e^{y}, \qquad x_2 = \frac{M}{\sqrt{s}} e^{-y}$$

Now, using the PDFs the lepton pair production cross section is given by

$$\sigma_{p(P_1)+p(P_2)\to \ell^+\ell^-} = \int_0^1 dx_1 \int_0^1 dx_2 \sum_f f_f(x_1) f_{\bar{f}}(x_2)\, \sigma_{q_f \bar{q}_{\bar{f}} \to \ell^+\ell^-}$$

Changing variables from $(x_1, x_2) \to (M^2, y)$, we obtain the Jacobian

$$\frac{\partial(M^2, y)}{\partial(x_1, x_2)} = \begin{vmatrix} x_2 s & x_1 s \\ \frac{1}{2x_1} & -\frac{1}{2x_2} \end{vmatrix} = s = \frac{M^2}{x_1 x_2}$$

It follows the differential cross section

$$\frac{d\sigma_{pp\to \ell^+\ell^- X}}{dM^2\, dy} = \sum_f x_1 f_f(x_1)\, x_2 f_f(x_2)\, \frac{1}{3} Q_f^2 \frac{4\pi\alpha^2}{3M^4}$$

3.5 PDFs, DGLAP & Violation of Bjorken Scaling

Since the proton consists of two valence u quarks and one valence d quark as well as no valence s and c quarks, we can readily infer the constraints

$$\int_0^1 dx \left[f_u(x) - f_{\bar{u}}(x) \right] = 2 \qquad \int_0^1 dx \left[f_d(x) - f_{\bar{d}}(x) \right] = 1$$

$$\int_0^1 dx \left[f_{s,c}(x) - f_{\bar{s},\bar{c}}(x) \right] = 0$$

Furthermore, the momenta carried by the partons must sum to the total momentum of the hadron, i.e.

$$\int dx\, x \sum_f f_f(x) = 1$$

We also remark that protons and neutrons are related by the isospin symmetry that exchanges u and d quarks and that is only mildly violated by the different charges and masses. One therefore obtains

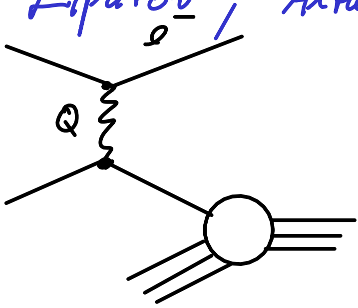
$$f_d^n(x) \approx f_u^p(x), \quad f_{\bar{d}}^n(x) \approx f_{\bar{u}}^p(x), \quad f_u^n(x) \approx f_d^p(x), \quad f_{\bar{u}}^p(x) \approx f_{\bar{d}}^p(x)$$

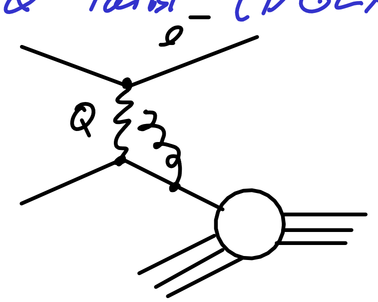
etc.

A plot of the proton PDFs at $Q^2 = 4\,\mathrm{GeV}^2$ is given by Figure 17.6 of Peskin & Schroeder. The scale of momentum transfer is a relevant piece of information, because for larger Q^2, we probe the structure of the proton on smaller scales. For e^-p DIS, we should therefore improve our previous formula to

$$\frac{d^2\sigma}{dx\,dy} = \left(\sum_i x f_i(x,Q)\, Q_i^2\right) \frac{2\pi\alpha^2 s}{Q^4}\left[1 + (1-y)^2\right]$$

Hence, there is a violation of Bjorken scaling that is isolated in the Q dependence of the f_i. While the PDFs encompass the perturbatively inaccessible soft regime of QCD, their Q dependence can actually be calculated perturbatively. The calculation has been performed by independent groups consisting of Dokshitzer, Gribov, Lipatov, Altarelli & Parisi (DGLAP).





The reasoning is that the second of these two processes is not suppressed compared to the first one, provided the transversal momentum of the gluon is smaller than Q. For such a collinear emission, the extra intermediate quark propagator is almost on shell, what compensates the perturbative suppression due to the extra vertex. Now, the collinear composition of the proton at transfer Q is already contained in the $f_i(x,Q)$, so we already have accounted for these collinear emissions. However, when increasing $Q \longrightarrow Q+\Delta Q$, we have to account for the increased likelihood of collinear emission, now up to the scale $Q+\Delta Q$.

One can calculate the <u>splitting functions</u>, which describe the likelihood of an incoming parton to radiate a collinear parton that carries away a fraction z of its energy prior to the hard scattering:

$$P_{q\leftarrow q}(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] \text{ where } \int_0^1 dx\, \frac{f(x)}{(1-x)_+} = \int_0^1 dx\, \frac{f(x)-f(1)}{1-x}$$

$$P_{g\leftarrow q}(z) = \frac{4}{3}\,\frac{1+(1-z)^2}{z}$$

$$P_{q\leftarrow g}(z) = \frac{1}{2}\left[z^2+(1-z)^2\right]$$

$$P_{g\leftarrow g}(z) = 6\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(1-z)\right]$$

The PDFs then evolve as

$$f_X(x,Q+\Delta Q) = f_X(x,Q) + \sum_Y \int_x^1 dz\, \frac{\alpha_s(Q)}{2\pi}\,\frac{\Delta P_\perp^2}{z P_\perp^2}\, P_{X\leftarrow Y}(z)\, f\left(\frac{x}{z},Q\right)$$

Now, $p_\perp \mapsto Q$ and $\Delta p_\perp^2 = 2 p_\perp \Delta p_\perp \mapsto 2 Q \Delta Q$.

$$\frac{d}{d\log Q} = \frac{dQ}{d\log Q}\frac{d}{dQ} = \frac{d\, e^{\log Q}}{d\log Q}\frac{d}{dQ} = Q\frac{d}{dQ}$$

such that

$$\frac{d}{d\log Q} f(x,Q) = \sum_y \frac{\alpha_s(Q)}{\pi}\int_x^1 \frac{dz}{z} P_{X\leftarrow y}(z)\, f\left(\frac{x}{z}, Q\right)$$

In terms of the splitting functions, the evolution of the PDFs with varying Q is therefore given by the <u>Altarelli-Parisi equations:</u>

$$\frac{d}{d\log Q} f_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Big\{ P_{g\leftarrow q}(z)\sum_f \Big[f_f\left(\frac{x}{z},Q\right) + f_{\bar f}\left(\frac{x}{z},Q\right)\Big] + P_{g\leftarrow g}(z)\, f_g\left(\frac{x}{z},Q\right)\Big\}$$

$$\frac{d}{d\log Q} f_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Big\{ P_{q\leftarrow q}(z)\, f_f\left(\frac{x}{z},Q\right) + P_{q\leftarrow g}(z)\, f_g\left(\frac{x}{z},Q\right)\Big\}$$

$$\frac{d}{d\log Q} f_{\bar f}(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Big\{ P_{q\leftarrow q}(z)\, f_{\bar f}\left(\frac{x}{z},Q\right) + P_{q\leftarrow g}(z)\, f_g\left(\frac{x}{z},Q\right)\Big\}$$

The intuitive explanation of the effect of these equations is that at larger Q, more individual partons are resolved, which implies that the individual f_i decrease for large x and increase for small x. This is experimentally verified and illustrated by Figures 17.21 & 17.22 of Peskin & Schroeder.

We finally note that collinear splitting also affects the products of the collision, in particular the hadrons. Individual hard quarks & gluons "hadronise" when leaving the inter= action region, i.e. they form colour singlet "jets" consisting of collinear radiation. In order to determine the production rate for jets beyond the leading order, one has to account for the inclusive rate that encompasses virtual corrections as well as contributions from soft and collinear radiation. Crucial is here the cancellation of IR divergences similar to what we encountered for Coulomb scattering in QED.