2. Quantum Electrodynamics QED

2.1 The QED Lagrangian, Gauge Symmetry, Feynman Rules

The Lagrangian for the Dirac-field

$$\mathcal{L}(x) = \overline{\psi}(x)(i\not{\partial} - m)\psi(x)$$

has a global $U(1)$ symmetry:

$$\psi(x) \longmapsto e^{i\varphi}\psi(x)$$
$$\mathcal{L}(x) \longmapsto \mathcal{L}(x)$$

When attempting to generalise this to a space-time dependent phase, we realise that

$$\psi(x) \longmapsto e^{i\alpha(x)}\psi(x)$$

$$\mathcal{L}(x) \longmapsto \mathcal{L}(x) - \overline{\psi}(x)(\not{\partial}\alpha(x))\psi(x)$$

what is due to the property

$$\partial_\mu \psi(x) \longmapsto i(\partial_\mu \alpha(x))e^{i\alpha(x)}\psi(x) + e^{i\alpha(x)}\partial_\mu \psi(x)$$

In order to compensate for the extra term (i.e. the first term), we introduce the vector potential A^μ and define its transformation property

$$A^\mu(x) \longmapsto A^\mu(x) - \frac{1}{e}\partial^\mu \alpha(x)$$

where e is a constant. The covariant derivative is

$$D_\mu \psi(x) = (\partial_\mu + ieA_\mu)\psi(x)$$

It transforms in the desired manner:

$$D_\mu \psi(x) \longmapsto e^{i\alpha(x)} D_\mu \psi(x)$$

In order to promote A_μ to a dynamical field, we add the kinetic term

$$-\frac{1}{4}\underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{=:F^{\mu\nu}}(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

This is second order in derivatives, and since under above transformation,

$$F^{\mu\nu} \longmapsto F^{\mu\nu},$$

it leads to an invariant Lagrangian, that we summarise as

$$\boxed{\begin{array}{c}\underline{\text{Lagrangian of Quantum Electrodynamics}}\\ \mathcal{L}_{QED}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi\end{array}}$$

(Even though to this end, the theory is classical, we follow the common practice of calling this the QED Lagrangian.)

The symmetry $\mathcal{L}_{QED}(x) \longmapsto \mathcal{L}_{QED}(x)$ under the local transformation through $\alpha(x)$ is called a <u>local symmetry</u> or a <u>gauge symmetry</u>. The transformation through $\alpha(x)$ is a gauge transformation and $A_\mu(x)$ is a gauge field.

Due to the gauge invariance, the quantisation of $A_\mu(x)$ is tricky. For QCD, which is a generalisation of QED, we will perform this task in a systematic manner, using the functional formalism.

For the time being, we make a <u>gauge fixing</u> by choosing the Lorentz gauge $\partial^\mu A_\mu = 0$. Through partial integration, we can then express the kinetic term as

$$\frac{1}{2} A_\mu \partial^2 A^\mu = \frac{1}{2} A^\mu g_{\mu\nu} \partial^2 A^\nu$$

The photon (that is what A^μ is) propagator then is

$$i\Delta^F_{\mu\nu}(p) = -\frac{i g_{\mu\nu}}{p^2 + i\varepsilon}$$

When compared to the scalar field, the LSZ reduction formula (that relates Green functions to S-matrix elements) must account for the normalisation of the wave-function. Recall from the last chapter, that we impose the photons to be transversely polarised:

$\varepsilon^\mu = (0, \vec{\varepsilon})$ and $\vec{k} \cdot \vec{\varepsilon} = 0$ and $\vec{\varepsilon} \cdot \vec{\varepsilon}^* = 1$ (normalisation)

Recall from our discussion of Lorentz-symmetry, that

$$J_3 = J^{12} = i\left[g^{\mu 1} \delta^2{}_\nu - g^{\mu 2} \delta^1{}_\nu\right] = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_3 \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$$

Hence, for a photon travelling in 3-direction, we can choose the basis of polarisation vectors $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$, what corresponds to eigenstates of spin, or, when formulated in a direction-independent manner, of the helicity $h = \frac{\vec{k}}{|\vec{k}|} \cdot \vec{J} = \pm 1$. Photons with $h = +1$ are right-, with $h = -1$ are left polarised. We can now state the rule that when calculating an invariant matrix element $i\mathcal{M}$ with an external photon, we must contract the amputated Green function with the polarisation vector ε_μ.

The Feynman rules for fermions can be explained in a similar manner. The Feynman propagator (which we conventionally denote by iS for fermions, instead of $i\Delta$ for scalars) is given by

$$iS^F_{\alpha\beta}(x_1,x_2) = \langle 0 | T[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)] | 0 \rangle$$

where α, β are spinor-indices. It satisfies the equation

$$(i\not{\partial}_{x_1} - m)\, iS^F(x_1,x_2) = i\,\delta^4(x_1-x_2) * \mathbb{1}_{spinor}$$

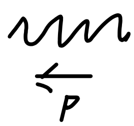
Taking account of the time-ordered boundary conditions, the momentum-space solution is

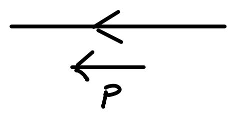
$$iS^F(x_1,x_2) = \frac{i(\not{p}+m)}{p^2+m^2+i\varepsilon}$$

In a Feynman graph, external, incoming (outgoing) Dirac fermions are contracted with $u(\vec{p},s)$ $(\overline{u}(\vec{p},s))$, incoming (outgoing) anti-fermions with $\overline{v}(\vec{p},s)$ $(v(\vec{p},s))$.
We now summarise these rules They apply to the interaction of charged spin-$\frac{1}{2}$ fermions (e.g. e, μ, τ and quarks as fundamental particles, the proton as a composite particle) with the electromagnetic (photon) field. As other fermions are either strongly interacting (QCD) or are heavy with a short life-time, the best fermions in order to study the properties of pure QED are electrons e and muons μ.

Feynman rules for QED

propagators:

photon:  $= -\dfrac{i g_{\mu\nu}}{p^2+i\varepsilon}$

charged fermion:  $= \dfrac{i(\not p + m)}{p^2+m^2+i\varepsilon}$

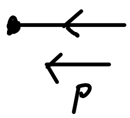
The arrow on the propagator indicates the direction from $\overline{\psi}$ to ψ.

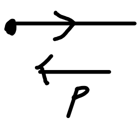
vertex:

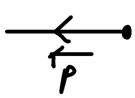
 $= -ie\gamma^\mu$, impose momentum conservation

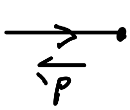
note that e is negative for the electron

external legs:

 $= u(\vec{p},s)$ incoming fermion

 $= \overline{v}(\vec{p},s)$ incoming anti-fermion

 $= \overline{u}(\vec{p},s)$ outgoing fermion

 $= v(\vec{p},s)$ outgoing anti-fermion

 $= \varepsilon^\mu$ incoming photon

 $= \varepsilon^{*\mu}$ outgoing photon

loops:

Integrate over loop momenta. Fermion loops pick up a factor of -1. This is because a fermion loop, that does not couple fermions to an external fermion-line originates from a set of functional derivatives

$$\frac{\delta}{\delta\bar{\eta}_{\alpha_1}(x_1)}\frac{\delta}{\delta\eta_{\alpha_1}(x_1)}\cdot\frac{\delta}{\delta\bar{\eta}_{\alpha_2}(x_2)}\frac{\delta}{\delta\eta_{\alpha_2}(x_2)}\cdot\ldots\cdot\frac{\delta}{\delta\bar{\eta}_{\alpha_n}(x_n)}\frac{\delta}{\delta\eta_{\alpha_n}(x_n)},$$

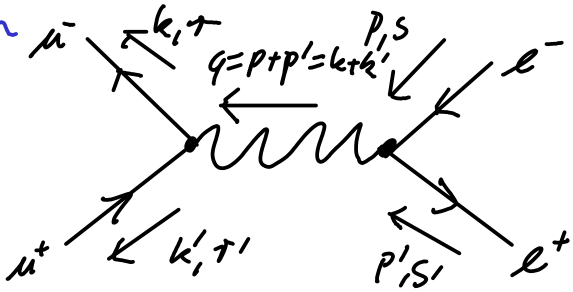
such that the field $\bar{\psi}_{\alpha_1}(x_1)$ appears on the left.
The functions η and $\bar{\eta}$ are currents that are used for variational purposes, similar to J for the scalar field. While J commutes, η and $\bar{\eta}$ anti-commute.
One refers to such anti-commuting complex numbers as Graßmann-numbers.
Now, we would rather like a term of the form

$$\langle 0|T[\psi_{\alpha_1}(x_1)\bar{\psi}_{\alpha_2}(x_2)]|0\rangle\cdot\ldots\cdot\langle 0|T[\psi_{\alpha_n}(x_n)\bar{\psi}_{\alpha_1}(x_1)]|0\rangle$$
$$= \mathrm{tr}\left[iS^F(x_1,x_2)\cdots iS^F(x_n,x_1)\right]$$

As it is necessary for this purpose, to commute the anti-commuting derivative $\frac{\delta}{\delta\bar{\eta}_{\alpha_1}(x_1)}$ for $2n-1$ times, we obtain the minus sign.
One should also be aware that minus signs from the anti-commuting spinors may already be important in the interference of tree-level diagrams.

2.2 Elementary Scattering Processes and Crossing Symmetries

We now aim to calculate the cross section for the process $e^+e^- \longrightarrow \mu^+\mu^-$. The leading contribution to the invariant matrix element is represented by the Feynman diagram



τ, τ' and s, s' stand for the spin

Application of the Feynman rules yields

$$i\mathcal{M}_{e^+e^-\to\mu^+\mu^-} = \overline{u}_\mu(k,\tau)(-ie\gamma^\mu)\, v_\mu(k',\tau')\, \frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\, \overline{v}_e(p',s')(-ie\gamma^\nu)\, u_e(p,s)$$

Now suppose (as it is the case in many experimental situations) that the electron and positron beams are unpolarised and that we do not detect the spin of the produced muons. (A muon chamber, for example, would indeed only measure the momentum, not the spin of a muon). Hence, our observable is

$$\underbrace{\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}}_{\text{average over in-coming spin}} \underbrace{\sum_\tau \sum_{\tau'}}_{\text{sum over outgoing spin}} \left| i\mathcal{M}_{e^+e^-\to\mu^+\mu^-}\right|^2 =: \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

Consider the piece (the Greek indices are spinorial)

$$\sum_{\tau,\tau'} \overline{u}_{\mu\alpha}(k,\tau)\gamma^\mu_{\alpha\beta} v_{\mu\beta}(k',\tau')\, \overline{v}_{\mu\gamma}(k',\tau')\gamma^\varrho_{\gamma\delta} u_{\mu\delta}(k,\tau)$$

<u>NB</u> γ^0, $\gamma^0\gamma^\mu$ are hermitian!

$$= (\not{k}' - m_\mu)_{\beta\gamma}\, \gamma^\varrho_{\gamma\delta}\, (\not{k} + m_\mu)_{\delta\alpha}\, \gamma^\mu_{\alpha\beta} = \mathrm{tr}\left[(\not{k}' - m_\mu)\gamma^\varrho(\not{k}+m_\mu)\gamma^\mu\right]$$

We have used here the identities for the spin sums of the basis spinors quoted in the previous chapter. The same manipulation can also be applied to the electron spinors. We therefore obtain

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^2}{4q^4}\, \mathrm{tr}\left[(\not{k}'-m_\mu)\gamma^\varrho(\not{k}'+m_\mu)\gamma^\mu\right] \mathrm{tr}\left[(\not{p}+m_e)\gamma_\varrho(\not{p}'-m_e)\gamma_\mu\right]$$

The evaluation of spinor traces is routinely encountered in phenomenological calculations ($\longrightarrow$ there are useful computer algebra tools for their evaluation). The present traces are manageable by the use of the following rules:

$$\mathrm{tr}\, \mathbb{1} = 4$$

$$\mathrm{tr}\,(\text{any odd \# of } \gamma\text{-matrices}) = 0$$

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\varrho\gamma^\sigma) = 4(g^{\mu\nu}g^{\varrho\sigma} - g^{\mu\varrho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\varrho})$$

A short calculation then gives

$$\mathrm{tr}\left[(\not{p}+m_e)\gamma_\varrho(\not{p}'-m_e)\gamma_\mu\right] = 4\left[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}(p\cdot p' + m_e^2)\right]$$

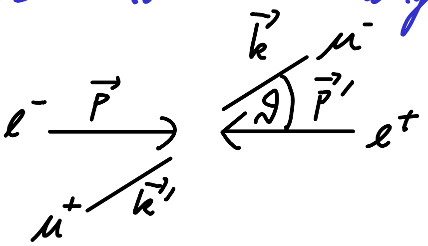
$$\mathrm{tr}\left[(\not{k}'-m_\mu)\gamma^\varrho(\not{k}'+m_\mu)\gamma^\mu\right] = 4\left[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k' + m_\mu^2)\right]$$

The muon mass $m_\mu = 105{,}7$ MeV is roughly 200 times larger than the electron mass $m_e = 511{,}0$ keV, such that we may neglect it in the present calculation (corrections of order $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{129}$ at next to leading order would be of roughly the same size).

The contraction of Minkowski indices then leads to

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2\,(p\cdot p')\right]$$

The total cross section Lorentz-transforms as an area, and differential expressions depend entirely on the chosen reference frame. Let us choose the center of mass (CMS) frame, where $\vec{p} = -\vec{p}\,'$. Consequently, also $\vec{k} = -\vec{k}\,'$, and let's denote the angle between electron and muon by ϑ:



The various Lorentz-invariants in the squared matrix element are:

$$p \cdot p' = p^0 p'^0 - \vec{p} \cdot \vec{p}\,' = 2\vec{p}^{\,2}$$

$$q^2 = (p+p')^2 = \underbrace{2m_e^2}_{\approx 0} + 2p \cdot p' = 4\vec{p}^{\,2}$$

$$p \cdot k = p' \cdot k' = p^0 k^0 - \vec{p} \cdot \vec{k} = \vec{p}^{\,2} - |\vec{p}|\sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta$$

$$p \cdot k' = p' \cdot k = p^0 k^0 + \vec{p} \cdot \vec{k} = \vec{p}^{\,2} + |\vec{p}|\sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta$$

$\omega(\vec{p}) \approx \omega(\vec{k}) \approx |\vec{p}|$

It follows

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{16|\vec{p}|^4}\Big[|\vec{p}|^2\big(|\vec{p}| - \sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta\big)^2 + |\vec{p}|^2\big(|\vec{p}| + \sqrt{\vec{p}^{\,2} - m_\mu^2}\,\cos\vartheta\big)^2 + 2\vec{p}^{\,2} m_\mu^2\Big]$$

$$= e^4\left[\left(1 + \frac{m_\mu^2}{\vec{p}^{\,2}}\right) + \left(1 - \frac{m_\mu^2}{\vec{p}^{\,2}}\right)\cos^2\vartheta\right]$$

For two final states in the center of mass frame, it is useful to integrate the differential cross section first over the moduli of the outgoing momenta, such that

$$\sigma = \frac{1}{2\omega_A 2\omega_B |v_A - v_B|}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{(2\pi)^4\delta^4(p_A + p_B - k - k')}{4\sqrt{\vec{k}^{\,2} + m^2}\sqrt{\vec{k}'^{\,2} + m'^2}}\,\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2$$

$$= \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \int d\Omega \, \frac{|\vec{k}|^2 d|\vec{k}|}{(2\pi)^3} \, \frac{(2\pi)\,\delta\left(p_A^0 + p_B^0 - \sqrt{\vec{k}^2+m^2} - \sqrt{\vec{k}'^2+m'^2}\right)}{4\sqrt{\vec{k}^2+m^2}\,\sqrt{\vec{k}'^2+m'^2}}$$

$\uparrow$ $\vec{p}_A + \vec{p}_B = 0 \Rightarrow \vec{k} = -\vec{k}'$ (CMS)

$$* \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2$$

$$= \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \frac{1}{16\pi^2} \int d\Omega \, \frac{\vec{k}^2 \, \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2}{\frac{|\vec{k}|}{\sqrt{\vec{k}^2+m^2}} + \frac{|\vec{k}|}{\sqrt{\vec{k}^2+m'^2}}} \, \frac{1}{\sqrt{\vec{k}^2+m^2}\,\sqrt{\vec{k}^2+m'^2}}$$

$$= \frac{1}{2\omega_A 2\omega_B |v_A - v_B|} \frac{1}{16\pi^2} \int d\Omega \, \frac{|\vec{k}| \, \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(\vec{p}_A \vec{p}_B \to \vec{k}\vec{k}')|^2}{p_A^0 + p_B^0}$$

$\rightarrow \sqrt{\vec{k}^2+m^2} + \sqrt{\vec{k}^2+m'^2} = p_A^0 + p_B^0$

We now identify $|v_A - v_B| = 2$ and $\omega_A = \omega_B = p_A^0 = p_B^0 = |\vec{p}|$, such that

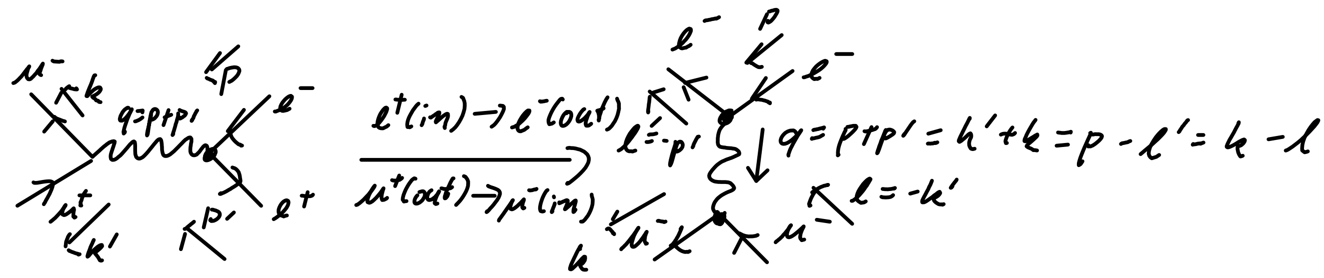
$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2} \frac{\sqrt{\vec{p}^2 - m_\mu^2}}{|\vec{p}|^3} \, \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2$$

$$= \frac{1}{256\pi^2} \frac{\sqrt{\vec{p}^2 - m_\mu^2}}{|\vec{p}|^3} \, e^4 \left[\left(1 + \frac{m_\mu^2}{\vec{p}^2}\right) + \left(1 - \frac{m_\mu^2}{\vec{p}^2}\right)\cos^2\vartheta \right]$$

The total cross section is

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} = 2\pi \int_{-1}^{1} d\cos\vartheta \, \frac{d\sigma}{d\Omega} = \frac{e^4}{48\pi} \frac{\sqrt{\vec{p}^2 - m_\mu^2}}{|\vec{p}|^3} \left(1 + \frac{1}{2}\frac{m_\mu^2}{\vec{p}^2}\right)$$

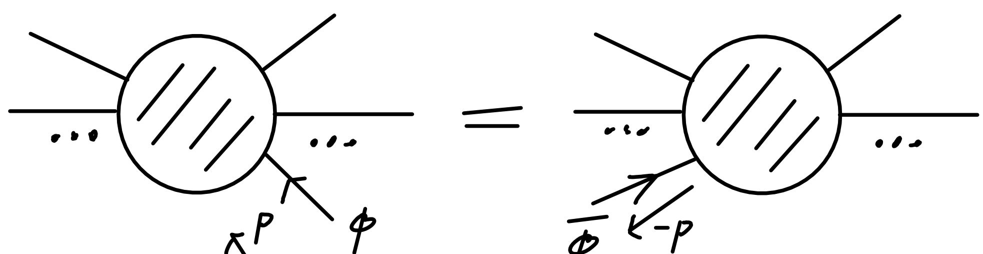
Now, we consider the process $e^-\mu^- \longrightarrow e^-\mu^-$. The Feynman diagram of this process is related to the one just considered above by replacing an incoming positron by an outgoing electron and an outgoing anti-muon by an incoming muon:



$e^+e^- \longrightarrow \mu^+\mu^-$ $\qquad$ $e^-\mu^- \longrightarrow e^-\mu^-$

We can easily convince ourselves that the spin-sums over the squared matrix elements for these processes are identical. It is possible to prove the more general crossing symmetry

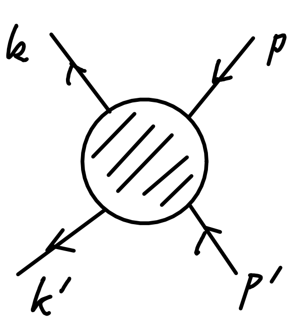
$$i\mathcal{M}\left(\phi(p) + \ldots \longrightarrow \ldots\right) = i\mathcal{M}\left(\ldots \longrightarrow \ldots \bar{\phi}(-p)\right)$$



Replacing fermions by antifermions can lead to an additional sign change, because

$$\sum_{\text{spins}} u(p)\bar{u}(p) = \not{p} + m = -(\not{k} - m) = -\sum_{\text{spins}} v(k)\bar{v}(k)$$

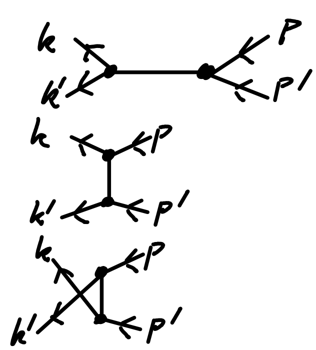
While for $e^+e^- \longrightarrow \mu^+\mu^-$, q is the sum of the incoming or equivalently the outcoming momenta, in $e^-\mu^- \to e^-\mu^-$, it parametrises the difference between incoming and outgoing momenta. For $2 \longleftrightarrow 2$ processes, it is useful to categorise these kinematic situations through the Mandelstam variables:



$$s = (p+p')^2 = (k+k')^2$$

$$t = (k-p)^2 = (k'-p')^2$$

$$u = (k'-p)^2 = (k-p')^2$$



One of the variables can always be eliminated by use of the relation

$$s+t+u=\sum_{i=1}^{4} m_i^2$$

As indicated by the sketched diagrams, the Mandelstam variables are the invariant momentum square of the exchanged particle.

For the process $e^+e^- \longrightarrow \mu^+\mu^-$, the Mandelstam variables are.

$$s=(p+p')^2 = 2p\cdot p' \quad \rightsquigarrow p\cdot p' = \frac{1}{2}s$$

$$t=(k-p)^2 = m_\mu^2 - 2k\cdot p \quad \rightsquigarrow p\cdot k = \frac{1}{2}(m_\mu^2 - t)$$

$$u=(k'-p)^2 = m_\mu^2 - 2k'\cdot p \quad \rightsquigarrow p\cdot k' = \frac{1}{2}(m_\mu^2 - u)$$

Knowing that

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{e^+e^-\to\mu^+\mu^-}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2 (p\cdot p')\right]$$

$$= \frac{2e^4}{s^2}\left[t^2 - 2t\,m_\mu^2 + m_\mu^4 + u^2 - 2u\,m_\mu + m_\mu^4 + 2m_\mu^2(2m_\mu^2 - u - t)\right]$$

$$= \frac{2e^4}{s^2}\left[t^2+u^2 - 4(t+u)\,m_\mu^2 + 6\,m_\mu^4\right]$$

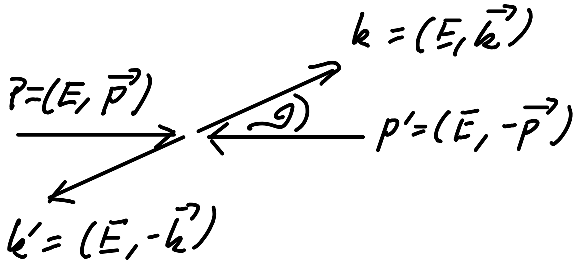
we can immediately express

$$s=(p+p')^2 \longrightarrow (p-l')^2 = t \qquad t=(k-p)^2 \longrightarrow (k-p)^2 = u$$

$$u=(k-p')^2 \longrightarrow (k+l')^2 = s$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{e^-\mu^-\to e^-\mu^-}|^2 = \frac{2e^4}{t^2}\left[s^2+u^2 - 4(s+u)m_\mu^2 + 6m_\mu^4\right]$$

In order to get a quick grasp of how the cross sections are related to the Mandelstam variables, we note their ϑ-dependence in case all four scattering particles are of mass m:



$|\vec{k}| = |\vec{p}|$ since all masses are the same

$$s = (p+p')^2 = 4E^2$$
$$t = (k-p)^2 = -2|\vec{p}|^2(1-\cos\vartheta)$$
$$u = (k'-p)^2 = -2|\vec{p}|^2(1+\cos\vartheta)$$

Therefore, in the ultrarelativistic limit, i.e. when we may approximate $m_e \approx m_\mu \approx 0$,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{e^-\mu^- \to e^-\mu^-}|^2 = \frac{2e^4}{4(1-\cos\vartheta)^2|\vec{p}|^4}\left[16|\vec{p}|^4 + 4|\vec{p}|^4(1+\cos\vartheta)^2\right]$$

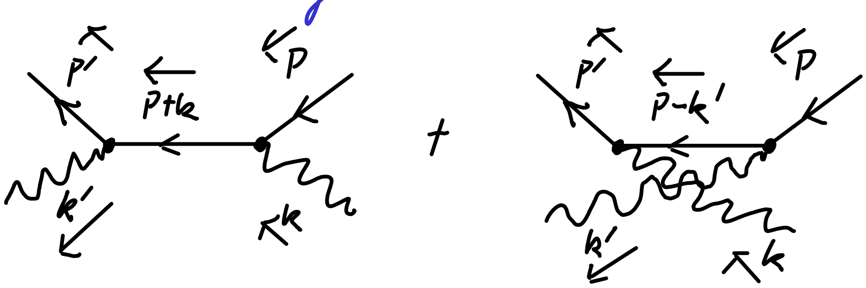
$$\frac{d\sigma_{e^-\mu^- \to e^-\mu^-}}{d\Omega} = \frac{1}{8\vec{p}^2}\frac{1}{16\pi^2}\frac{|\vec{p}|\,\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2}{2|\vec{p}|}$$

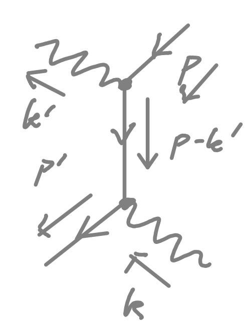
$$= \frac{e^4}{128\pi^2}\frac{1}{\vec{p}^2}\frac{4+(1+\cos\vartheta)^2}{(1-\cos\vartheta)^2}$$

Next, we study <u>Compton Scattering</u>, $e^-\gamma \to e^-\gamma$. The new calculational features that we encounter are the interference of two diagrams and the polarisation sum over external photons.

The two diagrams are:

The second diagram is of course t-channel





Let m denote the electron mass.
The Feynman rules yield the invariant Matrix element:

$$i\mathcal{M} = \bar{u}(p',s')(-ie\gamma^\mu)\,\varepsilon^*_\mu(k')\,\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\varepsilon}\,(-ie\gamma^\nu)\,\varepsilon_\nu(k)\,u(p,s)$$

$$+\,\bar{u}(p',s')(-ie\gamma^\nu)\,\varepsilon_\nu(k)\,\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2+i\varepsilon}\,(-i\gamma^\mu)\,\varepsilon^*_\mu(k')\,u(p,s)$$

$$= -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p',s')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2+i\varepsilon}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2+i\varepsilon}\right]u(p,s)$$

Make use of

$$\left.\begin{aligned}(p+k)^2-m^2 &= 2p\cdot k\\ (p-k')^2-m^2 &= -2p\cdot k'\end{aligned}\right\}\ \text{because } p^2=m^2 \text{ and } k^2=0$$

$$(\not{p}+m)\gamma^\nu u(p) \underset{\uparrow\atop \not{p}\gamma^\nu+\gamma^\nu\not{p}=2p^\nu}{=} (2p^\nu-\gamma^\nu\not{p}+\gamma^\nu m)\,u(p,s) \underset{\uparrow\atop \text{Dirac Eq.}}{=} 2p^\nu u(p,s)$$

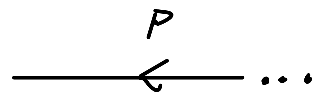
and $\varepsilon \longrightarrow 0$

$\Longrightarrow$

$$i\mathcal{M} = -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p',s')\left[\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2p\cdot k}+\frac{\gamma^\nu\not{k}'\gamma^\mu-2\gamma^\nu p^\mu}{2p\cdot k'}\right]u(p,s)$$

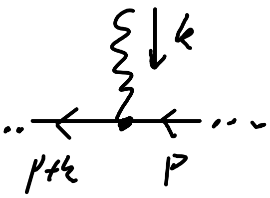
When squaring this amplitude, a simplification occurs again when averaging over polarisations. To argue how the polarisation sum is performed, we use the <u>Ward-identity</u>, which is a special case of the <u>Ward-Takahashi identity</u>. The latter follows from the property of Green functions under the insertion of photon vertices.
Consider a Green function that contains an electron propagator labeled by n

$$iG(\dots) = \dots \overset{\text{nth propagator}}{\frac{i}{\not p - m + i\epsilon}} \dots = \dots$$



Insert a photon-vertex into this propagator

$$iG^\mu_n(\dots, k) = \dots \frac{i}{(\not p + \not k) - m + i\epsilon} (-ie\gamma^\mu) \frac{i}{\not p - m + i\epsilon} = \dots$$



and dot with k_μ:

$$k_\mu \, iG^\mu_n(\dots, k) = \dots \frac{i}{(\not p + \not k) - m + i\epsilon} (-ie)[\not p + \not k - m - (\not p - m)] \frac{i}{\not p - m + i\epsilon} \dots$$

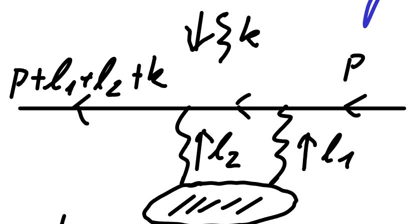
$$= \dots e \left[\frac{i}{\not p - m + i\epsilon} - \frac{i}{\not p + \not k - m + i\epsilon} \right] \dots$$

Now, define

$$iG^\mu(\dots, k) = \sum_n iG^\mu_n(\dots, k)$$

the sum over all photon insertions. Above formula can be applied to each propagator. Consider a fermion line with two photon insertions (the generalisation should be clear):

$$\sum_{\text{insertions}}$$



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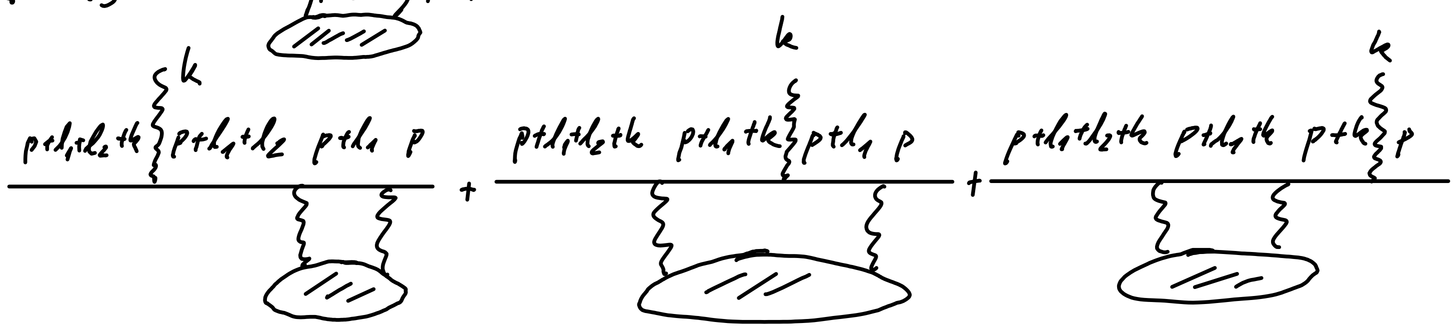

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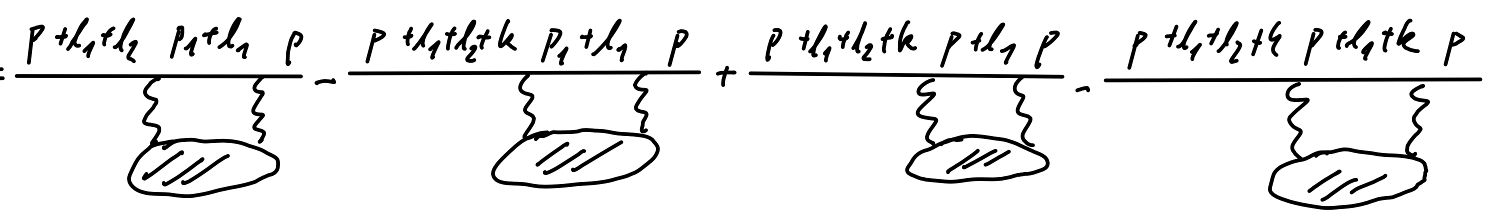
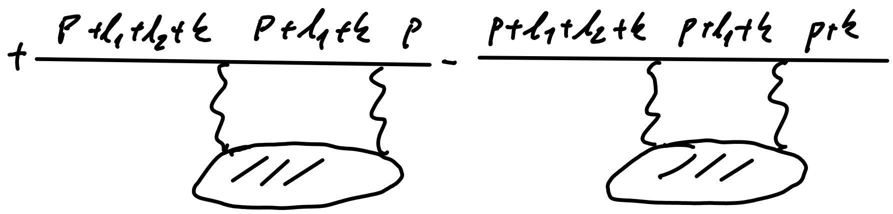




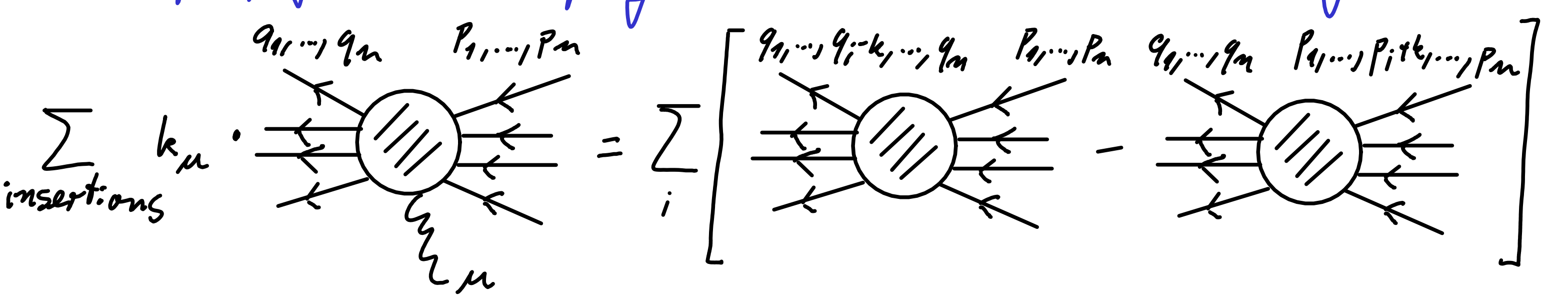
All, but the first and the last diagram cancel. If the line is a closed loop, then $p = p + l_1 + l_2 + k$, and the first and the last diagram cancel as well after redefining the integration variable $p \longrightarrow p + k$ in the first diagram.

When the end points are external, call $q_i = p + \sum_j l_j + k$ and $p = p_i$.

Then, we can observe the <u>Ward-Takahashi identity</u>

$$k_\mu \, i\mathcal{G}^\mu(\ldots, k) = e \sum \Big[i\mathcal{G}(q_1, \ldots, q_i - k; p_1, \ldots) - i\mathcal{G}(q_1, \ldots; p_1, \ldots, p_i + k, \ldots) \Big]$$

When thinking of it in terms of Feynman diagrams, it is important that to the photon vertex, no propagator is attached, whereas the external electron lines have propagators. Keeping this in mind, we may draw:



We have derived this in the present generality, as it will prove useful when we will discuss loop effects and renormalisation for gauge theory.

For the present purposes, instead of considering Green functions, it is sufficient to derive the consequence for invariant matrix elements:
Suppose all external propagators on the left-hand side are on shell. Then, one external fermion line in each of the diagrams on the right-hand side is off shell. When amputating the external lines using the LSZ reduction technique, the off-shell lines give factors of zero. Define $i\mathcal{M}^\mu$ as the invariant matrix element without the photon polarisation factor ε_μ. It follows the Ward identity

$$k_\mu i\mathcal{M}^\mu = 0$$

A simple argument that also leads to this important identity is that the poton attaches to a vertex $-ie\bar{\psi}\gamma^\mu\psi = -ie\, j^\mu$ (where j^μ is here understood as an operator). Current conservation implies that $\partial_\mu \langle \ldots j^\mu(x) \ldots \rangle = 0$, such that, when writing

$$i\mathcal{M}^\mu(k) = \int d^4x\, e^{ikx} \langle \ldots j^\mu(x) \ldots \rangle,$$

we immediately obtain the Ward identity.
Now back to the photon polarisation sum. Without loss of generality, take $k^\mu = (k, 0, 0, k)$, such that the two transverse polarisations are

$\varepsilon_1^\mu = (0, 1, 0, 0)$ and $\varepsilon_2^\mu = (0, 0, 1, 0)$.

Then

$$\sum_{\substack{\text{photon} \\ \text{pol.}}} |\varepsilon_\mu^* i\mathcal{M}^\mu(k)|^2 = \sum_{i=1,2} \varepsilon_{i\mu}^* \varepsilon_{i\nu} \mathcal{M}^\mu(k) \mathcal{M}^{*\nu}(k) = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2$$

$$= |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2 + \underbrace{|\mathcal{M}^3(k)|^2 - |\mathcal{M}^0(k)|^2}_{=0 \text{ by Ward identity}} = -g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{*\nu}(k)$$

Hence, the rule for performing a photon polarisation sum is

$$\sum_{\substack{\text{photon}\\ \text{pol.}}} \varepsilon^*_\mu \varepsilon_\nu \longmapsto -g_{\mu\nu}$$

It is now straightforward, but get tedious, to obtain cross section for Compton scattering. The spin/polarisation sums lead to

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = \frac{e^2}{4} \underset{\text{photon } k}{g_{\mu\varrho}} \underset{\text{photon } k'}{g_{\nu\sigma}} \operatorname{tr}\Big\{ (\not p' + m)\left[\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not k' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right]$$
$$* (\not p + m)\left[\frac{\gamma^\sigma \not k \gamma^\varrho + 2\gamma^\varrho p^\sigma}{2p\cdot k} + \frac{\gamma^\varrho \not k' \gamma^\sigma - 2\gamma^\sigma p^\varrho}{2p\cdot k'}\right]\Big\}$$

(Note the transposition of the Dirac matrices in the second term in square brackets).

Now, write

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = \frac{e^4}{4}\left[\frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathrm{III}}{(2p\cdot k')(2p\cdot k)} + \frac{\mathrm{IV}}{(2p\cdot k')^2}\right]$$

The numerators are traces of Dirac matrices. As a homework problem, evaluate these and express them in terms of the Mandelstam variables. For this purpose, you can use the rules for evaluating Dirac traces, but we recommend to use a computer algebra package, such as FORM.

The answers are

$$\mathrm{I} = 16\left[2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right]$$

$$\mathrm{II} = \mathrm{III} = -8\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right]$$

$$\mathrm{IV} = 16\left[2m^4 + m^2(u-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right]$$

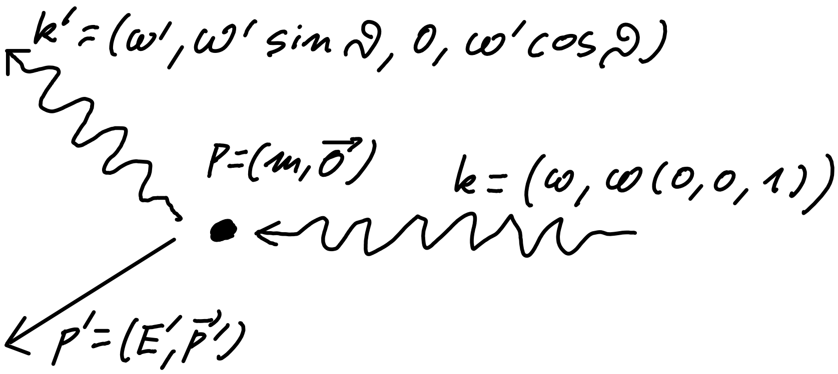
Notice that in order to obtain these, it is sufficient to compute e.g. I & II and then to take $k \leftrightarrow -k'$, what results in $s \leftrightarrow u$.

The final answer is

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

This formula applies to the scattering of photons on effectively free electrons, i.e. when the electron is bound, its binding energy is much less than the photon energy. We recall that the opposite case, the scattering on bound electrons, is called Rayleigh scattering.

We take the electron to be at rest and parametrise the kinematic situation as follows:



We aim to express the differential cross section as a function of ω and ϑ. Compton's formula relates the final photon wavelength to these parameters:

$$m^2 = p'^2 = (p+k-k')^2 = p^2 + 2p\cdot(k-k') - 2k\cdot k'$$

$$= m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\vartheta) \implies \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1-\cos\vartheta}{m}$$

For the photon energy, it follows

$$\omega' = \left(\frac{1}{\omega} + \frac{1-\cos\vartheta}{m} \right)^{-1} = \left(\frac{m + \omega(1-\cos\vartheta)}{\omega m} \right)^{-1} = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\vartheta)}$$

Now, integrate the final phase-space except for $d\vartheta$:

$$\int \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega'} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} (2\pi)^4 \delta^4(k'+p'-k-p)$$

$$= \int_0^\infty \frac{\omega'^2 d\omega\, d\Omega}{(2\pi)^3} \frac{1}{4\omega' E'} 2\pi\, \delta\left(\omega' + \sqrt{\omega^2 + \omega'^2 - 2\omega\omega'\cos\vartheta + m^2} - \omega - m \right)$$

$\vec{p}\,' = \vec{k} - \vec{k}'$

$p'^{0\,2} = (\vec{k} - \vec{k}')^2 + m^2 = E'^2$

$$= 2\pi \int_{-1}^{1} d\cos\vartheta \frac{1}{(2\pi)^2} \frac{\omega'}{4E'} \left| 1 + \frac{\omega' - \omega\cos\vartheta}{E'} \right|^{-1}$$

$$= \frac{1}{8\pi} \int_{-1}^{1} d\cos\vartheta \frac{\omega'}{\underbrace{E' + \omega'}_{=\omega+m} - \omega\cos\vartheta} = \frac{1}{8\pi} \int_{-1}^{1} d\cos\vartheta \frac{\omega'}{m + \omega(1-\cos\vartheta)}$$

$$= \frac{1}{8\pi} \int_{-1}^{1} d\cos\vartheta \frac{\omega'^2}{\omega m}$$

Substituting the matrix element and this into the cross-section yields

$$\frac{d\sigma}{d\cos\vartheta} = \frac{1}{4\, m\omega} \frac{1}{8\pi} \frac{\omega'^2}{\omega m} \frac{1}{4} \sum_{\text{spins}} |i\mathcal{M}|^2$$

Now, $p\cdot k = m\,\omega$ and $p\cdot k' = m\,\omega'$. We can substitute

$$\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right) + m^4 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right)^2$$

$$= \frac{m\omega'}{m\omega} + \frac{m\omega}{m\omega'} + 2m^2\left(\frac{1}{m\omega} - \frac{1}{m\omega'} \right) + m^4 \left(\frac{1}{m\omega} - \frac{1}{m\omega'} \right)^2$$

$$= \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2(1-\cos\vartheta) + (1-\cos\vartheta)^2$$

$$= \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 1 + \cos^2\vartheta = \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\vartheta$$

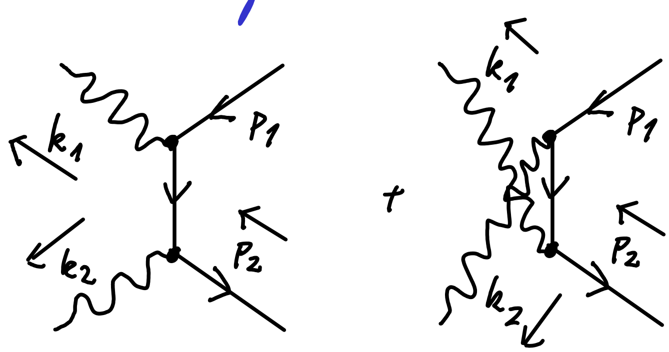
such that we obtain the <u>Klein-Nishima formula</u>

$$\frac{d\sigma}{d\cos\vartheta} = \underbrace{\frac{e^4}{16\pi}}_{=\pi\alpha^2} \frac{1}{m^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\vartheta\right]$$

The non-relativistic limit is when $\omega \to 0$ such that $\omega \to \omega'$. We obtain then the <u>Thomson cross section</u>

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} d\cos\vartheta\,(1+\cos^2\vartheta) = \frac{8\pi\alpha^2}{3m^2}$$

As another example for the use of crossing symmetry, we consider <u>pair annihilation</u>, $e^+e^- \to \gamma\gamma$. The leading-order diagrams are



The amplitude for this process can be obtained from the one for Compton scattering by making the replacements

$$p \mapsto p_1 \qquad k' \mapsto k_1 \qquad p' \mapsto -p_2 \qquad k \mapsto -k_2$$

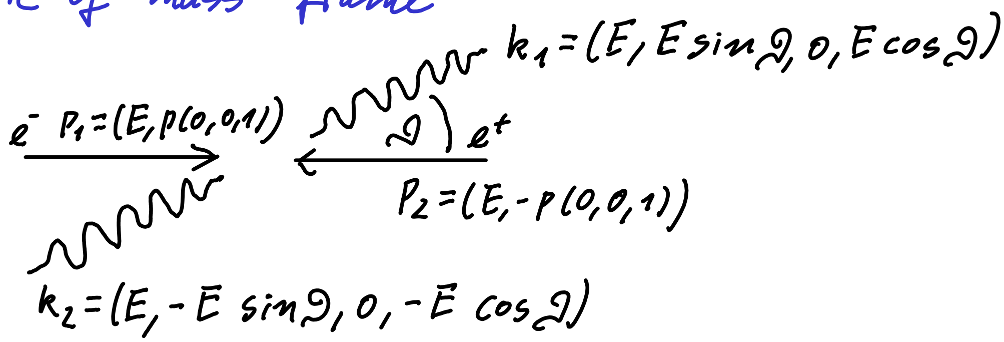
such that

$$\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2 = 2e^4\left[\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} + 2m^2\left(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\right) - m^4\left(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\right)^2\right]$$

Note the additional minus sign due to the replacement of a fermion by an antifermion.

Recall that pair creation is an important interaction of high energetic ($\omega > 2m$) photons with matter. In above diagrams, one of the photons is then an off-shell photon from the field of an electron or a nucleus.

Pair annihilation is routinely observed in e^+e^- colliders. Therefore, it is useful to parametrise the kinematics in the centre-of-mass frame



$$p_1 \cdot k_1 = E^2 - Ep\cos\vartheta \qquad p_1 \cdot k_2 = E^2 + Ep\cos\vartheta$$

$$\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} = 2\frac{E^4 + E^2p^2\cos^2\vartheta}{E^4 - E^2p^2\cos^2\vartheta} = 2\frac{E^2 + p^2\cos^2\vartheta}{E^2 - p^2 + p^2\sin^2\vartheta} = 2\frac{E^2 + p^2\cos^2\vartheta}{m^2 + p^2\sin^2\vartheta}$$

$$\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2} = \frac{2E^2}{E^4 - E^2p^2\cos^2\vartheta} = \frac{2}{m^2 + p^2\sin^2\vartheta}$$

The phase-space integral is

$$\int \frac{d^3k_1}{(2\pi)^3}\frac{1}{2E}\int\frac{d^3k_2}{(2\pi)^3}\frac{1}{2E}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$$

$$= \frac{1}{4E^2}\int_0^\infty \frac{k_1^2\,dk_1\,d\Omega}{(2\pi)^3}\,2\pi\,\delta(p_1+p_2-2E) = \frac{1}{8\pi}\int_{-1}^{1} d\cos\vartheta$$

$$v_1 - v_2 = \frac{2p}{E}$$

Substituting these pieces into the cross section yields

$$\frac{d\sigma}{d\cos\vartheta} = \frac{e^4}{32\pi pE}\left[\frac{E^4 + E^2p^2\cos^2\vartheta}{E^4 - E^2p^2\cos^2\vartheta} + \frac{2m^2}{m^2 + p^2\sin^2\vartheta} - \frac{2m^4}{(m^2 + p^2\sin^2\vartheta)^2}\right]$$

For $E \gg m$, this gives

$$\frac{d\sigma}{d\cos\vartheta} = \frac{e^4}{32\pi E^2}\,\frac{1+\cos^2\vartheta}{\sin^2\vartheta}$$

Finally, notice that since the two photons are identical, in order to obtain the total cross section, $d\cos\vartheta$ should only be integrated from 0 to 1.

2.3 Rutherford & Mott Scattering, Bremsstrahlung

The cross section for the process $e^-\mu^- \longrightarrow e^-\mu^-$

$$\frac{d\sigma_{e^-\mu^-\longrightarrow e^-\mu^-}}{d\Omega} = \frac{e^4}{128\pi^2}\,\frac{1}{\vec{p}^{\,2}}\,\frac{4+(1+\cos\vartheta)^2}{(1-\cos\vartheta)^2} \quad \text{(ultrarelativistic limit)}$$

diverges when $\vartheta \longrightarrow 0$. Clearly, the interpretation is that all electrons get deflected by the Coulomb potential of the muon (or vice versa). If the distance (or the impact parameter $\vec{b}$) is large, the deflection angle ϑ is however very small. In the analysis of collider data, these small ϑ events correspond to a small transfer of transverse momentum and are discarded. Only events with large ϑ correspond to central collisions and therefore probe the microscopic structure of the colliding particles. Nonetheless, small ϑ processes are of great importance in Nuclear-, Particle and Astrophysics, because these describe the energy loss of charged particles in matter (e.g. the atmosphere, the rock above an underground detector or scintillation detectors that collect the light caused by atomic recombination when ionising radiation traverses). The charged particles lose their energy by a large number of small ϑ scattering events with atomic electrons, therefore causing the atoms to ionise. An important question is therefore, how far muons get within matter, or what scintillation signal they may cause within a detector.

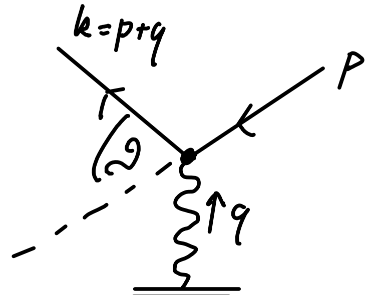
We therefore expand the above scattering cross-section for small ϑ:

$$(1+\cos\vartheta)^2 = 1+2\sqrt{1-\sin^2\vartheta}+1-\sin^2\vartheta \approx 1+2\left(1-\frac{1}{2}\sin^2\vartheta\right)+1-\sin^2\vartheta$$
$$= 4-2\sin^2\vartheta \approx 4-8\sin^2\frac{\vartheta}{2}$$

$$(1-\cos\vartheta)^2 = 1-2\sqrt{1-\sin^2\vartheta}+1-\sin^2\vartheta$$
$$\approx 1-2\left(1-\frac{1}{2}\sin^2\vartheta-\frac{1}{8}\sin^4\vartheta\right)+1-\sin^2\vartheta = \frac{1}{4}\sin^4\vartheta$$
$$\approx 4\sin^4\frac{\vartheta}{2}$$

$$\longrightarrow \frac{d\sigma_{e^-\mu^-\to e^-\mu^-}}{d\Omega} \underset{\vartheta\ll 1}{\approx} \frac{e^4}{128\pi^2}\frac{1}{\vec{p}^{\,2}}\frac{8\left(1-\sin^2\frac{\vartheta}{2}\right)}{4\sin^4\frac{\vartheta}{2}} \underset{\alpha=\frac{e^2}{4\pi}}{=} \frac{\alpha^2}{4\vec{p}^{\,2}}\frac{1-\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}}$$

If our above reasoning is correct, this simplified formula for small ϑ accounts for the Coulomb potential but not for the spin-interactions (interactions between magnetic dipole fields) of the scattering particles. To check this, we calculate the following process:



The photon is not an external particle here, but part of the Coulomb field of one of the scattering particles.

The Coulomb potential of a point charge Ze is

$$A_0(\vec{x}) = -\frac{Ze}{4\pi|\vec{x}|} \longrightarrow$$

$$A_0(\vec{q}) = -\frac{Ze}{4\pi}\int \frac{d^3x}{|\vec{x}|} e^{-i\vec{q}\cdot\vec{x}} = -\frac{Ze}{4\pi} 2\pi \int_{-1}^{1} d\cos\vartheta \int_0^\infty |\vec{x}|\, d|\vec{x}|\, e^{-i|\vec{q}||\vec{x}|\cos\vartheta}$$

$$= \frac{Ze}{2}\int_0^\infty d|\vec{x}| \left[\frac{e^{i|\vec{q}||\vec{x}|}}{i|\vec{q}|} - \frac{e^{-i|\vec{q}||\vec{x}|}}{i|\vec{q}|} \right]$$

$$= -\frac{Ze}{|\vec{q}|^2}\left[\cos(|\vec{q}||\vec{x}|)\right]_0^\infty = \frac{Ze}{\vec{q}^2} \quad \& \quad \vec{A}(\vec{q}) = \vec{0}$$

(The boundary term for $|\vec{x}| \to \infty$ can be neglected by replacing $e^{-i\vec{q}\cdot\vec{x}} \mapsto e^{i\vec{q}\cdot\vec{x} - \varepsilon|\vec{x}|}$.) The four-momentum is $q \approx (0, \vec{q})$, what corresponds to a negligible energy transfer (elastic scattering). The invariant matrix element is

$$i\mathcal{M} = -\bar{u}(p+q, r)\, ie^2 Z \gamma^0 u(p,s) \frac{1}{\vec{q}^2}$$

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \frac{1}{2}g_{\mu\nu}\mathbb{1}_4 2g^{\mu\nu} = 4\mathbb{1}_4$$

$$\not{p}\gamma_\mu = g^{\varrho\nu}p_\varrho\gamma_\nu\gamma_\mu = -\gamma_\mu\gamma_\nu p_\varrho g^{\varrho\nu} + \{\gamma_\mu, \gamma_\nu\} p_\varrho g^{\varrho\nu}$$

$$= -\gamma_\mu\gamma_\nu p_\varrho g^{\varrho\nu} + 2g_{\mu\nu}\mathbb{1}_4 p_\varrho g^{\varrho\nu} = -\gamma_\mu\not{p} + 2p_\mu$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{2}e^4Z^2 \operatorname{tr}\left[(\not{p}+\not{q}+m)\gamma^0(\not{p}+m)\gamma_0\right]\frac{1}{|\vec{q}|^4}$$

$$= \frac{1}{2}e^4Z^2\operatorname{tr}\left[-(\not{p}+\not{q})\not{p} + 2p^0(p^0+q^0) + m^2\right]\frac{1}{|\vec{q}|^4}$$

$$= 2e^2Z^4\left[2p^0(p^0+q^0) - p\cdot(p+q) + m^2\right]\frac{1}{|\vec{q}|^4}$$

$$\vec{q}^{\,2} = \left[(\vec{p}+\vec{q}) - \vec{p}\right]^2 = \underbrace{(\vec{p}+\vec{q})^2}_{=\vec{p}^{\,2},\ \text{energy conservation}} + \vec{p}^{\,2} - 2|\vec{p}||\vec{p}+\vec{q}|\cos\vartheta = 2\vec{p}^{\,2}(1-\cos\vartheta)$$

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$$

$$\longrightarrow \vec{q}^{\,2} = 4\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}$$

$$p\cdot(p+q) = \vec{p}^{\,2}+m^2-\vec{p}^{\,2}\cos\vartheta = m^2+2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2} \Rightarrow \vec{p}\cdot\vec{q} = -2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4Z^2\left(2\vec{p}^{\,2}+2m^2-m^2-2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}+m^2\right)\frac{1}{|q|^4}$$

$$\underset{\uparrow\ \frac{|\vec{p}|}{|p^0|}=\beta}{=} e^4Z^2\left(\vec{p}^{\,2}+m^2\right)\left(1-\beta^2\sin^2\frac{\vartheta}{2}\right)\frac{1}{4|\vec{p}|^4\sin^4\frac{\vartheta}{2}}$$

Now dropping the Lorentz-factors for one of the incoming particles in the cross-section formula:

$$\sigma = \frac{1}{v\,2\sqrt{\vec{p}^{\,2}+m^2}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{\vec{k}^{\,2}+m^2}}\,2\pi\,\delta(k^0-p^0)\ \frac{1}{2}\sum_{\text{Spins}}|\mathcal{M}|^2$$

$$=\frac{1}{4v(\vec{p}^{\,2}+m^2)}\frac{1}{(2\pi)^2}\int_{-1}^{1}d\cos\vartheta\int|\vec{k}|^2dk\,2\pi\,\delta\left(\sqrt{\vec{k}^{\,2}+m^2}-\sqrt{\vec{p}^{\,2}+m^2}\right)\frac{1}{2}\sum_{\text{Spins}}|\mathcal{M}|^2$$

$$=\frac{1}{8\pi}\int_{-1}^{1}d\cos\vartheta\ \frac{1}{2}\sum_{\text{Spins}}|\mathcal{M}|^2 = \frac{\vec{p}^{\,2}+m^2}{32\pi|\vec{p}|^4}e^4Z^2\int_{-1}^{1}d\cos\vartheta\,\frac{1-\beta^2\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}}$$

$$=\frac{\pi}{\beta^2|\vec{p}|^2}\frac{\alpha^2}{2}Z^2\int_{-1}^{1}d\cos\vartheta\,\frac{1-\beta^2\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}}$$

We obtain the <u>Mott cross-section:</u>

$$\frac{d\sigma}{d\cos\vartheta} = 2\pi\left(\frac{\alpha Z}{2\beta|\vec{p}|}\right)^2\frac{1-\beta^2\sin^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}} = 2\pi\,\frac{d\sigma}{d\Omega},$$

which is, by the way, the relativistic generalisation of the <u>Rutherford cross-section</u>, that describes, e.g. the scattering of α-particles of gold nuclei, which led to the discovery of the nucleus in the celebrated <u>Geiger-Marsden experiment</u> at Manchester.

Furthermore, we note the agreement with electron-muon scattering at small angles.
In order to calculate the energy loss in a material, we replace ϑ by the momentum transfer $|\vec{q}|$,

$$\frac{d\sigma}{d\cos\vartheta} = 2\pi\left(\frac{\alpha Z}{2\beta|\vec{p}|}\right)^2 \frac{16|\vec{p}|^4}{|\vec{q}|^4}\left(1-\beta^2\frac{\vec{q}^{\,2}}{4\vec{p}^{\,2}}\right)$$

$$\frac{d\vec{q}^{\,2}}{d\cos\vartheta} = 4\vec{p}^{\,2}\,\frac{d\,\frac{1}{2}(1-\cos\vartheta)}{d\cos\vartheta} = -2\vec{p}^{\,2}$$

$$\longrightarrow \frac{d\sigma}{d\vec{q}^{\,2}} = 2\,\frac{d\sigma}{d\cos\vartheta}\,\frac{d\cos\vartheta}{d\vec{q}^{\,2}} = \left(\frac{\alpha Z}{\beta}\right)^2\frac{8\pi}{|\vec{q}|^4}\left(1-\beta^2\frac{\vec{q}^{\,2}}{4\vec{p}^{\,2}}\right)$$

$\hookrightarrow \vec{p}\cdot\vec{q}$ can have either sign

Now in each scattering, an electron in the material picks up the energy

$$\Delta E = \frac{1}{2}\frac{\vec{q}^{\,2}}{m_e} \Rightarrow d\vec{q}^{\,2} = 2m_e\,d\Delta E$$

This is also the energy, that the traversing particle loses.

$$\frac{d\sigma}{2m_e\,d\Delta E} = 8\pi\left(\frac{Z\alpha}{\beta\,2m_e\Delta E}\right)^2\left(1-\beta^2\frac{2m_e\Delta E}{4(m_e\gamma\beta)^2}\right)$$

$$\frac{d\sigma}{d\Delta E} = 16\pi m_e\left(\frac{Z\alpha}{\beta\,2m_e\Delta E}\right)^2\left(1-\frac{\Delta E}{2m_e\gamma^2}\right)$$

$$= \frac{4\pi}{m_e}\left(\frac{Z\alpha}{\beta\Delta E}\right)^2\left[1-\frac{\Delta E}{2m_e}(1-\beta^2)\right]$$

Now the energy loss per unit distance is

$$-\frac{dE}{dx} = \varrho_{el}\int_{\Delta E_{min}}^{\Delta E_{max}} \Delta E\,\frac{d\sigma}{d\Delta E}\,d\Delta E$$

where ϱ_{el} is the number density of electrons

$$-\frac{dE}{dx} = e_{el}\left[\frac{4\pi}{m_e}\left(\frac{Z\alpha}{\beta}\right)^2\left(\log \Delta E - \frac{\Delta E}{2m_e}(1-\beta^2)\right)\right]_{\Delta E_{min}}^{\Delta E_{max}}$$

$$= e_{el}\,\frac{4\pi}{m_e}\left(\frac{Z\alpha}{\beta}\right)^2\left(\log\frac{\Delta E_{max}}{\Delta E_{min}} - \frac{\Delta E_{max}-\Delta E_{min}}{2m_e}(1-\beta)^2\right)$$

The maximum energy transfer is twice the electron energy in the center of mass system minus the rest energy

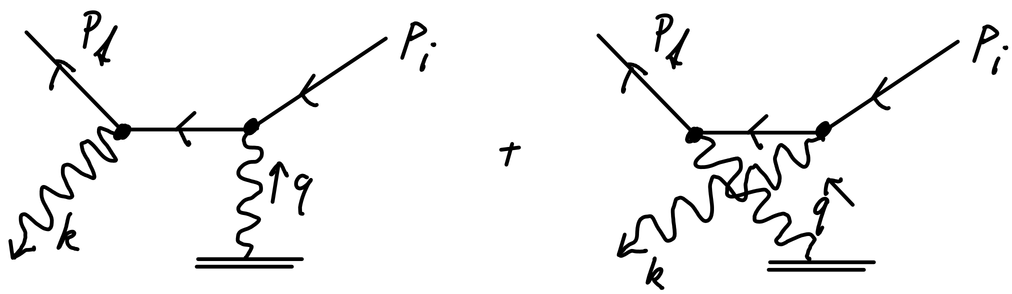
$$2\,m_e\gamma^2 - 2m_e = 2m_e\,\frac{1-1+\beta^2}{1-\beta^2} = 2m_e\,\frac{\beta^2}{1-\beta^2} = 2m_e\gamma^2\beta^2$$

The lower bound is given by the mean excitation potential $I \simeq 10\,eV\,Z$, where Z is the atomic number (not to be confused with the Z that occurs in the remaining formulae and that denotes the charge of the ionising particle). We finally obtain the Bethe formula:

$$-\frac{dE}{dx} = e_{el}\,\frac{4\pi}{m_e}\left(\frac{Z\alpha^2}{\beta}\right)^2\left[\log\frac{2m_e\beta^2}{I(1-\beta^2)} - \beta^2\right]$$

Note that for low velocities, the stopping power sharply increases. A beam of a certain energy therefore loses most of its energy after traveling a certain distance into the material. This makes ionising particle beams interesting for medical reasons.

The Bethe formula applies to particles that are much heavier than electrons, which have a more complicated stopping behaviour. In matter, besides scattering with atomic electrons, they get deflected in the field of nuclei. Due to the large charge of nuclei of large atomic number, electrons can get strongly accelerated, what leads to the emission of Bremsstrahlung typically in the form of X-rays. The diagrams associated with that process are



4-momentum conservation: $p_i + q = k + p_f$

Invariant matrix element:

$$i\mathcal{M} = \bar{u}(\vec{p}_f, s_f)\, \varepsilon^*_\mu (-ie\gamma^\mu) \frac{i(\not{p}_f + \not{k} + m)}{(p_f+k)^2 - m^2 + i\varepsilon} (-ie\gamma^0)\left(-\frac{eZ}{\vec{q}^{\,2}}\right) u(\vec{p}_i, s_i)$$

$$+ \bar{u}(\vec{p}_f, s_f)(-ie\gamma^0)\left(-\frac{eZ}{\vec{q}^{\,2}}\right) \frac{i(\not{p}_i - \not{k} + m)}{(p_i-k)^2 - m^2 + i\varepsilon} (-ie\gamma^\mu)\, \varepsilon^*_\mu\, u(\vec{p}_i, s)$$

$$\frac{1}{2}\sum_{pol} |i\mathcal{M}|^2 = \frac{e^6}{2}\frac{Z^2}{|\vec{q}|^4}\sum_\varepsilon$$

$$* \operatorname{tr}\Big\{ (\not{p}_f + m)\Big[\not{\varepsilon}^* \frac{(\not{p}_f + \not{k} + m)}{(p_f+k)^2 - m^2 + i\varepsilon}\gamma^0 + \gamma^0 \frac{(\not{p}_i - \not{k} + m)}{(p_i-k)^2 - m^2 + i\varepsilon}\not{\varepsilon}^* \Big](\not{p}_i + m)$$

$$* \Big[\gamma^0 \frac{(\not{p}_f + \not{k} + m)}{(p_f+k)^2 - m^2 + i\varepsilon}\not{\varepsilon} + \not{\varepsilon}\frac{(\not{p}_i - \not{k} + m)}{(p_i-k)^2 - m^2 + i\varepsilon}\gamma^0 \Big]\Big\}$$

$$=: \frac{e^2}{2}\frac{Z^2}{|\vec{q}|^4}\sum_\varepsilon \frac{1}{4}(T_1 + T_2 + T_3)$$

can skip the traces on the blackboard

$$(p_i - k)^2 - m^2 = -2p_i\cdot k \qquad (p_f + k)^2 - m^2 = 2p_f\cdot k$$

$$T_1 = \frac{1}{(p_f\cdot k)^2}\operatorname{tr}\left[(\not{p}_f + m)\not{\varepsilon}^*(\not{p}_f + \not{k} + m)\gamma^0(\not{p}_i + m)\gamma^0(\not{p}_f + \not{k} + m)\not{\varepsilon}\right]$$

$$\underset{\uparrow}{=} \frac{8}{(p_f\cdot k)^2}\Big[2(\varepsilon\cdot p_f)^2(m^2 + 2p_i^0 p_f^0 + 2p_i^0 k^0 - p_i\cdot p_f - p_i\cdot k)$$

choose basis with real ε^μ

$$+ 2\,\varepsilon\cdot p_f\,\varepsilon\cdot p_i\, k\cdot p_f + 2p_i^0 k^0\, k\cdot p_f - p_i\cdot k\, p_f\cdot k\Big]$$

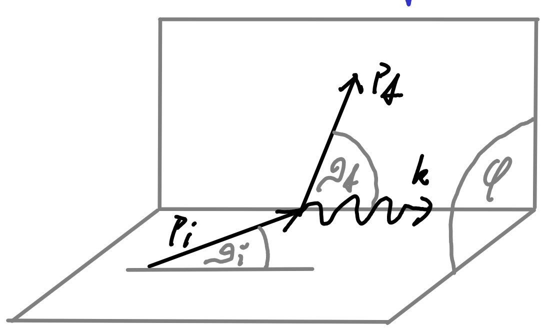
$$T_2 = \frac{1}{(p_i\cdot k)^2}\,\mathrm{tr}\left[(\not p_i + m)\not\varepsilon(\not p_i - \not k + m)\gamma^0(\not p_f + m)\gamma^0(\not p_i - \not k + m)\not\varepsilon^*\right]$$

$$= T_1(p_i \leftrightarrow -p_f)$$ (and $m \leftrightarrow -m$, in principle, but this sign does not matter in the present trace)

$$T_3 = -\frac{1}{(p_i\cdot k)}\frac{1}{(p_f\cdot k)}\,\mathrm{tr}\Big[\gamma^0(\not p_i - \not k + m)\not\varepsilon^*(\not p_i + m)\gamma^0(\not p_f + \not k + m)\not\varepsilon(\not p_f + m) + p_i \leftrightarrow -p_f\Big]$$

$$= \frac{16}{(p_i\cdot k)(p_f\cdot k)}\Big[\varepsilon\cdot p_i\,\varepsilon\cdot p_f(p_i\cdot k - p_f\cdot k + 2p_i\cdot p_f - 4p_i^0 p_f^0 - 2m^2) + (\varepsilon\cdot p_f)^2 k\cdot p_i - (\varepsilon\cdot p_i)^2 k\cdot p_f + p_i\cdot k\, p_f\cdot k - m^2 k^{0\,2} + k^0(k^0 p_i\cdot p_f - p_i^0 p_f\cdot k - p_f^0 p_i\cdot k)\Big]$$

Parametrisation of the momenta



$$k^\mu = (\omega, 0, 0, \omega)$$
$$p_f^\mu = (E_f, 0, |\vec{p}_f|\sin\vartheta_f, |\vec{p}_f|\cos\vartheta_f)$$
$$p_i^\mu = (E_i, |\vec{p}_i|\sin\vartheta_i\sin\varphi, |\vec{p}_i|\sin\vartheta_i\cos\varphi, |\vec{p}_i|\cos\vartheta_i)$$

Now, a real basis for transverse photon polarisations is

$$\varepsilon^{(1)} = (0,1,0,0), \quad \varepsilon^{(2)} = (0,0,1,0)$$

Such that

$$\sum_i (\varepsilon^{(i)}\cdot p_f)^2 = \vec{p}_f^{\,2}\sin^2\vartheta_f, \qquad \sum_i(\varepsilon^{(i)}\cdot p_i)^2 = \vec{p}_i^{\,2}\sin^2\vartheta_i$$

$$\sum_i(\varepsilon^{(i)}\cdot p_f)(\varepsilon^{(i)}\cdot p_i) = |\vec{p}_i||\vec{p}_f|\sin\vartheta_i\sin\vartheta_f\cos\varphi$$

$$k\cdot p_{i,f} = \omega(E_{i,f} - |\vec{p}_{i,f}|\cos\vartheta_{i,f})$$

The differential cross section is now

$$d\sigma = \frac{1}{v_i\,2E_i}\,\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega}\,\frac{d^3p_f}{(2\pi)^3}\,\frac{1}{2E_f}\,2\pi\,\delta(k^0 + p_f^0 - q^0 - p_i^0)\,\frac{1}{2}\sum_{\text{pol}}|i\mathcal{M}|^2$$

We can perform the integration over $|\vec{p}_f|$ using the δ-function and, as usual, express

$$d\Omega = \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi$$

Substituting above results then yields the Bethe-Heitler formula:

$$d\sigma = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{|\vec{p}_f|}{|\vec{p}_i||\vec{q}|^4}\frac{d\omega}{\omega}\, d\Omega_\gamma\, d\Omega_e$$

$$* \Bigg[\frac{|\vec{p}_f|^2 \sin^2\vartheta_f}{(E_f - |\vec{p}_f|\cos\vartheta_f)^2}\left(4E_i^2 - |\vec{q}|^2\right) + \frac{|\vec{p}_i|^2\sin^2\vartheta_i}{(E_i - |\vec{p}_i|\cos\vartheta_i)^2}\left(4E_f^2 - |\vec{q}|^2\right)$$

$$+ 2\omega^2 \frac{|\vec{p}_i|^2\sin^2\vartheta_i + |\vec{p}_f|^2\sin^2\vartheta_f}{(E_f - |\vec{p}_f|\cos\vartheta_f)(E_i - |\vec{p}_i|\cos\vartheta_i)} - 2\frac{|\vec{p}_f||\vec{p}_i|\sin\vartheta_i\sin\vartheta_f\cos\varphi}{(E_f - |\vec{p}_f|\cos\vartheta_f)(E_i - |\vec{p}_i|\cos\vartheta_i)}$$

$$* \left(4E_iE_f - |\vec{q}|^2 + 2\omega^2\right)\Bigg]$$

Now consider the limit of soft photon emission (small ω, $|\vec{q}|^2 = 4|\vec{p}|^2\sin^2\frac{\vartheta_{if}}{2}$, $|\vec{p}_i| \approx |\vec{p}_f| \approx |\vec{p}|$):

$$\left(\frac{d\sigma}{d\Omega_e}\right)_{elastic,0} = \left(\frac{\alpha Z}{2\beta|\vec{p}|}\right)^2 \frac{1 - \beta^2\sin^2\frac{\vartheta_{if}}{2}}{\sin^4\frac{\vartheta_{if}}{2}} = (\alpha Z E_{if})^2 \frac{1 - \frac{|\vec{q}|^2}{4E_{if}^2}}{|\vec{q}|^4}$$

And therefore

$$\left(\frac{d\sigma}{d\Omega_e}\right)_\gamma = \left(\frac{d\sigma}{d\Omega_e}\right)_{elastic,0} e^2 \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\sum_\varepsilon \left(\frac{\varepsilon\cdot p_f}{k\cdot p_f} - \frac{\varepsilon\cdot p_i}{k\cdot p_i}\right)^2 \sim \frac{d|\vec{k}|}{|\vec{k}|}$$

Without further ado, this is bad, because even for large $|\vec{q}|$, we would obtain an infinite cross section due to soft photon emission.

Before dealing with this issue, we note that through crossing of p_i and k, above amplitude can be employed to calculate the important process of pair creation from a high-energetic photon in the field of a nucleus.

Now back to the divergence in the cross section:
It is logarithmic and it occurs for $\omega \to 0$, which is why it is referred to as an infrared (IR) divergence.
The problem is due to the electron propagator from which the external photon is emitted and which is almost on shell when the photon is soft:

$$\frac{1}{(p-k)^2-m^2} \sim \frac{1}{-2p\cdot k} \sim \frac{1}{\omega}$$

We extract this divergence by giving a small, fictuous mass λ to the photon and integrating the cross section up to some energy $\overline{\omega}$:

$$\frac{d\sigma}{d\Omega_e} = \left(\frac{d\sigma}{d\Omega_e}\right)_{el,0} \frac{\alpha}{4\pi^2}\int_0^{\overline{\omega}} \frac{\vec{k}^2 d|\vec{k}|}{\underbrace{\omega}_{\omega=\sqrt{\vec{k}^2+\lambda^2}}} \int d\Omega \underbrace{\sum_\varepsilon}_{-g_{\mu\nu}} \left(\frac{\varepsilon\cdot p_f}{k\cdot p_f} - \frac{\varepsilon\cdot p_i}{k\cdot p_i}\right)^2$$

$$= \left(\frac{d\sigma}{d\Omega_e}\right)_{el,0} \frac{\alpha}{4\pi^2}\int_0^{\overline{\omega}} \frac{\vec{k}^2 d|\vec{k}|}{\omega} \int d\Omega \left(\frac{2p_i\cdot p_f}{k\cdot p_f\, k\cdot p_i} - \frac{m^2}{(k\cdot p_f)^2} - \frac{m^2}{(k\cdot p_i)^2}\right) =: \left(\frac{d\sigma}{d\Omega}\right)_{el,0} B$$

First for the last two terms:

$$\frac{d\Omega}{4\pi}\frac{m^2}{(k\cdot p_f)^2} = \frac{1}{2}\int_{-1}^{1} d\cos\vartheta \frac{m^2}{(\omega E_f - \beta|\vec{k}|E_f\cos\vartheta)^2} = \frac{1}{2}\frac{m^2}{\beta|\vec{k}|E_f^2}\left[\frac{1}{\omega-\beta|\vec{k}|\cos\vartheta}\right]_{-1}^{1}$$

$$= \frac{1}{2}\frac{m^2}{\beta|\vec{k}|E_f^2}\left(\frac{1}{\omega-\beta|\vec{k}|} - \frac{1}{\omega+\beta|\vec{k}|}\right)$$

$$\longrightarrow \frac{\alpha}{4\pi^2}\int_0^{\overline{\omega}} \frac{\vec{k}^2 d|\vec{k}|}{\omega}\int d\Omega \left(-\frac{m^2}{(k\cdot p_f)^2} - \frac{m^2}{(k\cdot p_i)^2} \right)$$

$$= -\frac{\alpha}{\pi}\int_0^{\overline{\omega}} \frac{|\vec{k}|\, d|\vec{k}|}{\sqrt{\vec{k}^2+\lambda^2}} \frac{m^2}{\beta E_f^2}\left(\frac{1}{\omega-\beta|\vec{k}|} - \frac{1}{\omega+\beta|\vec{k}|} \right)$$

$$= \frac{\alpha}{\pi}\frac{m^2}{\beta E_f^2}\frac{2}{1-\beta^2}\left[\frac{1}{i}\arctan\left(i\frac{\beta|\vec{k}|}{\omega} \right) - \beta \log\left(2(\vec{k}+\omega) \right) \right]_{|\vec{k}|=0}^{|\vec{k}|=\overline{\omega}}$$

$$\uparrow = \frac{\alpha}{\pi}\frac{2}{\beta}\left[\frac{1}{2}\log\frac{1+\beta}{1-\beta} - \beta\log\frac{2\overline{\omega}}{\lambda} \right]$$

$$\frac{1}{i}\arctan iz = \frac{1}{2}\log\frac{1+z}{1-z}$$

$$= \frac{\alpha}{\pi}\left[\frac{E_{i,f}}{|\vec{p}_{i,f}|}\log\frac{E_{i,f}+|\vec{p}_{i,f}|}{E_{i,f}-|p_{i,f}|} - 2\log\frac{2\overline{\omega}}{\lambda} \right]$$

For the first term, we introduce a Feynman parameter z as:

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax+b(1-x)]^2} \underset{z=2x-1}{\overset{\uparrow}{=}} \frac{1}{2}\int_{-1}^{1} dz \frac{1}{\left[\frac{1}{2}a(1+z)+\frac{1}{2}b(1-z)\right]^2}$$

$$k\cdot p_f\,(1+z) + k\cdot p_i\,(1-z) = k\cdot(p_i+p_f) + kz(p_f-p_i) = 2\,k\cdot l$$

$$l = \frac{1}{2}\left[(p_i+p_f) + z(p_f-p_i)\right]$$

Besides, parametrise the momentum transfer by the hyperbolic angle φ:

$$q^2 = -\vec{q}^{\,2} = -4\vec{p}^{\,2}\sin^2\frac{\vartheta}{2} = -4m^2\sinh^2\varphi$$

$$\sin^2\frac{\vartheta}{2} = \frac{m^2}{\vec{p}^{\,2}}\sinh^2\varphi$$

$$p_i\cdot p_f = p_i\cdot(p_i+q) = \vec{p}^{\,2}+m^2-\vec{p}^{\,2}\cos\vartheta = m^2+2\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}$$
$$= m^2+2m^2\sinh^2\varphi = m^2+2m^2\frac{1}{4}\left(e^{2\varphi}-2+e^{-2\varphi}\right)$$
$$= m^2\cosh 2\varphi$$

$$l^2 = \frac{1}{4}\left[p_i^2+p_f^2+2p_i\cdot p_f+2z(p_f^2-p_i^2)+z^2(p_f^2-2p_f\cdot p_i+p_i^2)\right]$$
$$= \frac{1}{4}\left[2m^2(1+\cosh 2\varphi)+2z^2m^2(1-\cosh 2\varphi)\right]$$
$$= m^2\left[\cosh^2\varphi - z^2\sinh^2\varphi\right]$$

Then,

$$\frac{\alpha}{4\pi^2}\int_{\omega<\overline{\omega}} d^3k\,\frac{1}{\omega\,(k\cdot p_f)(k\cdot p_i)} = \frac{\alpha}{4\pi^2}\int_{-1}^{1}dz\int_{\omega<\overline{\omega}}d^3k\,\frac{1}{\omega(k\cdot l)^2}$$

$$(k\cdot l)^2 = (\omega l_0-\vec{k}\cdot\vec{l})^2 = (\vec{k}^{\,2}+\lambda^2)l_0^2+(\vec{k}\cdot\vec{l})^2-2\omega l_0\,\vec{k}\cdot\vec{l}$$

$$\int_{\omega<\overline{\omega}}\frac{d^3k}{\omega(k\cdot l)^2} = 2\pi\int_{-1}^{1}d\cos\vartheta\int_0^{\overline{\omega}}\frac{d|\vec{k}|}{\omega}\,\frac{\vec{k}^{\,2}}{(\omega l^0-|\vec{k}||\vec{l}|\cos\vartheta)^2}$$
$$= 2\pi\int_0^{\overline{\omega}}\frac{|\vec{k}|}{|\vec{l}|}\frac{d|\vec{k}|}{\omega}\left[\frac{1}{\omega l^0-|\vec{k}||\vec{l}|}-\frac{1}{\omega l^0+|\vec{k}||\vec{l}|}\right]$$
$$= 4\pi\int_0^{\overline{\omega}}\vec{k}^{\,2}\,d|\vec{k}|\,\frac{1}{\omega}\,\frac{1}{\vec{k}^{\,2}l^2+\lambda^2 l^{0\,2}}$$

$$= 4\pi\left[\frac{1}{l^2}\left(\frac{-l^0}{i|\vec{l}|}\arctan\left(\frac{k}{\omega}\frac{i|\vec{l}|}{l^0}\right)+\log\left(2(|\vec{k}|+\omega)\right)\right)\right]_0^{\overline{\omega}}$$

$$= \frac{4\pi}{l^2}\left(-\frac{1}{2}\frac{l^0}{|\vec{l}|}\log\frac{l^0+|\vec{l}|}{l^0-|\vec{l}|}+\log\frac{2\overline{\omega}}{\lambda}\right)\Bigg]$$

$$\frac{1}{i}\arctan iz = \frac{1}{2}\log\frac{1+z}{1-z}$$

So we obtain for B (defined above):

$$B=\frac{\alpha}{\pi}\left\{2\log\frac{2\overline{\omega}}{\lambda}\left[-1+\frac{p_i\cdot p_f}{2}\int_{-1}^{1}dz\,\frac{1}{\ell^2}\right]+\frac{E_i}{2|\vec{p}_i|}\log\frac{E_i+|\vec{p}_i|}{E_i-|\vec{p}_i|}+\frac{E_i}{2|\vec{p}_i|}\log\frac{E_i+|\vec{p}_i|}{E_i-|\vec{p}_i|}\right.$$

$$\left.-\frac{p_i\cdot p_f}{2}\int_{-1}^{1}dz\,\frac{1}{\ell^2}\,\frac{\ell^0}{|\vec{\ell}|}\log\frac{\ell^0+|\vec{\ell}|}{\ell^0-|\vec{\ell}|}\right\}$$

$$\frac{p_i\cdot p_f}{2}\int_{-1}^{1}dz\,\frac{1}{\ell^2}=\frac{1}{2}m^2\cosh 2\varphi\int_{-1}^{1}dz\,\frac{1}{m^2[\cosh^2\varphi-z^2\sinh^2\varphi]}$$

$$=\cosh 2\varphi\int_{0}^{1}dz\,\frac{1}{\cosh^2\varphi-z^2\sinh^2\varphi}$$

$\sinh\varphi\cosh\varphi$
$=\frac{1}{2}\sinh(2\varphi)$

$$=\cosh 2\varphi\,\frac{1}{\sinh\varphi\cosh\varphi}\left[\operatorname{arctanh}\left(\frac{\sinh\varphi}{\cosh\varphi}z\right)\right]_0^1=2\varphi\coth 2\varphi$$

For the remaining integral, define $\xi=\frac{|\vec{\ell}|}{\beta\ell^0}$ and apply the soft radiation approximation $E_i\approx E_f\approx E$ and $|\vec{p}_i|\approx|\vec{p}_f|\approx|\vec{p}|$.
Then

$$\frac{1}{\beta^2}-\xi^2=\frac{1}{\beta^2}\left(1-\frac{|\vec{\ell}|^2}{\ell^{02}}\right)=\frac{1}{\beta^2\ell^{02}}\ell^2=\frac{m^2}{\beta^2\underbrace{\ell^{02}}_{\approx E^2}}[\cosh^2\varphi-z^2\sinh^2\varphi]$$

$$\approx\frac{m^2}{\beta^2E^2}[\cosh^2\varphi-z^2\sinh^2\varphi]$$

$\longrightarrow$

$$\xi^2=\frac{1}{\beta^2}\left[1-\frac{m^2}{E^2}(\cosh^2\varphi-z^2\sinh^2\varphi)\right]$$

$\sin^2\frac{\vartheta}{2}=\frac{m^2}{|\vec{p}|^2}\sinh^2\varphi$
$=\frac{1-\beta^2}{\beta^2}\sinh^2\varphi$

$$=\frac{1}{\beta^2}\left[1-(1-\beta^2)\left(1+\frac{\beta^2}{1-\beta^2}\sin^2\frac{\vartheta}{2}-z^2\frac{\beta^2}{1-\beta^2}\sin^2\frac{\vartheta}{2}\right)\right]$$

$$=\left[\cos^2\frac{\vartheta}{2}+z^2\sin^2\frac{\vartheta}{2}\right]\Rightarrow z^2=\frac{\xi^2-\cos^2\frac{\vartheta}{2}}{\sin^2\frac{\vartheta}{2}}$$

$$\ell^2=\ell^{02}(1-\beta^2\xi^2)\approx E^2(1-\beta^2\xi^2)$$

$$\frac{d\xi}{dz} = \frac{1}{\xi} z \sin^2\frac{\vartheta}{2} = \frac{1}{\xi} \sin\frac{\vartheta}{2}\sqrt{\xi^2 - \cos^2\frac{\vartheta}{2}}$$

$$\frac{p_i \cdot p_f}{2} \int_{-1}^{1} dz \, \frac{1}{l^2} \frac{l^0}{|\vec{l}|} \log \frac{l^0 + |\vec{l}|}{l^0 - |\vec{l}|}$$

$$= \frac{m^2}{E^2} \cosh 2\varphi \int_0^1 dz \, \frac{1}{1-\beta^2\xi^2} \frac{1}{\beta\xi} \log \frac{1+\beta\xi}{1-\beta\xi}$$

$$= \frac{1-\beta^2}{\beta} \frac{\cosh 2\varphi}{\sin\frac{\vartheta}{2}} \int_{\cos\frac{\vartheta}{2}}^{1} d\xi \, \frac{1}{1-\beta^2\xi^2} \frac{1}{\sqrt{\xi^2 - \cos^2\frac{\vartheta}{2}}} \log \frac{1+\beta\xi}{1-\beta\xi}$$

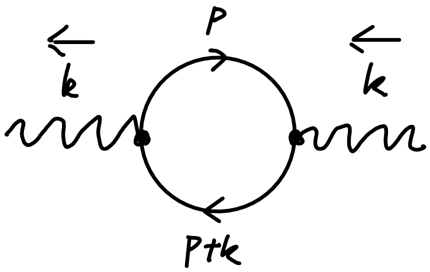
We finally have brought the cross section for soft photon emission to the form:

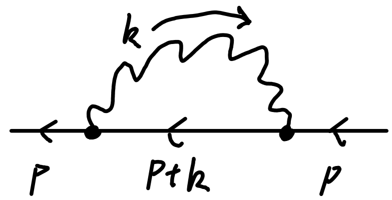
$$\left(\frac{d\sigma}{d\Omega_e}\right)_\gamma = \left(\frac{d\sigma}{d\Omega_e}\right)_{el,0} \frac{2\alpha}{\pi} \Bigg\{ (2\varphi \coth 2\varphi - 1) \log \frac{2\overline{\omega}}{\lambda} + \frac{1}{2\beta} \log \frac{1+\beta}{1-\beta}$$
$$- \frac{1}{2} \frac{1-\beta^2}{\beta} \frac{\cosh 2\varphi}{\sin\frac{\vartheta}{2}} \int_{\cos\frac{\vartheta}{2}}^{1} d\xi \, \frac{1}{1-\beta^2\xi^2} \frac{1}{\sqrt{\xi^2 - \cos^2\frac{\vartheta}{2}}} \log \frac{1+\beta\xi}{1-\beta\xi} \Bigg\}$$

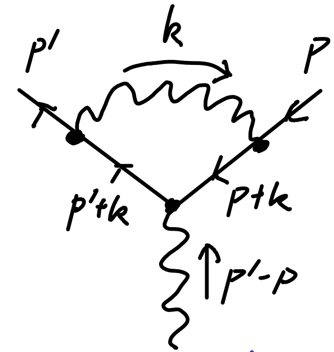
This expression explicitly exhibits the logarithmic IR-divergence in the unphysical photon mass λ. The resolution to this problem occurs when accounting for the loop effects, that occur at the same order in perturbation theory.

2.4 QED to one Loop

There are three elementary one-loop corrections to the two- and three-point functions of QED:







vacuum polarisation $i\Pi_{\mu\nu}(k)$ electron self-energy $-i\Sigma(p)$ vertex correction $\Lambda_\mu(p',p)$

Vacuum Polarisation

$$i\Pi_{\mu\nu}(k) = \underset{\substack{\uparrow \\ \text{fermion} \\ \text{loop}}}{-}(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left[\gamma_\mu \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\gamma_\nu \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\varepsilon}\right]$$

$$= -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{N_{\mu\nu}(p,k)}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)}$$

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2+p\cdot k-m^2)$$

From power counting, we conclude that the ultraviolet (UV) divergence of this integral is quadratic. However, there are crucial cancellations that render the divergence logarithmic only. Regulation by a momentum-cutoff would have the disadvantage of breaking Lorentz-invariance. In the Pauli-Villers scheme, a heavy particle that couples with the negative loop contribution than the original photon/electron loop is introduced. For the case of the photon loop, this has the disadvantage of breaking gauge invariance. Within <u>dimensional regularisation</u>, we perform the loop integral in

$d = 4 - \varepsilon$

dimensions, such that the divergence is isolated within terms that become infinite as $\varepsilon \longrightarrow 0$. Clearly, this is just a calculational trick and it is not how the effects of quantum fluctuations are UV-regulated in nature. Fortunately, the theory remains predictive, independent of the regulation scheme, once the electron charge and its mass are determined experimentally, but the down side is, that this also obscures our view on what ultimately regulates the theory in the UV. ("UV" completion, e.g. through asymptotic freedom or within string theory.)

That being said, we obtain

$$i\Pi_{\mu\nu}(k) = -4e^2\mu^{\varepsilon}\int\frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p,k)}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)}$$

The parameter μ is of mass-dimension one, and its purpose is to retain the mass-dimension two for $\Pi_{\mu\nu}$.
Its value will be fixed by a renormalisation condition (prediction of a physically observeable quantity by the theory).
Again, we get the Feynman parameter trick out of the box:

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax+b(1-x)]^2}$$

such that

$$\begin{aligned}
i\Pi_{\mu\nu}(k) &= -4e^2\mu^{\varepsilon}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}}{[x(p+k)^2-xm^2+(1-x)(p^2-m^2)+i\varepsilon]^2}\\
&= -4e^2\mu^{\varepsilon}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}}{[p^2+x\,2k\cdot p+xk^2-m^2+i\varepsilon]^2}\\
&= -4e^2\mu^{\varepsilon}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}}{[(p+kx)^2+k^2x(1-x)-m^2+i\varepsilon]^2}
\end{aligned}$$

We complete the square by introducing

$l = p + kx \iff p = l - kx$

$\Delta = m^2 - k^2 x(1-x)$

$$N_{\mu\nu} = 2 l_\mu l_\nu + 2x^2 k_\mu k_\nu - 2x k_\mu k_\nu - g_{\mu\nu}(l^2 + x^2 k^2 - x \cdot k - m^2)$$
$$-2x l_\mu k_\nu - 2x l_\nu k_\mu + l_\mu k_\nu + l_\nu k_\mu - g_{\mu\nu}(-2x k \cdot l + l \cdot k)$$

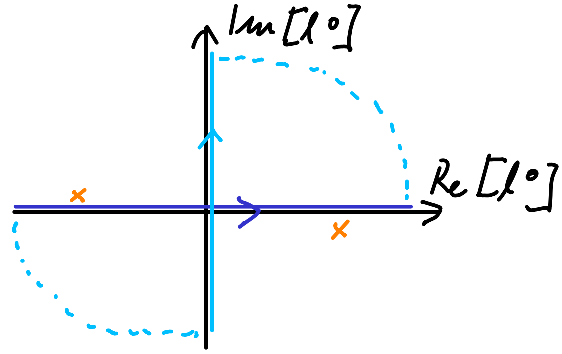
The terms odd in l do not contribute to the integral, such that we drop these in the following. We obtain

$$i\Pi_{\mu\nu}(k) = -4e^2 \mu^\varepsilon \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{N_{\mu\nu}}{[l^2 - \Delta + i\varepsilon]^2} = -4e^2 \mathcal{N}_{\mu\nu}$$

It will prove useful to note that

$$\int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{[l^2 - \Delta + i\varepsilon]^2} = \frac{1}{d} g_{\mu\nu} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2 - \Delta + i\varepsilon]^2}$$

what follows from the facts that the term must be Lorentz-invariant, that it is zero for $\mu \neq \nu$ and that $g^{\mu\nu} g_{\mu\nu} = d$. Now, we could perform the integral straightforwardly by first doing the l^0 integral and making use of the residue theorem. The disadvantage is then, that we lose the manifest Lorentz invariance at an intermediate step of the calculation. To get away without breaking the Lorentz symmetry, consider the poles in the Feynman propagators (this is the first instance in these lectures where the ε-prescription becomes effective) and deform the integration contour as:



This is the celebrated <u>Wick-rotation.</u> When we introduce

$l_E = (-il^0, \vec{l})$, then we note that we can replace

$$\int_{-\infty}^{\infty} dl^0 \longrightarrow i \int_{-\infty}^{\infty} dl_E^0 \quad \text{and} \quad l^2 = l^{02} - \vec{l}^{\,2} = -l_E^{02} - \vec{l}^{\,2} = -l_E^2 \,, \quad \text{where}$$

$l_E = (l_E^0, \vec{l}\,)$ is a <u>Euclidean</u> four momentum.

In general, we may write

$$I_{r,m} = \int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^r}{[l^2 - \Delta + i\varepsilon]^m} = i(-1)^{r-m} \int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)^r}{[l_E^2 + \Delta]^m}$$

The integral over the solid angle in d dimensions is

$$\int d\Omega_{d-1} = 2 \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \quad \text{such that} \quad \int d^d l_E = \int_0^\infty d|l_E| \, |l_E|^{d-1} d\Omega_{d-1}$$

where $|l_E| = \sqrt{l_E^2}$. In order to abbreviate our notation, in the following, we write $l_E \equiv |l_E|$, without inducing any confusion. Now,

$$\int_0^\infty dx \frac{x^M}{[x^2 + A]^N} = \frac{\Gamma(\frac{M+1}{2}) \Gamma(N - \frac{M+1}{2})}{A^{N - \frac{M+1}{2}} \, 2\Gamma(N)}$$

such that

$$I_{r,m} = i(-1)^{r-m} \frac{2}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty d l_E \frac{l_E^{2r+d-1}}{[l_E^2 + \Delta]^m}$$

$$= i(-1)^{r-m} \frac{1}{(4\pi)^{2-\frac{\varepsilon}{2}}} \left(\Delta \right)^{-m+r+2-\frac{\varepsilon}{2}} \frac{\Gamma(r+2-\frac{\varepsilon}{2}) \Gamma(m-r-2+\frac{\varepsilon}{2})}{\Gamma(2-\frac{\varepsilon}{2}) \Gamma(m)}$$

For our special case, we can now express

$$N_{\mu\nu} = \int_0^1 dx \left(\frac{2}{d} - 1 \right) g_{\mu\nu} \mu^\varepsilon I_{1,2} + \left[-2x(1-x) k_\mu k_\nu + x(1-x) k^2 g_{\mu\nu} + g_{\mu\nu} m^2 \right] \mu^\varepsilon I_{0,2}$$

The singularities are now contained in

$$\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon) \quad \text{for} \quad r,m = 0,2$$

$$\Gamma\left(-1 + \frac{\varepsilon}{2}\right) = -\frac{2}{\varepsilon} + \gamma_E - 1 + \mathcal{O}(\varepsilon) \quad \text{for} \quad r,m = 1,2$$

In order to get the finite terms right, the remaining factors must be expanded to order ε, e.g.

$$x^{N+\varepsilon} = x^N x^\varepsilon = x^N e^{\varepsilon \log x} = x^N \left(1 + \varepsilon \log x + \mathcal{O}(\varepsilon^2)\right)$$

It is then useful to define

$$\Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi$$

This way, one obtains that

$$\mu^\varepsilon I_{0,2} = \frac{i}{16\pi^2}\left(\Delta_\varepsilon - \log\frac{\Delta}{\mu^2} + \mathcal{O}(\varepsilon)\right)$$

$$\mu^\varepsilon I_{1,2} = \frac{i}{16\pi^2}\,\Delta\left(1 + 2\Delta_\varepsilon - 2\log\frac{\Delta}{\mu^2} + \mathcal{O}(\varepsilon)\right)$$

When collecting all terms, a number of cancellations occurs, leaving us with

$$N_{\mu\nu} = \int_0^1 dx\, \frac{i}{16\pi^2}\left(\Delta_\varepsilon - \log\frac{\Delta}{\mu^2}\right)\left(g_{\mu\nu} k^2 - k_\mu k_\nu\right) 2x(1-x)$$

and

$$\Pi_{\mu\nu}(k) = -\left(g_{\mu\nu} k^2 - k_\mu k_\nu\right)\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\varepsilon - \log\frac{m^2 - x(1-x)k^2}{\mu^2}\right]$$

$$=: -\left(g_{\mu\nu} k^2 - k_\mu k_\nu\right)\Pi(k^2)$$

As a consistency check, we note that this result observes the Ward identity $k^\mu \Pi_{\mu\nu}(k) = 0$.

Notice also that the factor $\left(\frac{2}{d} - 1\right)$ cancels the singularity in $I_{1,2}$ for $d=2$ ($\varepsilon = 2$). This indicates that the divergence, that was indicated by power counting to be quadratic, is in fact only logarithmic.

Now for the meaning of $\Pi_{\mu\nu}$. It appears as a correction to the two-point function as a geometric series:

$$= -\frac{i g_{\mu\nu}}{k^2} + \frac{-i g_{\mu\rho}}{k^2}\left[-i\left(k^2 g^{\rho\sigma} - k^\rho k^\sigma\right)\Pi(k^2)\right]\frac{-i g_{\sigma\nu}}{k^2} + \dots$$

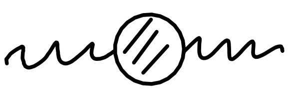
$$= -\frac{i g_{\mu\nu}}{k^2} + \frac{-i g_{\mu\rho}}{k^2}\underbrace{\left(\delta^\rho_\nu - \frac{k^\rho k_\nu}{k^2}\right)}_{=:\Delta^\rho{}_\nu}\left(-\Pi(k^2)\right) + \dots = -\frac{i g_{\mu\nu}}{k^2} + \frac{-i g_{\mu\rho}}{k^2}\Delta^\rho{}_\nu \sum_{n=1}^{\infty}\left(-\Pi(k^2)\right)^n$$

note $\Delta^\rho{}_\mu \Delta^\mu{}_\nu = \Delta^\rho{}_\nu$

$$= -\cancel{\frac{i g_{\mu\nu}}{k^2}} + \frac{-i}{k^4}\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right)\frac{1}{1+\Pi(k^2)} + \frac{i}{k^4}\left(\cancel{k^2 g_{\mu\nu}} - k_\mu k_\nu\right)$$

This identity is useful for our one-loop expression for $\Pi_{\mu\nu}$, but more generally for any one-particle-irreducible (1PI) expression for $\Pi_{\mu\nu}$, in particular those valid to higher orders in perturbation theory. The expression 1PI means that no diagram contributing to $\Pi_{\mu\nu}$ decomposes into two sub-diagrams when cutting just one line.

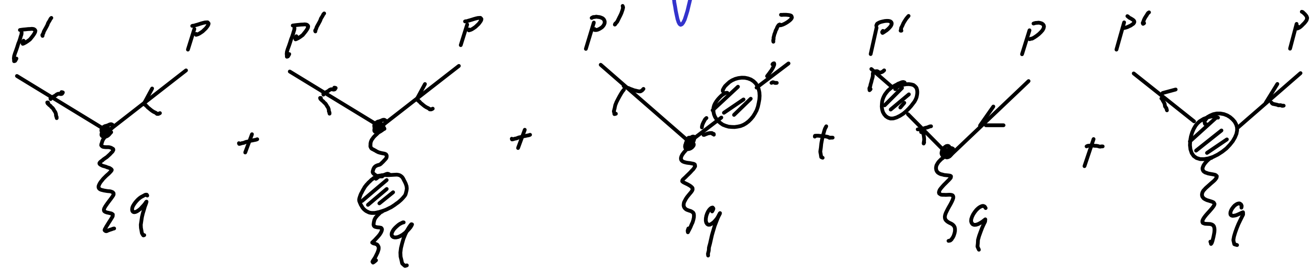
At least one end of this propagator attaches to a Feynman diagram. When we consistently sum over all possible insertions, all terms $\propto k_\mu$ cancel due to the Ward identity. We can therefore write

 $= \dfrac{-i g_{\mu\nu}}{k^2\left(1+\Pi(k^2)\right)}$

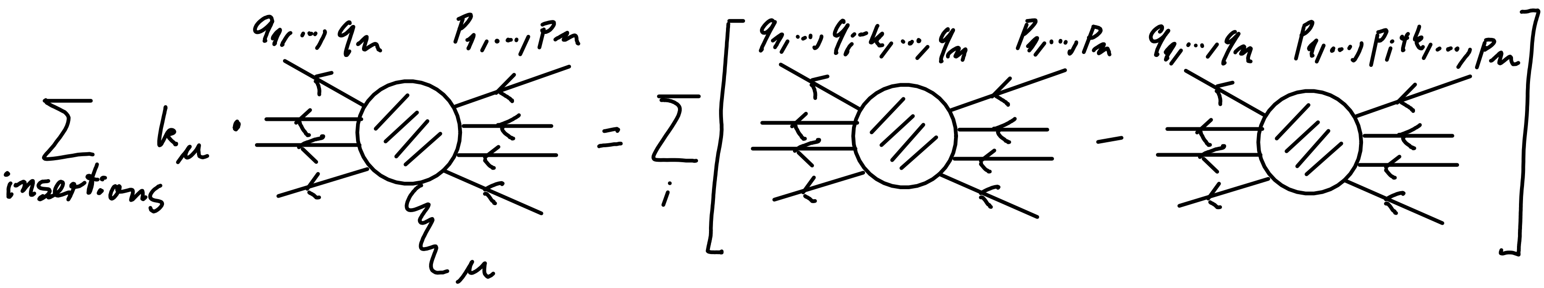
This propagator still has a pole at $k^2=0$, which is a consequence of the fact that gauge invariance requires the photon to be massless, a property that remains valid after inclusion of quantum corrections. The effect of $\Pi(k^2=0)$ is to shift the residue of the pole. It is customary to define $\delta Z_3 = -\Pi(0)$, such that the residue is multiplied by $(1-\delta Z_3)^{-1}$.

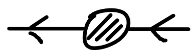
Now suppose that we determine the electron charge e

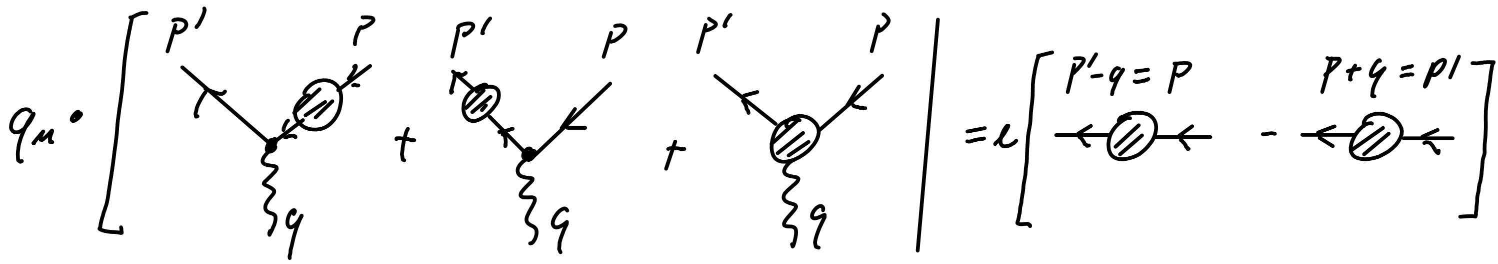
through the interaction the particle experiences in an external field. Up to next-to-leading order, we must include



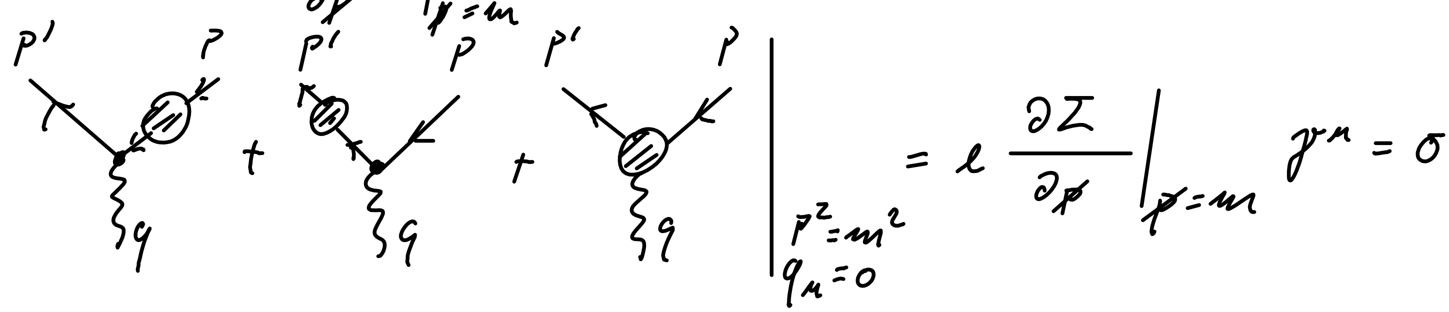
The last three diagrams do not contribute to the determination of the charge. The demonstration of this is an opportunity for the Ward-Takahashi identity to shine:



When applied to , this equation reads



Suppose we measure the electric charge of an on-shell electron ($p^2 = m^2$) in a slowly varying electric field ($q_\mu \approx 0$). Furthermore (as we explain later), impose the renormalisation condition $\left.\frac{\partial \Sigma(p)}{\partial \not{p}}\right|_{\not{p}=m} = 0$. Then,



The coupling that we measure therefore gets divided by $\sqrt{Z_3}$. Since we would still like to call the measured quantity e, we replace it in the Lagrangian with the "bare" charge

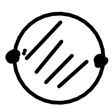
$$-ie A_\mu \bar{\psi} \gamma^\mu \psi \longrightarrow -i \underbrace{\frac{e}{\sqrt{Z_3}}}_{\text{bare charge}} A_\mu \bar{\psi} \gamma^\mu \psi$$

The factor Z_3 is therefore called the electric <u>charge renormalisation</u>.
We come to a systematic redefinition of the Lagrangian suitable for renormalised perturbation theory at the end of this Chapter. For now, we note that

$$\delta Z_3 = -\Pi(0) = -\frac{\alpha}{3\pi}\left[\Delta_\varepsilon - \log\frac{m^2}{\mu^2}\right]$$

<u>Electron Self-Energy</u>

For the fermion field, we define the self-energy with the opposite sign as for the gauge bosons:

$i\Sigma = -$ 

The resummed propagator is then

$$\int d^4x \, \langle \Omega | T \psi_H(x) \bar{\psi}_H(y) | \Omega \rangle e^{ip\cdot(x-y)}$$

$$= \frac{i}{\not p - m + i\varepsilon} + \frac{i}{\not p - m + i\varepsilon}\left(-i\Sigma(p)\right)\frac{i}{\not p - m + i\varepsilon} + \ldots = \frac{i}{\not p - m + i\varepsilon}\sum_{n=0}^{\infty}\left[\frac{\Sigma}{\not p - m + i\varepsilon}\right]^n$$

$$= \frac{i}{\not p - m + i\varepsilon}\,\frac{1}{1 - \frac{\Sigma(p)}{\not p - m + i\varepsilon}} = \frac{i}{\not p - m - \Sigma(p) + i\varepsilon}$$

The one-loop contribution has an IR-divergence that we again fix by a photon mass λ, such that

$$-i\Sigma(p) = -\mu^{\varepsilon} e^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-\lambda^2+i\varepsilon} \gamma^\mu \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\varepsilon} \gamma_\mu$$

The same calculational strategies as above lead to the result

$$\Sigma(p) = A(p^2) + B(p^2)\not{p}$$

$$A(p^2) = \frac{e^2 m}{4\pi^2} \int_0^1 dx \left(\Delta_\varepsilon - \frac{1}{2} - \log \frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2} \right)$$

$$B(p^2) = -\frac{e^2}{8\pi^2} \int_0^1 dx\,(1-x) \left(\Delta_\varepsilon - 1 - \log \frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2} \right)$$

The self-energy now multiplies the residue by a factor

$$\frac{1}{1-\frac{\partial\Sigma(p)}{\partial\not{p}}\Big|_{\not{p}=m}} = (1-\delta Z_2)^{-1}, \text{ because}$$

$$\not{p}-m-\Sigma(p)\Big|_{\not{p}\approx m} = (\not{p}-m)\left(1-\frac{\partial\Sigma(p)}{\partial\not{p}}\Big|_{\not{p}=m}\right)$$

Ultimately, we will add a counterterm that cancels this field-strength renormalisation.

Besides, the mass is shifted by

$$-\delta m = \Sigma(\not{p}=m) = A(m^2) + m\,B(m^2)$$

$$= \frac{3\alpha m}{4\pi}\left[\Delta_\varepsilon - \frac{1}{3} - \frac{2}{3}\int_0^1 dx\,(1+x)\log\frac{m^2x^2}{\mu^2}\right]$$

The field-strength renormalisation is

$$\delta Z_2 = \frac{\partial\Sigma}{\partial\not{p}}\Big|_{\not{p}=m} = \frac{\partial A(p^2)}{\partial\not{p}}\Big|_{\not{p}=m} + B(m^2) + m\frac{\partial B(p^2)}{\partial\not{p}}\Big|_{\not{p}=m}$$

$$= \frac{\alpha}{4\pi}\left[-\Delta_\varepsilon - 4 + \log\frac{m^2}{\mu^2} - 2\log\frac{\lambda^2}{m^2}\right]$$

Vertex Correction

$$-ie\mu^{\frac{\varepsilon}{2}}\Lambda_\mu(p',p)$$

$$= \left(-ie\mu^{\frac{\varepsilon}{2}}\right)^3 \int \frac{d^dk}{(2\pi)^d}(-i)\frac{g_{\varrho\sigma}}{k^2-\lambda^2+i\varepsilon}\gamma^\sigma \frac{i[\not p'+\not k+m]}{(p'+k)^2-m^2+i\varepsilon}\gamma_\mu \frac{i[\not p+\not k+m]}{(p+k)^2-m^2+i\varepsilon}\gamma^\varrho$$

The reason for the normalisation is as follows: in the d-dimensional Lagrangian, all terms must have mass-dimension d (such that the action is dimensionless).

From $\dfrac{i(\not p+m)}{p^2+m^2+i\varepsilon} = \int d^dx\, e^{ip\cdot x}\langle 0|\psi(x)\bar\psi(0)|0\rangle$,

we conclude that the dimension of ψ is $[\psi] = \frac{d-1}{2}$.

Similarly, $[A_\mu] = \frac{d-2}{2}$. The interaction term should therefore be $-\mu^{\frac{\varepsilon}{2}}e\bar\psi\gamma^\mu A_\mu\psi$ which for dimensionless e has the correct dimension $\frac{4-d}{2}+2\frac{d-1}{2}+\frac{d-2}{2}=d$. Note the consistency with the prefactors attached to the vacuum polarisation and self-energy diagrams.

We will eventually sandwich Λ_μ between on-shell spinors. As this also makes the result more manageable, (i.e. use the identities $(\not p-m)u(p)=0$, $(\not p+m)v(p)=0$) we calculate

$$i\bar u(p')\Lambda_\mu u(p) = e^2\mu^\varepsilon \int \frac{d^dk}{(2\pi)^d}\frac{\bar u(p)\gamma_\varrho(\not p'+\not k+m)\gamma_\mu(\not p+\not k+m)\gamma^\varrho u(p)}{D_0D_1D_2}$$

$$= e^2\mu^\varepsilon\int\frac{d^dk}{(2\pi)^d}\frac{N_\mu}{D_0D_1D_2}$$

$$D_0 = k^2-\lambda^2+i\varepsilon$$

$$D_1 = (k+p')^2-m^2+i\varepsilon$$

$$D_2 = (k+p)^2-m^2+i\varepsilon$$

$$N_\mu = \bar{u}(p)\Big[(-2+d)k^2\gamma_\mu + 4p\cdot p'\gamma_\mu + 4(p+p')\cdot k\,\gamma_\mu + 4mk_\mu - 4\not{k}(p+p')_\mu + 2(2-d)\not{k}\,k_\mu\Big]u(p)$$

What can be evaluated to

$$i\bar{u}(p')\Lambda_\mu u(p) = i\bar{u}(p')\left[G(q^2)\gamma_\mu + H(q^2)(p+p')_\mu\right]u(p)$$

where

$$G(q^2) = \frac{\alpha}{4\pi}\Bigg[\Delta_\varepsilon - 2 - 2\int_0^1 dx_1\int_0^{1-x_1} dx_2\,\log\frac{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}{\mu^2}$$
$$+ \int_0^1 dx_1\int_0^{1-x_1} dx_2\,\frac{-2(x_1+x_2)^2m^2 - x_1x_2q^2 - 4m^2 + 2q^2 + 2(x_1+x_2)(4m^2-q^2)}{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}\Bigg]$$

$$H(q^2) = \frac{\alpha}{4\pi}\int_0^1 dx_1\int_0^{1-x_1} dx_2\,\frac{-2m(x_1+x_2) + 2m(x_1+x_2)^2}{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}$$

As we are often interested in soft momentum exchange, $p-p' = q \approx 0$, it is useful to introduce $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$ (do not confuse with the tensor for two-spinors) and to note the <u>Gordon identity</u>

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + i\,\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

Such that

$$\bar{u}(p')\Lambda_\mu(p,p')u(p) = \bar{u}(p')\left[\left(G(q^2) + 2mH(q^2)\right)\gamma_\mu - iH(q^2)\sigma_{\mu\nu}q^\nu\right]u(p)$$
$$= \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]u(p)$$

where we have introduced the <u>form factors</u>

$F_1(q^2) = G(q^2) + 2mH(q^2)$

$F_2(q^2) = -2m\,H(q^2)$

For soft momentum exchange, one obtains the comparably simple results

$$G(0) = \frac{\alpha}{4\pi}\left[\Delta_\varepsilon + 6 - \log\frac{m^2}{\mu^2} - 2\log\frac{\lambda^2}{m^2}\right]$$

$$H(0) = -\frac{\alpha}{4\pi}\,\frac{1}{m}$$

The corrections to the form factor $F_1(0)$ are conventionally denoted as

$$\delta Z_1 = -G(0) - 2m\,H(0) = \frac{\alpha}{4\pi}\left[-\Delta_\varepsilon - 4 + \log\frac{m^2}{\mu^2} - 2\log\frac{\lambda^2}{m^2}\right]$$

Renormalised Perturbation Theory

In the NLO corrections to QED, we have found various UV-singular contributions. These must now be dealt with in order for QED to remain a predictive theory. Moreover, the NLO corrections are singular and therefore by no means small. This raises the important question whether perturbation theory is a valid approch at all. The <u>renormalisation</u> program successfully addresses these issues.

For this purpose, we modify the QED Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overline{\psi}(i\partial\!\!\!/ - e A\!\!\!/ - m)\psi$$

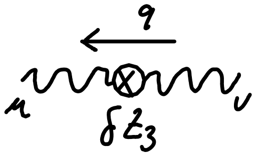
as

$$\mathcal{L}_{QED} \longmapsto \mathcal{L}_{QED} + \Delta\mathcal{L}$$

where

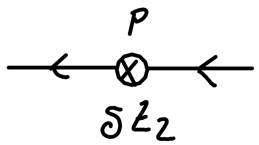
$$\Delta\mathcal{L} = -\frac{1}{4}\delta Z_3 F_{\mu\nu}F^{\mu\nu} + i\,\delta Z_2\,\overline{\psi}\partial\!\!\!/\psi - \delta Z_2\, m\overline{\psi}\psi + \delta m\overline{\psi}\psi + \delta Z_2\,\delta m\overline{\psi}\psi$$
$$- \underbrace{\delta Z_1}_{=\delta Z_2 \text{ (to be shown below)}}\, e\,\overline{\psi}\gamma^\mu\psi A_\mu$$

This is called the bare Lagrangian, $m+\delta m$ the bare mass & $\sqrt{1+\delta Z_2}\,\psi$, $\sqrt{1+\delta Z_3}\,A_\mu$ the bare fields. We obtain additional Feynman rules:

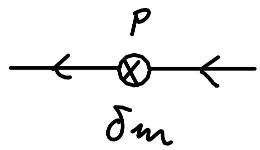


$$= -i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta Z_3$$

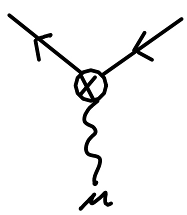
(recall our earlier argument that we are effectively allowed to drop the second term).



$$= i\left(\not{p}\,\delta Z_2 - m\,\delta Z_2\right)$$



$$= -i\left(\delta Z_2 + 1\right)\delta m \simeq -i\,\delta m\left(1+\mathcal{O}(\alpha)\right)$$



$$= -ie\gamma^\mu\,\delta Z_1$$

We call δm, $\delta Z_{1,2,3}$ counterterms. They are counted as order $\alpha = \frac{e^2}{4\pi}$ and should therefore consistently be added to in perturbation theory to the one-loop corrections.

Now experimentally, we would measure the electron mass on-shell ($p^2 = m^2$), i.e. by observing the energy-levels in an atom or a trajectory in an electromagnetic field. The charge can e.g. by measured by soft momentum transfer in a static field ($q = p - p' \simeq 0$). These observables include the tree-level LO as well as the NLO loop and counter-term interactions. This leads us to the renormalisation conditions:

$$\left.\frac{\partial \Sigma}{\partial \not{p}}\right|_{\not{p}=m} = 0 \quad \text{(unit residue for on-shell electron)}$$

$\Sigma(\not p = m) = 0$ (the observable electron mass is m)

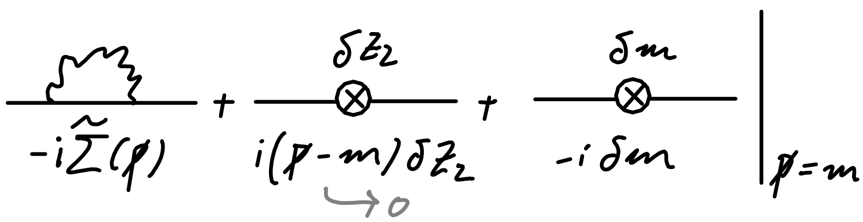
Since the Σ, Π and Λ computed above contain only the loops and not the counterterms, we now decorate the loop-only quantities with a tilde.

Now indeed, with our definitions

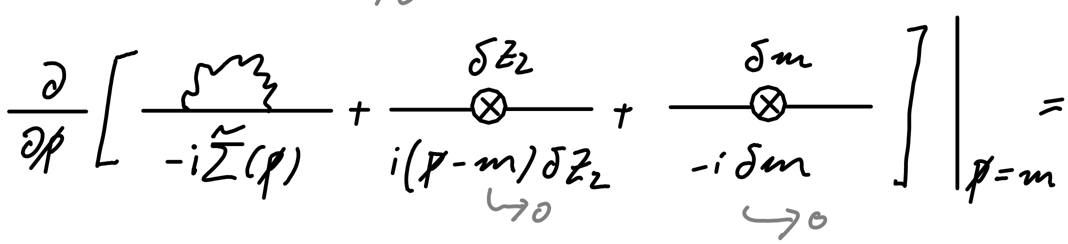
$$\delta Z_2 := \frac{\partial \tilde\Sigma}{\partial \not p}\Big|_{\not p = m} = \frac{\alpha}{4\pi}\left[-\Delta_\varepsilon - 4 - \log\frac{m^2}{\mu^2} - 2\log\frac{\lambda^2}{m^2}\right]$$

$$\delta m := -\tilde\Sigma\Big|_{\not p = m} = \frac{3\alpha m}{4\pi}\left[\Delta_\varepsilon - \frac{1}{3} - \frac{2}{3}\int_0^1 dx\,(1+x)\log\frac{m^2x^2}{\mu^2}\right]$$

we find that



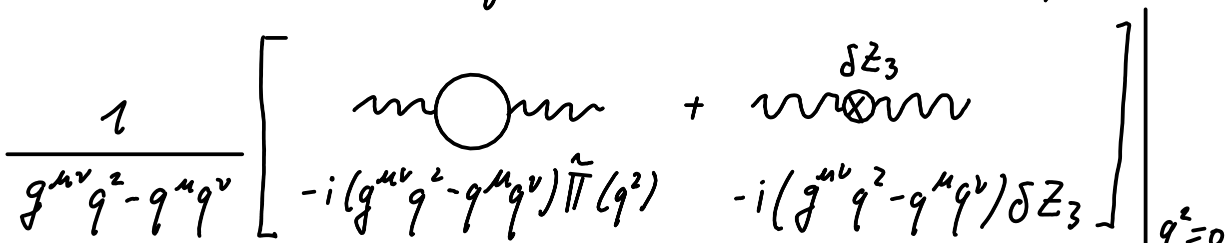
$= 0$



$= 0$

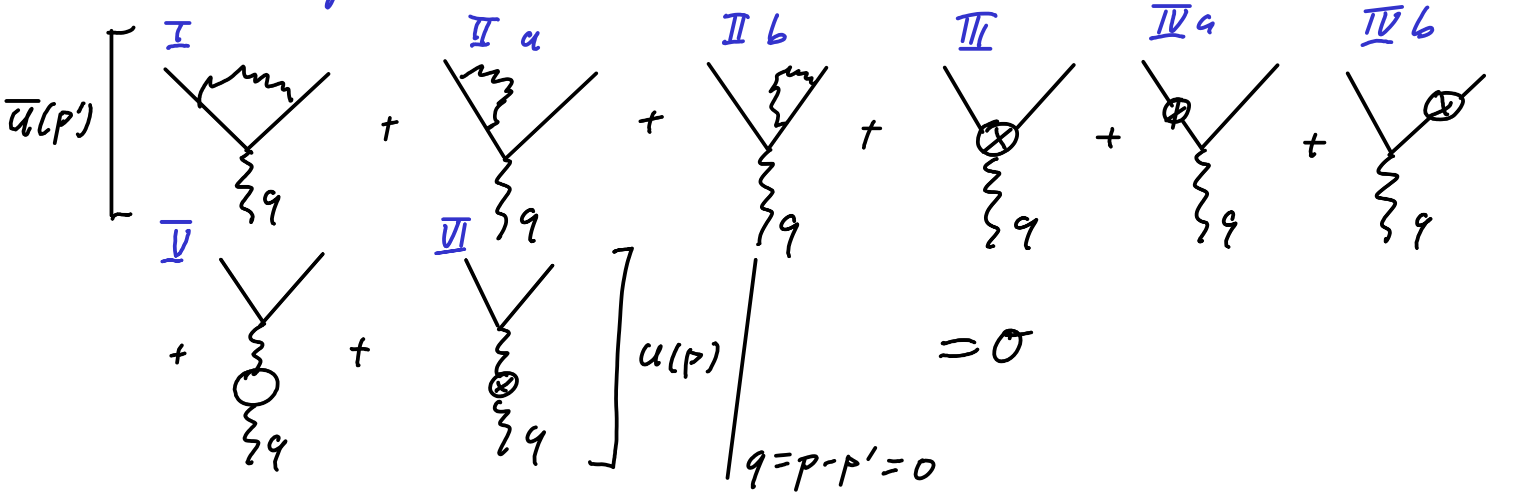
The vanishing photon mass is protected by the gauge symmetry. Still, in order to obtain a unit residue at the on-shell ($q^2=0$) pole, the condition $\Pi(q^2=0)=0$ must be imposed. It holds, because

$$\delta Z_3 = -\tilde\Pi(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\Delta_\varepsilon - \log\frac{m^2 - x(1-x)}{\mu^2}\right]$$

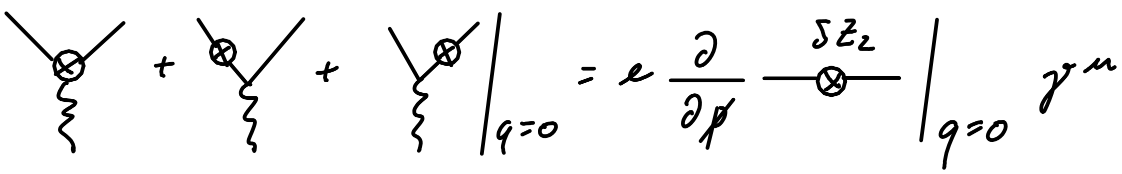


$= 0$

So we have used the field normalisations and particle masses as renormalisation conditions. Now for the electric charge. Using the Ward identity, we have argued that it is only corrected by the photon polarisation. Now we will see how this works in renormalised perturbation theory. The renormalisation condition is the cancellation of the NLO contributions to the measurement of the electron charge:



We have already seen that the diagrams I, II a,b, III & IV a,b cancel due to the Ward identity, because $-i\Sigma = -i\tilde{\Sigma} + $ —⊗—, such that $\left.\frac{\partial \Sigma}{\partial \not{p}}\right|_{\not{p}=m} = 0$, as necessary for the cancellation.
It is instructive to write down the separate piece for the counterterms:



$$-ie\gamma^\mu \delta Z_1 + i\delta Z_2 (\not{p}' - m)\frac{i(\not{p}' + m)}{p'^2 - m^2}(-ie\gamma^\mu) + (-ie\gamma^\mu)\frac{i(\not{p}+m)}{p^2-m^2} i\delta Z_2 (\not{p} - m)$$
$$= ie\gamma^\mu \delta Z_2$$

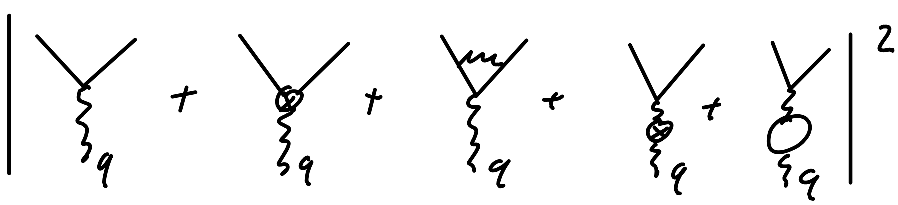
$\Longrightarrow$ $\boxed{\delta Z_1 = \delta Z_2}$ as we have verified explicitly.
Alternatively, it is of course possible to compute the vertex diagram for $q=0$ from $\Sigma(p)$ and application of the Ward identity, which is an easy way to obtain δZ_1.
Diagrams V and VI cancel due to the renormalisation

condition $\Pi(q^2=0)=0$.

Finally, we point out that instead of the counterterm $-(1+\delta Z_1)e\bar{\psi}\gamma^\mu A_\mu \psi$, we may have written $-e_0(1+\delta Z_2)\sqrt{1+\delta Z_3}\,\bar{\psi}\gamma^\mu A_\mu\psi$, where e_0 is the <u>bare charge.</u> The physical charge is $e=e_0\sqrt{1+\delta Z_3}$. This makes the role of the field strength renormalisations explicit and is in line with the interpretation of the vacuum polarisation as a charge renormalisation.

2.5 Finite Inclusive Cross Section

We still have to resolve the embarrassment that we found the tree-level cross-section for Coulomb scattering and finite momentum transfer $|\vec{q}|$ infinite, due to soft photon emission. The IR divergence was isolated in a logarithm of a photon mass. The same logarithms occur in the loop diagrams, where virtual electrons and photons are exchanged. We will now demonstrate that these logarithms precisely match and cancel, such that the theory makes correct and finite predictions also in the infrared. In particular, we want to account for the interference terms in



The counterterms are chosen such that the NLO corrections cancel for $q=0$, but there are non-trivial effects for $q\neq 0$. Note that the wave-function corrections for the electron are irrelevant here, as the electrons are on shell and the corrections are cancelled by the counterterms. The analytical expressions for the vertex corrections for $q\neq 0$

are given by

$$\bar{u}(p')\Lambda_\mu(p,p')u(p) = \gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)$$

$$F_1(q^2) = \frac{\alpha}{4\pi}\left\{\left(4\log\frac{\lambda}{m}+4\right)(2\varphi\coth 2\varphi - 1) - 2\varphi\tanh\varphi - 8\coth 2\varphi\int_0^\varphi \theta\, d\theta\tanh\theta\right\}$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\frac{2\varphi}{\sinh 2\varphi} \qquad \left(\text{recall that } \frac{q^2}{4m^2} = -\sinh^2\varphi\right)$$

Note that $F_1(0)=0$.

The correction from F_1 has the same Dirac structure as the tree-level vertex and will lead in the cross section to a correction with the same angular dependence. For the correction from F_2, this is not the case. Note first that

$$\frac{i}{2m}\sigma_{\mu\nu}q^\nu = -\frac{1}{4m}[\gamma_\mu,\gamma_\nu]q^\nu = -\frac{1}{4m}[\gamma_\mu\gamma_\nu q^\nu - \gamma_\nu\gamma_\mu q^\nu]$$

$$= -\frac{1}{4m}\left[2\gamma_\mu\gamma_\nu q^\nu - \{\gamma_\mu,\gamma_\nu\}q^\nu\right] = -\frac{1}{4m}\left[2\gamma_\mu \not{q} - 2q_\mu\right]$$

Since we choose the gauge for the external field where $A^i=0$, only the $\mu=0$ terms contribute. Furthermore, recall that $q^\mu=(0,\vec{q})$. When interfering with the tree-level diagram, we obtain the trace

$$\mathrm{tr}\left[(\not{p}'+m)\gamma_0\not{q}(\not{p}+m)\gamma_0\right] = 4m\left[q\cdot(p+p')\right] = -4m\,\vec{q}\cdot(\vec{p}+\vec{p}')$$

$$= 8m\,\vec{p}^{\,2}\sin^2\frac{\vartheta}{2}$$

$$\vec{p}\cdot\vec{q} = -2p^2\sin^2\frac{\vartheta}{2}$$

The angular dependence differs explicitly from the tree-level. The finite contribution to the vacuum polarisation is given by

$$\Pi(q^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\varepsilon - \log\frac{m^2 - x(1-x)q^2}{\mu^2} + \delta Z_3\right]$$

$$= -\frac{\alpha}{3\pi}\left\{2\left(1+\frac{2m^2}{q^2}\right)\left[\sqrt{\frac{4m^2}{q^2}-1}\ \operatorname{arccot}\sqrt{\frac{4m^2}{q^2}-1}-1\right]+\frac{1}{3}\right.$$

$$= -\frac{\alpha}{3\pi}\left\{2\left(1-\frac{1}{2\sinh^2\varphi}\right)\left[\sqrt{1+\frac{1}{\sinh^2\varphi}}\ \operatorname{arccoth}\sqrt{1+\frac{1}{\sinh^2\varphi}}-1\right]+\frac{1}{3}\right\}$$

$$= -\frac{\alpha}{3\pi}\left\{2\left(\frac{3}{2}-\frac{1}{2}\coth^2\varphi\right)\left[\varphi\coth\varphi-1\right]+\frac{1}{3}\right\}$$

It appears as a multiplicative factor to the tree-diagram, and hence leads to the same angular dependence in the cross-section.

Putting these pieces together, we obtain

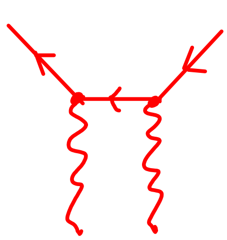
$$\left(\frac{d\sigma}{d\Omega_e}\right)_{el} = \left(\frac{d\sigma}{d\Omega_e}\right)_{el,0}\left\{1+\frac{2\alpha}{\pi}\left[\left(1+\log\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi-1)\right.\right.$$

$$-2\coth 2\varphi\int_0^\varphi d\varphi'\,\varphi'\tanh\varphi' - \frac{\varphi}{2}\tanh\varphi$$

$$\left.+\left(1-\frac{\coth^2\varphi}{3}\right)(\varphi\coth\varphi-1)+\frac{1}{9}-\frac{\varphi}{\sinh 2\varphi}\,\frac{\beta^2\sin^2\frac{\vartheta}{2}}{1-\beta^2\sin^2\frac{\vartheta}{2}}\right]$$

$$\left.+Z\alpha\pi\,\frac{\beta\sin\frac{\vartheta}{2}\left(1-\sin\frac{\vartheta}{2}\right)}{1-\beta^2\sin^2\frac{\vartheta}{2}}\right\}$$

We have highlighted here the contributions from leading order, $F_1(q)$, $F_2(q)$, $\Pi(q)$, and from second order interactions with the external field. A derivation of the latter can be found in the book by Itzykson & Zuber. They are due to the diagram



The decisive point is that the IR-divergent term $(2\varphi \coth 2\varphi - 1) \log \lambda$ is cancelled against a contribution in $\left(\frac{d\sigma}{d\Omega_e}\right)_\gamma$, so the inclusive rate (with or without photons in the final state) is finite. It makes perfect sense to consider up to a certain photon energy only the inclusive cross section, since arbitrarily soft photons cannot be detected. The proper treatment of soft radiation is not only crucial for the present example of bremsstrahlung. In collider physics the soft radiation of strongly interacting particles is of great importance and must be accounted for accurately.

2.6 Anomalous Magnetic Moment

Another fruit that is hanging now in front of us within our reach are the corrections to the gyromagnetic ratio g of the electron. It is defined through the relation

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}$$

where $\vec{\mu}$ is the magnetic moment and $\vec{S}$ the angular momentum (here: the electron spin $|\vec{S}| = \frac{1}{2}$). For a classical charge on a circular orbit, $g=1$. For the intrinsic spin of particles, this is typically not true, and one speaks of the anomalous magnetic moment.

We consider now a magnetic field

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} \vec{B}(\vec{q}) = \vec{\nabla} \times \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} \vec{A}(\vec{q})$$

$$= -\int \frac{d^3q}{(2\pi)^3} i\vec{q} \times \vec{A}(\vec{q})$$

$$\Longrightarrow B^i(\vec{q}) = -i\,\varepsilon^{ijk} q^j A^k(\vec{q})$$

Thus, we can express the matrix element for the tree-level plus vertex-corrected interaction with the magnetic field as

$$i\mathcal{M} = ie\,\bar{u}(p')\left[\gamma^i(1+F_1(q^2)) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p)\,A^i(\vec{q})$$

We expand this for soft momentum transfer (small $|\vec{q}|$) and non-relativistic electrons.

$$u(\vec{p},s) = \sqrt{m}\begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\xi_s \\ e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\xi_s \end{pmatrix} = \sqrt{m}\begin{pmatrix} \left(\cosh\left|\frac{\vec{\varphi}}{2}\right| - \sinh\left|\frac{\vec{\varphi}}{2}\right|\frac{\vec{\varphi}\cdot\vec{\sigma}}{|\vec{\varphi}|}\right)\xi_s \\ \left(\cosh\left|\frac{\vec{\varphi}}{2}\right| + \sinh\left|\frac{\vec{\varphi}}{2}\right|\frac{\vec{\varphi}\cdot\vec{\sigma}}{|\vec{\varphi}|}\right)\xi_s \end{pmatrix}$$

$$\approx \sqrt{m}\begin{pmatrix} \left(1-\frac{\vec{p}}{2m}\cdot\vec{\sigma}\right)\xi_s \\ \left(1+\frac{\vec{p}}{2m}\cdot\vec{\sigma}\right)\xi_s \end{pmatrix}$$

Then,

$$\bar{u}(\vec{p}',s')\gamma^i u(\vec{p},s) \approx 2m\,\xi^\dagger_{s'}\left[\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\sigma^i + \sigma^i\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right]\xi_s$$

$$\underset{\uparrow}{=} 2m\,\xi'_{s'}\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\xi_s + \text{spin-independent term}$$

$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

and

$$\bar{u}(\vec{p}',s')\left(\frac{i}{2m}\sigma^{i\nu}q_\nu\right)u(p,s) \approx 2m\,\xi^\dagger_{s'}\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\xi_s$$

Such that

$$\bar{u}(\vec{p}',s')\left[\gamma^i(1+F_1(q^2)) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(\vec{p},s)$$

$$\approx 2m\,\xi^\dagger_{s'}\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\left[1+\underbrace{F_1(0)}_{=0} + F_2(0)\right]\right]\xi_s$$

and

$$i\mathcal{M} = -i(2m)\,e\,\xi^\dagger_{s'}\frac{-1}{2m}\sigma^k\left[1+F_2(0)\right]\xi_s\,B^k(\vec{q})$$

This matrix element can be perceived as the NLO approximation to $2\omega = 2m$ times the expectation value of the interaction Hamiltonian (recall that $\sqrt{2\omega(\vec{p})}\, a^\dagger(\vec{p}, s)$ creates a particle)

$$\langle H_I \rangle = - \langle \vec{\mu} \rangle \cdot \vec{B}(\vec{x})$$

with

$$\langle \vec{\mu} \rangle = \frac{e}{m} \left[1 + F_2(0)\right] \xi_s^\dagger \frac{\vec{\sigma}}{2} \xi_s = \frac{e}{m} \left[1 + F_2(0)\right] \langle \vec{S} \rangle$$

With our definition of the gyromagnetic ratio above, we find

$$\boxed{g = 2\left[1 + F_2(0)\right] = 2\left(1 + \frac{\alpha}{2\pi}\right)}$$

This famous relation provides an important experimental test of QED. Given the experimental accuracy, higher loop calculations are necessary and have been performed.
The gyromagnetic ratio of the muon is also sensitive to hadronic and electroweak corrections and is potentially sensitive to the presence of new degrees of freedom beyond the Standard Model. Both, its theoretical calculation and its experimental determination are therefore an active field of research.

2.7 Electric Charge as a Running Coupling Constant

Our choice of renormalisation conditions defines the electric charge due to the force the electron experiences in a weak electric field (low momentum transfer). Of course, this is a very reasonable definition. Due to the loop effects, the electron couples at larger momentum transfer with a different strength to the electromagnetic field. Not taking account of the loops would therefore lead to an erroneous measurement of what we have defined as e or $\alpha = \frac{e^2}{4\pi}$.

In turn, our choice of counterterms is such that these cancel the loop effects for $q=0$. When we perform calculations at larger momentum transfer, the NLO contributions become large, what may result in a worse convergence behaviour of perturbation theory.

Both, the question what effective coupling strengths are observed at different momentum transfers as well as the control of the perturbative behaviour are addressed by the concept of running couplings and the renormalisation group.

Inspecting the expressions for $\tilde{\Sigma}(q)$, $\tilde{\Lambda}(q=p'-p)$, $\tilde{\Pi}(q)$, we see that μ appears in terms of the form

$\log \dfrac{A_1 \lambda^2 + A_2 m^2 + A_3 q^2}{\mu^2}$, where $A_{1/2/3}$ are some factors of Feynman parameters. In order to keep these logarithms small and perturbation theory well-behaved, the choice $\mu^2 \sim q^2$ serves the purpose. When we vary μ, the renormalised couplings g must be rescaled, in order to fully compensate the change in the physical result. We therefore define the β-function:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}$$

Now solve for $g(\mu)$. First, we rewrite the definition

$$\frac{d\mu}{\mu} = \frac{dg}{\beta(g)}$$

such as to formally obtain

$$\log \frac{\mu}{\mu_0} = \int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)}$$

with μ_0 some reference point. Now assume that for small g, we can expand

$$\beta = b\, g^n + \dots$$

$$\Rightarrow \log\frac{\mu}{\mu_0} = -\frac{1}{b}\left[\frac{1}{(n-1)\,g^{n-1}}\right]_{g(\mu_0)}^{g(\mu)}$$

$$\Rightarrow -b\log\frac{\mu}{\mu_0} + \frac{1}{(n-1)\,g^{n-1}(\mu_0)} = \frac{1}{(n-1)g(\mu)}$$

$$\Rightarrow g^{n-1}(\mu) = \frac{g^{n-1}(\mu_0)}{1-b\,(n-1)\,g^{n-1}(\mu_0)\log\frac{\mu}{\mu_0}}$$

Now, we can apply this reasoning to QED. The effective charge to one-loop order is

$$e_{eff} = e\left(1-\frac{\alpha}{6\pi}\left[\Delta_\varepsilon - \log\frac{m^2}{\mu^2}\right] - \delta Z_3\right)$$

Now, shift $\mu \longrightarrow \mu'$, such that

$$e_{eff} = e'\left(1-\frac{\alpha}{6\pi}\left[\Delta_\varepsilon - \log\frac{m^2}{\mu'^2}\right] - \delta Z_3\right)$$

We must impose that $e' = e\left(1+\frac{e^2}{12\pi^2}\log\frac{\mu'}{\mu}\right)$, such that

$$\beta = \mu'\,\frac{\partial e'}{\partial \mu'} = \frac{e^3}{12\pi^2}$$

We obtain the running coupling in QED

$$e^2(\mu) = \frac{e^2(\mu_0)}{1-\frac{1}{6\pi^2}\,e^2(\mu_0)\log\frac{\mu}{\mu_0}}$$

Note that this function increases with μ and eventually hits a Landau pole. However, this occurs only above electroweak symmetry breaking, where QED is no longer valid anyway.

In order to make all predictions independent of the subtraction scale μ, also the field strengths and m need to be redefined. This task is addressed by the renormalisation group. It is interesting to notice that by the rescaling, we can keep logarithmic corrections $\sim e^2 \log \frac{\mu^2}{q^2}$, that appear beyond the given order in perturbation theory into account. At a fixed resummation scale μ, this would correspond to a "resummation" of a series of diagrams at arbitrary loop order, which would be challenging, if possible at all.

For the purpose of these phenomenological lectures, we however consider only the running coupling and the β-function, which contain crucial information on the physical behaviour of a theory, as we shall see on the example of QCD.