5. Hadrons & Flavour

5.1 Discrete Symmetries

We have already discussed that a complex scalar field ϕ transforms under charge conjugation C as

$$\phi(x) \overset{C}{\longmapsto} \phi^*(x)$$

We can also formally introduce C as a unitary operator that takes

$$C\,\phi(x)\,C^{-1} = \phi^*(x).$$

The next important discrete symmetry is parity P, that takes the effect of spatial reflections. For a scalar field, we have

$$P\,\phi(t,\vec{x})\,P^{-1} = \phi(t,-\vec{x})$$

whereas for a pseudoscalar

$$P\,\phi(t,\vec{x})\,P^{-1} = -\phi(t,-\vec{x})$$

There is nothing awkward about being a pseudoscalar. For example, given three vector fields $\vec{a}(\vec{x})$, $\vec{b}(\vec{x})$ and $\vec{c}(\vec{x})$ in three dimensions, we obtain the pseudoscalar field

$$p(\vec{x}) = \vec{a}(x)\cdot\left(\vec{b}(\vec{x})\times\vec{c}(\vec{x})\right)$$

A spatial reflection clearly takes $p(\vec{x}) \mapsto -p(-\vec{x})$. Spin-0 particles may inherit their parity properties from their constituents, if they are composite. (Just as $p(\vec{x})$ inherits its property from $\vec{a}(\vec{x})$, $b(\vec{x})$, $c(\vec{x})$). Such composite particles are a main topic of this chapter. The parity property of elementary particles may be considered a matter of definition. It is reasonable to assign parity in such a manner that QED and QCD interactions remain parity invariant. Due to their sensitivity to fermionic chirality, we recall that weak interactions do not conserve parity.

The third important discrete symmetry is the time reversal T, for which we encounter a complication. Chiral gauge theories violate T, but for QED, QCD and of course the free theory, we insist that an observer could not tell the difference between a film being played forward or backward (for systems that that are not thermodynamical).

Consider the Heisenberg equation

$$[H(t,\vec{x}), \phi(t,\vec{x})] = -i \frac{d}{dt} \phi(t,\vec{x})$$

Reversing the sign of t yields

$$[H(-t,\vec{x}), \phi(-t,\vec{x})] = i \frac{d}{dt} \phi(-t,\vec{x})$$

If we consider e.g. the free Klein-Gordon field, the Hamiltonian H is even in t. The sign flip of t hence yields a different theory in a sense that we explain below.

We can fix this problem by defining T such that besides a sign flip for t, it implies a complex conjugation, i.e.

$$T\phi(t,\vec{x}) = \phi^*(-t,\vec{x})\,T$$

Then, the Heisenberg equation transforms into

$$T[H(t,\vec{x}), \phi(t,\vec{x})]T^{-1} = T(-i)\frac{d}{dt}\phi(t,\vec{x})T^{-1}$$

$\longmapsto$

$$[H(-t,\vec{x}), \phi^*(-t,\vec{x})] = \frac{d}{dt}\phi^*(-t,\vec{x})$$

This is the same equation as before, up to the complex conjugation, which is unobservable. Notice that because complex conjugation is a non-linear transformation, T cannot be a unitary operator. However, it satisfies the defining property of an antiunitary operator

$$\langle T\phi | T\psi \rangle = \langle \psi | \phi \rangle$$

We now consider in more detail the implication of the sign flip in the Heisenberg equation. It derives from the canonical formalism in classical field theory, where

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \qquad \text{Lagrangian}$$

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \qquad \text{canonical momentum}$$

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L} \qquad \text{Hamiltonian density}$$

The Poisson bracket for fields

$$\{A(t,\vec{x}), B(t,\vec{y})\} = \int d^3z \left(\frac{\delta A(t,\vec{x})}{\delta\Pi(t,\vec{z})}\frac{\delta B(t,\vec{y})}{\delta\phi(t,\vec{z})} - \frac{\delta B(t,\vec{x})}{\delta\Pi(t,\vec{z})}\frac{\delta A(t,\vec{y})}{\delta\phi(t,\vec{z})} \right)$$

leads to the Hamilton-Jacobi equations

$$\{\mathcal{H}, \phi\} = \frac{\delta\mathcal{H}}{\delta\Pi} = \dot{\phi}$$

$$\{\mathcal{H}, \Pi\} = -\frac{\delta\mathcal{H}}{\delta\phi} = \underbrace{-\frac{\partial\mathcal{H}}{\partial\phi} - \partial_t\frac{\partial\mathcal{H}}{\partial\dot{\phi}} + \partial^i\frac{\partial\mathcal{H}}{\partial(\partial^i\phi)}}_{=0 \text{ (Euler-Lagrange)}} = -\dot{\Pi}$$

In view of quantisation, it is useful to note also the trivial classical relation

$$\{\phi(t,\vec{x}), \Pi(t,\vec{y})\} = \int d^3z \left(\frac{\delta\phi(t,\vec{x})}{\delta\Pi(t,\vec{z})} \frac{\delta\Pi(t,\vec{y})}{\delta\phi(t,\vec{z})} - \frac{\delta\Pi(t,\vec{y})}{\delta\Pi(t,\vec{z})} \frac{\delta\phi(t,\vec{x})}{\delta\phi(t,\vec{z})} \right) = -\delta^3(\vec{x}-\vec{y})$$

Quantisation is performed by replacing the Poisson brackets with commutators:

$$\{A,B\} \longmapsto i\,[A,B]$$

The sign is in principle a matter of convention, as it can always be compensated by a hermitian conjugation. The operators

$$\phi(t=0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \left(a(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} + a^+(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\Pi(t=0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{2\omega(\vec{p})} \left(a(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} - a^+(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

where $[a(\vec{p}), a^+(\vec{p}\,')] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')$,

satisfy the equal-time commutation relation

$$[\phi(t=0,\vec{x}), \Pi(t=0,\vec{x}\,')] = i\,\delta^3(\vec{x}-\vec{x}\,')$$

and the Heisenberg equations

$$[H,\phi] = -i\frac{d}{dt}\phi \Rightarrow \phi(t,\vec{x}) = e^{iHt}\phi(t=0,\vec{x})\,e^{-iHt}$$

$$[H,\Pi] = i\frac{d}{dt}\Pi \Rightarrow \Pi(t,\vec{x}) = e^{-iHt}\,\Pi(t=0,\vec{x})\,e^{-iHt}$$

When we now flip the sign of t in these equations but stick with the definition $\Pi = \frac{\partial \mathcal{L}}{\partial\dot{\phi}}$, we get the "wrong" sign in the equal-time commutation relation. It may be useful to compare with the quantum-mechanical commutator

$$[x, p] = [x, \frac{\hbar}{i}\frac{\partial}{\partial x}] = i\hbar$$

A sign change would lead to a complex conjugated quantum mechanics, which is different from the original, even though the sign was a convention to start with. So we may interpret the complex conjugation as a useful convention in order to keep the free theory, QED and QCD time-reversal invariant, even at the level of unobservable complex phases.

For a gauge field, we can directly write down the transformations

$$P A_\mu(t, \vec{x}) P^{-1} = A^\mu(t, \vec{x}) \quad \text{(note the positions of the indices)}$$

$$C A_\mu(t, \vec{x}) C^{-1} = -A_\mu(t, \vec{x}) \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$T A_\mu(t, \vec{x}) T^{-1} = A^\mu(-t, \vec{x}) \qquad \vec{E} = -\vec{\nabla} A^0 - \dot{A}^i$$

Notice in particular that the $\vec{E}$ field is even, the $\vec{B}$ field is odd under T, such that the motion of a charged object in the electromagnetic field is T even.

Finally, we consider Dirac fermions with the field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \sum_s \{ a(\vec{p}, s) u(\vec{p}, s) e^{-ip\cdot x} + b^+(\vec{p}, s) v(\vec{p}, s) e^{ip\cdot x} \}$$

Parity should take $\vec{p} \longmapsto -\vec{p}$ (vector) but leave the spin (pseudovector) invariant. So we have

$$P a(\vec{p}, s) P^{-1} = \eta_a a(-\vec{p}, s)$$

$$P b(\vec{p}, s) P^{-1} = \eta_b b(-\vec{p}, s) \qquad \text{where } \eta_{a,b} = \pm 1$$

The condition on $\eta_{a,b}$ is due to the fact that a two-fold application of parity reversal should keep the state invariant.
Recall that

$$u(\vec{p},s) = \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}} \xi_s \\ e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}} \xi_s \end{pmatrix} \qquad v(\vec{p},s) = \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}} \eta_s \\ -e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}} \eta_s \end{pmatrix}$$

and notice that the rapidity $\vec{\varphi}$ reverses its sign as well. The matrix γ^0 exchanges the upper and the lower two-spinors, such that we may write

$$u(\vec{p},s) = \gamma^0 u(-\vec{p},s) \qquad v(\vec{p},s) = -\gamma^0 v(-\vec{p},s)$$

Introducing $\tilde{p} = (p^0, -\vec{p})$, $\tilde{x} = (t, -\vec{x})$ and $\vec{\tilde{p}} = -\vec{p}$,
the transformed field operator can thus be expressed as

$$P\psi(x)P = \int \frac{d^3\vec{\tilde{p}}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{\tilde{p}})}} \sum_s \left(\eta_a a(\vec{\tilde{p}},s) \gamma^0 u(\vec{\tilde{p}},s) e^{-i\tilde{p}\cdot\tilde{x}} - \eta_b^* b^\dagger(\vec{\tilde{p}},s) \gamma^0 v(\vec{\tilde{p}},s) e^{i\tilde{p}\cdot\tilde{x}} \right)$$

We should choose $\eta_b^* = -\eta_a$, because this allows to write

$$P\psi(t,\vec{x})P = \eta_a \gamma^0 \psi(t,-\vec{x})$$

By convention, $\eta_a = 1$, such that single fermion states have positive parity and single anti-fermion states negative parity. This brings the important consequence that the parity of mesons, which are quark anti-quark pairs is minus one times the parity of their wave function. The latter can be inferred from the orbital angular momentum.

Time reversal should flip spin & momentum,

$$T a(\vec{p},s) T = a(-\vec{p},-s), \qquad T b(\vec{p},s) T = b(-\vec{p},-s)$$

In addition, when acting on a $\mathbb{C}$-number, we do not forget that T leads to a complex conjugation.

We will make use of the matrix

$$-\gamma^1\gamma^3 = -\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3 \end{pmatrix} = -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

$$-i\sigma^2 = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It is also useful to recall that $\vec{\sigma}\sigma^2 = -\sigma^2\vec{\sigma}^*$.

We define the flipped spinor as $\xi_{-s} = -i\sigma^2\xi_s^*$. This is appropriate, because if $\vec{n}\cdot\vec{\sigma}\,\xi = s\,\xi$, then

$$(\vec{n}\cdot\vec{\sigma})(-i\sigma^2\xi^*) = -i\sigma^2(-\vec{n}\cdot\vec{\sigma})^*\xi^* = i\sigma^2 s\,\xi^* = -s(-i\sigma^2\xi^*)$$

Notice that a two-fold application of the spin flip yields the negative of the original spinor.

$$-\gamma^1\gamma^3 u(-\vec{p},-s) = -\gamma^1\gamma^3\sqrt{m}\begin{pmatrix} e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\,\xi_{-s} \\ e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\,\xi_{-s} \end{pmatrix}$$

$$= \sqrt{m}\begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}^*}(-i\sigma^2)(-i\sigma^2)\,\xi_s^* \\ e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}^*}(-i\sigma^2)(-i\sigma^2)\,\xi_s^* \end{pmatrix} = -u^*(\vec{p},s)$$

$\Rightarrow$ $u(-\vec{p},-s) = -\gamma^1\gamma^3 u^*(\vec{p},s)$ and likewise

$v(-\vec{p},-s) = -\gamma^1\gamma^3 v^*(\vec{p},s)$

In summary, we may express

$$T\psi(t,\vec{x})T = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_s T\left(a(\vec{p},s)u(\vec{p},s)e^{-ip\cdot x} + b^\dagger(\vec{p},s)v(\vec{p},s)e^{ip\cdot x}\right)T$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \sum_s \left(a(-\vec{p},-s)\, u^*(\vec{p},s) e^{ip\cdot x} + b^\dagger(-\vec{p},-s)\, v^*(\vec{p},s) e^{-ip\cdot x} \right)$$

$$= -\gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \sum_s \left(a(-\vec{p},-s)\, u(-\vec{p},-s) e^{ip\cdot x} + b^\dagger(-\vec{p},-s)\, v(-\vec{p},-s) e^{-ip\cdot x} \right)$$

$$= -\gamma^1\gamma^3 \psi(-t,\vec{x})$$

Charge conjugation does

$C\, a(\vec{p},s)\, C = b(\vec{p},s)$ and $C\, b(\vec{p},s)\, C = a(\vec{p},s)$

such that we need a relation between the spinors u and v. We therefore note that

$$v^*(\vec{p},s) = \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}^*} \xi^*_{-s} \\ -e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}^*} \xi^*_{-s} \end{pmatrix} = \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}^*} (-i\sigma^2)\xi_s \\ -e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}^*} (-i\sigma^2)\xi_s \end{pmatrix}$$

$$= \sqrt{m}\; -i\gamma^2 \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}} \xi_s \\ e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}} \xi_s \end{pmatrix} = -i\gamma^2 u(\vec{p},s)$$

$\Rightarrow u(\vec{p},s) = -i\gamma^2 v^*(\vec{p},s)$ and likewise $v(\vec{p},s) = -i\gamma^2 u^*(\vec{p},s)$

The charge conjugation of a Dirac spinor therefore takes the effect

$$C\psi(x)C = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \sum_s \left(-i\gamma^2 b(\vec{p},s)\, v^*(\vec{p},s) e^{-ip\cdot x} - i\gamma^2 a^\dagger(\vec{p},s)\, u^*(\vec{p},s) e^{ip\cdot x} \right)$$

$$= -i\gamma^2 \psi^*(x) = -i(\bar{\psi}\gamma^0\gamma^2)^T$$

It is straightforward to work out the effect of the discrete transformations on the bilinears. We give the following table for reference:

	scalar $\bar{\psi}\psi$	pseudoscalar $i\bar{\psi}\gamma^5\psi$	vector $\bar{\psi}\gamma^\mu\psi$	pseudovector $\bar{\psi}\gamma^\mu\gamma^5\psi$	antisymmetric tensor $\bar{\psi}\sigma^{\mu\nu}\psi$	derivative ∂^μ
P	+1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	+1	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	+1	+1	-1	+1	-1	+1
CPT	+1	+1	-1	-1	+1	-1

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $(-1)^\mu = 1$ for $\mu = 0$, $(-1)^\mu = -1$ for $\mu = 1,2,3$.

Notice that the vector transforms like the vector potential A^μ and the antisymmetric tensor like the field strength tensor $F^{\mu\nu}$.

Weak interactions violate C and P maximally, because

$$\bar{\psi}\gamma^\mu A_\mu \frac{1-\gamma^5}{2}\psi \xrightarrow{P} \bar{\psi}\gamma^\mu A_\mu \frac{1+\gamma^5}{2}\psi$$

$$\bar{\psi}\gamma^\mu A_\mu \frac{1-\gamma^5}{2}\psi \xrightarrow{C} \bar{\psi}\gamma^\mu A_\mu \frac{1+\gamma^5}{2}\psi$$

Note however that they conserve the combined charge-parity CP. We will see that in the SM, CP is violated by the Yukawa interactions, what requires however the presence of at least three flavours. The combined symmetry CPT is however conserved in Quantum Field Theory, which is known as the CPT theorem.

From above table, we notice that odd-rank tensors are odd, even rank tensors even under CPT. Since the Lagrangian is a rank 0 tensor (i.e. a scalar), we immediately notice its invariance

$$CPT\, \mathcal{L}(x)\, (CPT)^{-1} = \mathcal{L}(-x).$$

One interesting application is that for the invariant matrix element

$$i\mathcal{M}_{A\to B} = i\mathcal{M}_{CPB(CP)^{-1}\to CPA(CP)^{-1}}$$

where A and B are some multi-particle states. Now if we take $A=B$, assume that this is a state of a single, unstable particle and take twice the imaginary part, the optical theorem tells us that this is the decay rate. Since the same applies to $CP\,A(CP)^{-1}$, we find that particles and anti-particles have the same life-time.

5.2 The CKM Matrix

While in the previous chapter, we have not indicated the weak interaction eigenstates with a prime, in order to keep the notation compact, we will introduce the prime from now on. The common choice is to perform the rotation into the weak interaction basis for the down type quarks

$$\underbrace{\begin{pmatrix} d_L' \\ s_L' \\ b_L' \end{pmatrix}}_{\text{weak eigenstates}} = V \underbrace{\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}}_{\text{mass eigen= states}}$$

Recall that rotations of the up-type quarks are under= stood to be readily absorbed within the CKM matrix V. We also recall that V is unitary and denote its components as

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Yet, there are ambiguities from field redefinitions, and rather than determining the individual entries,

it is useful to think first of a parametrisation.
An $n \times n$ unitary matrix can depend on

$$2n^2 - \underset{\substack{\uparrow \\ \text{normalisations}}}{n} - 2\underset{\substack{\uparrow \\ \text{orthogonalities}}}{\frac{n(n-1)}{2}} = n^2 \quad \text{real parameters.}$$

On the other hand, an orthogonal matrix has
$n^2 - n - \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$ parameters. Hence, we have $\frac{1}{2}n(n-1)$ mixing angles and $\frac{1}{2}n(n+1)$ phases. Most phases can be removed by the redefinitions

$$u_{L\alpha} \longmapsto e^{i\vartheta^u_\alpha} u_{L\alpha}, \quad d_{L\alpha} \longmapsto e^{i\vartheta^d_\alpha} d_{L\alpha},$$

which take the effect

$$V_{\alpha\beta} \longmapsto e^{i(\vartheta^d_\beta - \vartheta^u_\alpha)} V_{\alpha\beta}$$

An overall common rephasing takes no effect, such that $2n-1$ of the phases can be removed, leaving $\frac{1}{2}n^2 - \frac{3}{2}n + 1$. Hence, for $n=3$, there are 3 angles and one physical phase. The common choice for the parametrisation is ($s_{ij} = \sin\vartheta_{ij}$, $c_{ij} = \cos\vartheta_{ij}$)

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

We will explain how to determine the elements of this matrix in the following chapters. In order to understand the various processes involved, and because of the particular hierarchical structure of the CKM matrix, we first state the results.

Often quoted are the magnitudes of the CKM matrix elements $|V_{\alpha\beta}|$ [PDG 2012 (11.26)]

$$\begin{pmatrix} 0{,}97428 \pm 0{,}00015 & 0{,}2253 \pm 0{,}0007 & 0{,}00347^{+0{,}00016}_{-0{,}00012} \\ 0{,}2252 \pm 0{,}0007 & 0{,}97345^{+0{,}00015}_{-0{,}00016} & 0{,}0410^{+0{,}0011}_{-0{,}0007} \\ 0{,}00862^{+0{,}00026}_{-0{,}00020} & 0{,}0403^{+0{,}0011}_{-0{,}0007} & 0{,}999152^{+0{,}000030}_{-0{,}000045} \end{pmatrix}$$

While the origin of these mixings is not known and consequently, their sizes have not been predicted, we may recognise the hierarchy

$$1 \gg s_{12} \gg s_{23} \gg s_{13}$$

This gives rise to the often employed phenomenological Wolfenstein parametrisation. We define

$$s_{12} = \lambda = \frac{V_{us}}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}} \qquad s_{23} = A\lambda^2 = \lambda \left| \frac{V_{cb}}{V_{us}} \right|$$

$$s_{13} e^{i\delta} = V_{ub}^* = A\lambda^3(\varrho + i\eta) = \frac{A\lambda^3(\bar{\varrho} + i\bar{\eta})\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1 - A^2\lambda^4(\bar{\varrho} + i\bar{\eta})]}$$

where $$\bar{\varrho} + i\bar{\eta} = -\frac{V_{ud} V_{ub}^*}{V_{cd} V_{cb}^*}$$

Expanding in λ, we obtain

$$V = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\varrho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \varrho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

We can make at this point the remark that the small mixings $\sim \lambda, \lambda^2, \lambda^3$ suppress the CP-violation originating from the remaining phase. The experimental values for the Wolfenstein parameters are [PDG 2012 (11.26)]

$\lambda = 0{,}2253 \pm 0{,}0007$ $\qquad$ $A = 0{,}808^{+0{,}022}_{-0{,}015}$

$\bar{\rho} = 0{,}132^{+0{,}022}_{-0{,}014}$ $\qquad$ $\bar{\eta} = 0{,}341 \pm 0{,}013$

While above numbers are from global fits, the CKM elements can be determined individually. It is then interesting to test the SM with the unitarity constraints

$$\sum_i V_{ij} V_{ik}^* = \delta_{jk} \quad \text{and} \quad \sum_j V_{ij} V_{kj}^* = \delta_{ik}$$

There are six different sums that should vanish.

These relations can be geometrically interpreted as triangles in the complex plane. All six of these triangles have the same area $\left|\frac{J}{2}\right|$, where J is the Jarlskog invariant

exercise

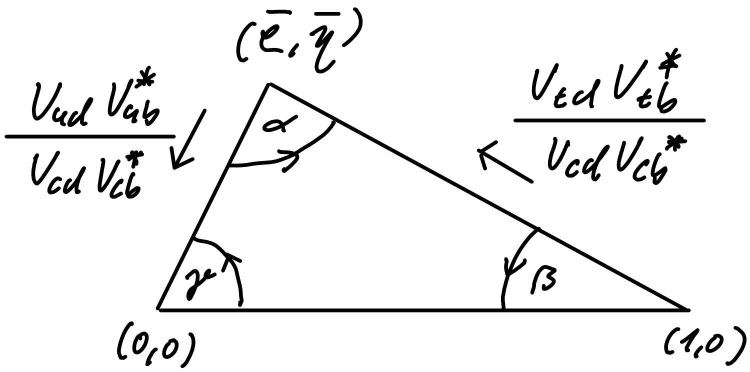
$$J = \mathrm{Im}\left[V_{td}^* V_{tb} V_{ub}^* V_{ud}\right] = c_{12} c_{23} c_{13}^2 s_{12} s_{23} s_{13} \sin\delta_{13} = A^2 \lambda^6 \eta$$

One can show that J is independent of the parametrisation. The smallness of the mixing angles thus implies that CP-violating effects are small in the SM.

From the Wolfenstein parametrisation, we see that four of the six triangles are squashed, while the remaining two take the same shape. The commonly considered combination is

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$$

The standard unitarity triangle is obtained when dividing each side by $V_{cd} V_{cb}^*$. The vertices are then at $(0,0)$, $(1,0)$ and $(\bar{\rho}, \bar{\eta})$.



$\alpha = (89{,}0^{+4{,}4}_{-4{,}2})^\circ$

$\gamma = (68^{+10}_{-11})^\circ$

The lengths of the complex sides of this triangle are given by

$$R_u = \left| \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right| = \sqrt{\bar{\rho}^2 + \bar{\eta}^2}, \quad R_t = \left| \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} \right| = \sqrt{(1-\bar{\rho})^2 + \bar{\eta}^2}$$

and the angles are defined

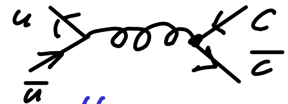
$$\alpha = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*} \right], \quad \beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right], \quad \gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right]$$

Notice that in the standard parametrisation, $\gamma = \delta$.
In terms of these lengths and angles, we can reexpress the standard unitarity relation as

$$R_u e^{i\gamma} + R_t e^{-i\beta} = 1$$

One of the important branches of particle physics research is to overconstrain the unitarity triangle in order to test or to falsify the CKM mechanism, that is a key element of the SM.

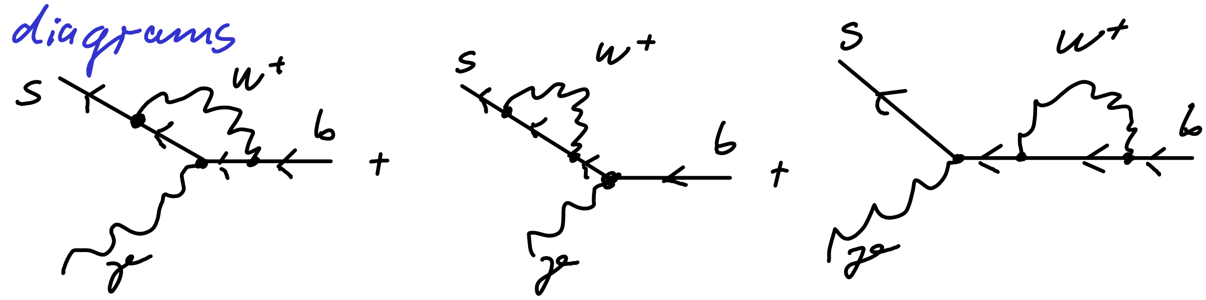
5.2 Heavy Quarks and Quarkonia

Light quark form bound states tied together by the strong interactions in the non-perturbative regime. For heavy quarks, in particular c and b, the Compton wave length $m_{c,b}^{-1} \ll \Lambda_{QCD}^{-1}$, such that we may expect their bound states to be described by a Coulomb potential, at least at short distances. Moreover, since strong interactions are weak at short distances, hadronic decays (e.g. 

) should be suppressed and consequently, these quarkonia should be narrow (have a comparably long life-time).

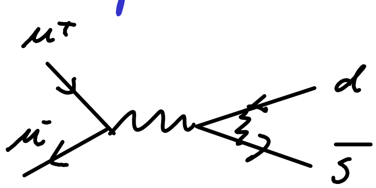
It is instructive to recall that historically, the first of the heavy quarks was a theoretical prediction, known as the GIM (Glashow-Iliopoulos-Maiani) mechanism. It was observed from Kaon decays that strange quark number violating neutral currents are tiny. As we have discussed, the absence of FCNCS at tree level is elegantly described by the CKM mechanism. The latter requires that the quarks can be arranged in weak doublets, and the missing partner of the s was predicted to be the c.

The GIM mechanism tells us in addition that FCNCs are also suppressed when including loop effects. To see how this works, consider the important loop decay $b \longrightarrow s\gamma$. While forbidden at tree level, the leading amplitudes are represented by the diagrams



Each of these is $\propto \sum_i f(m_i) V_{ib} V_{is}^*$, where $f(m_i)$ is a function of the internal line quark mass m_i. It is natural to expand f in terms of the parameter $\frac{m_i^2}{M_W^2}$. Clearly, the 0th order contribution vanishes, because $\sum_i V_{ib} V_{is}^* = 0$. Therefore, the present amplitude is dominated by top-quark exchange and can be used to probe $|V_{ts} V_{tb}|$, which is of order λ^2 (besides additional loop-suppression factors).

By the same token, we can also conclude that the rare decay [branching ratio $(6{,}84 \pm 0{,}11) \times 10^{-9}$] of a long lived Kaon into a muon pair via diagrams such as



(The decay to e^+e^- is helicity suppressed, in addition $\longrightarrow$ exercise).

is suppressed as $\lambda \frac{m_c^2}{m_W^2}$. This reasoning has lead to the prediction of the heavy c-quark before its discovery.

Before we proceed with discussing the heavy quarks and their bound states, we mention that the existence of the third generation (t, b) was inferred by Kobayashi and Maskawa before the discovery of these particles from realising that they can explain the CP-violation in the neutral Kaon system: As we have discussed above, we need to have at least three generations in order to obtain one physical CP phase in the CKM matrix that cannot be removed by field redefinitions.

5.2.1 B and D mesons

Confinement forces the heavy quarks into bound states. Combinations of c (b) with lighter quarks and vanishing orbital momentum are called D (B) mesons. Combinations of $c\bar{c}$ or $b\bar{b}$ and their angular excitations are called quarkonia, to which we will come back a bit later.

The masses and the wave-functions for the heavy quark mesons have not yet been obtained analytically, but obviously the main contribution to the mass comes from the heavy quark.

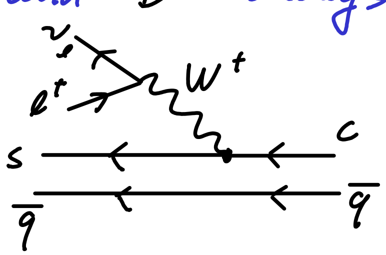
The table of D mesons is

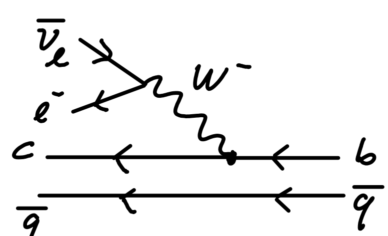
	quark content	mass [MeV]	J^P
D^+/D^-	$c\bar{d}$	$1869{,}62 \pm 0{,}15$	0^-
$D^0/\bar{D}^0$	$c\bar{u}$	$1864{,}86 \pm 0{,}13$	0^-
D_s^+/D_s^-	$c\bar{s}$	$1968{,}47 \pm 0{,}33$	0^-
D^{*+}/D^{*-}	$c\bar{d}$	$2010{,}28 \pm 0{,}13$	1^-
$D^{*0}/\bar{D}^{*0}$	$c\bar{u}$	$2006{,}98 \pm 0{,}15$	1^-

The B mesons are

	quark content	mass [MeV]	J^P
B^+/B^-	$u\bar{b}$	$5279{,}25 \pm 0{,}17$	0^-
$B^0/\bar{B}^0$	$d\bar{b}$	$5279{,}58 \pm 0{,}17$	0^-
B_s^0/B_s^0	$s\bar{b}$	$5366{,}77 \pm 0{,}24$	0^-
B^{*0}/B^{*0}	$u\bar{d}$	$5325{,}2 \pm 0{,}4$	1^-
$B_s^{*0}/\bar{B}_s^{*0}$	$s\bar{b}$	$5415^{+2{,}4}_{-2{,}1}$	1^-
B_c^+/B_c^-	$c\bar{b}$	6277 ± 6	0^-

We should already mention that the neutral mesons are not mass eigenstates. Rather, there are two separate eigenstates with a tiny mass difference. This has important consequences that we are going to discuss in the context of mixing and CP violation.
As we do not know analytically the structure of the mesons, we cannot straightforwardly compute their decays. However, often the simple <u>spectator model</u> serves as a sufficiently good approximation or we can obtain precise predictions from <u>hadronic matrix elements</u>.
The spectator model is useful when considering semi-leptonic decays. The CKM favoured processes for the D and B decays are





i.e. $D^0 \to K^- e^+ \nu_e$
$D^+ \to \bar{K}^0 e^+ \nu_e$

i.e. $\bar{B}^0 \to D^+ e^- \bar{\nu}_e$
$B^- \to D^0 e^- \bar{\nu}_e$

and accordingly for B_s/D_s

The effective Lagrangian terms giving rise to these CKM-favoured decays are:

$$\mathcal{L}^{(c)} = -\frac{G_F}{\sqrt{2}} V_{cs}\, \bar{s}\gamma^\mu(1-\gamma_5)c\, \bar{\nu}\gamma_\mu(1-\gamma_5)e$$

$$\mathcal{L}^{(b)} = -\frac{G_F}{\sqrt{2}} V_{cb}\, \bar{c}\gamma^\mu(1-\gamma_5)b\, \bar{e}\gamma_\mu(1-\gamma_5)\nu$$

The spectator model simply assumes that the antiquark $\bar{q}$ does not participate in the decay.

When we denote with $b, c, s, \nu, \bar{\nu}, e^+, e^-$ the four momenta of the involved particles, we obtain for the polarisation summed/averaged squared matrix elements

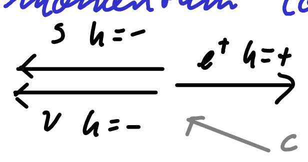
$$\overline{\sum}|\mathcal{M}^{(c)}|^2 = 64\, G_F^2\, |V_{cs}|^2\, c\cdot e^+\, s\cdot\nu$$

(exercise)

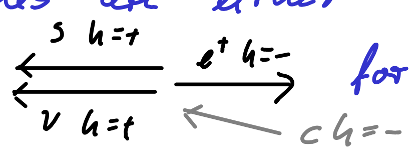
$$\overline{\sum}|\mathcal{M}^{(b)}|^2 = 64\, G_F^2\, |V_{cb}|^2\, b\cdot\bar{\nu}\, c\cdot e^-$$

The calculation can be performed along the same lines as for the $W^\pm/Z$ decays in Chapter 4.3. The above dependence on the scalar products of the four momenta can be qualitatively understood as follows:

Since this decay is mediated by weak interactions, the participating fermions are predominantly left-handed, the antifermions predominantly right-handed. The preferred configuration in the D decay should therefore be when s and ν are anti-parallel or, conversely, the configuration with parallel s and ν being anti-parallel with e^+ is disfavoured (due to angular momentum conservation, all particles are either left handed



or right-handed



for back-to-back decays). The positron spectrum in the decays is therefore soft and the neutrino spectrum hard. For B decays, the electron spectrum is hard and the neutrino spectrum is soft.

It is useful to define $x = \frac{2E_e}{m_Q}$, $y = \frac{2E_\nu}{m_Q}$, where $Q = c, b$ and the energies are defined in the rest frame of the heavy quark. For the three-body semi-leptonic decay rate, it is then found exercise

$$\Gamma_{sl}^{(Q)} = \frac{m_Q}{2^8 \pi^3} \int dx\, dy\, \vartheta(x+y-x_m)\,\vartheta(x_m - x - y + xy) \overline{\sum} |\mathcal{M}^{(Q)}|^2$$

where $x_m = 1 - \left(\frac{m_q}{m_Q}\right)^2$ defines the kinematic end points. Inserting the squared matrix element yields

$$\frac{d\Gamma_{sl}^{(c)}}{dx\,dy} = |V_{cs}|^2 \frac{G_F^2 m_c^5}{192\pi^3} \left[12\, x\,(x_m - x)\right]$$

$$\frac{d\Gamma_{sl}^{(b)}}{dx\,dy} = |V_{cb}|^2 \frac{G_F^2 m_b^5}{192\pi^3} \left[12\, y\,(x_m - y)\right]$$

such that the charged lepton spectra are

$$\frac{d\Gamma_{sl}^{(c)}}{dx} = |V_{cs}|^2 \frac{G_F^2 m_c^5}{192\pi^3} \frac{12\, x^2 (x_m - x)^2}{(1-x)}$$

cf. Figure

$$\frac{d\Gamma_{sl}^{(b)}}{dx} = |V_{cb}|^2 \frac{G_F^2 m_b^5}{192\pi^3} \frac{2x^2 (x_m - x)^2}{(1-x)^3} \left(6 - 6x + x x_m + 2x^2 - 3x_m\right)$$

Notice that as for the decays $B \to X_c e^- \bar{\nu}$, the electron spectrum in the CKM disfavoured processes $B \to X_u e^- \bar{\nu}$ is hard, but it extends beyond the kinematic threshold due to the c-quark mass. The measurement of electrons beyond that kinematic limit is therefore the method by which V_{ub} is determined.

For hadronic decays, the spectator model turns out to be entirely insufficient to describe the data. In particular, the obvious prediction that the decay rates of all of the different B or D mesons is not what is observed.

The calculation of the semi-leptonic rates can be improved by the use of hadronic matrix elements, that are obtained experimentally, from numerical lattice calculations, phenomenological models and combinations of these methods. For example, for D^0 decays, the matrix elements can be expressed in terms of form factors

$$\langle K^-(\vec{p}\,')|\,\bar{s}\gamma^\mu c\,|\,D^0(\vec{p}\,)\rangle = f_+(p+p')_\mu + f_-(p-p')_\mu$$

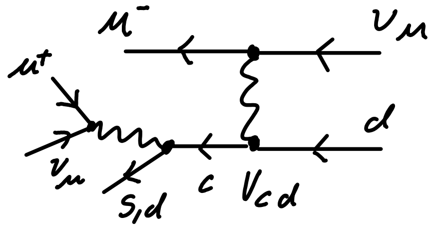
$$\langle K^{*-}(\vec{p}\,')|\,\bar{s}\gamma^\mu c\,|\,D^0(\vec{p}\,)\rangle = i\,g\,\varepsilon_{\mu\nu\alpha\beta}\,\varepsilon^{*\nu}(p+p')^\alpha(p-p')^\beta$$

$$\langle K^{*-}(\vec{p}\,')|\,\bar{s}\gamma^\mu\gamma^5 c\,|\,D^0(\vec{p}\,)\rangle = f_1\,\varepsilon^*_\mu + \varepsilon^*\cdot q\,[f_2(p+p')_\mu + f_3 q_\mu]$$

The form factors are functions of the four-momentum transfer $q^2 = (p-p')^2$, and ε_μ is the polarisation of K^{*-}. These matrix elements can be convoluted with the weak interaction vertices and propagators to obtain the semi-leptonic decay rates of the D^0.

A reason for the importance of the semileptonic decays of the B and D mesons is that these give rise to the most precise determinations of $|V_{cb}|$, $|V_{ub}|$ and $|V_{cs}|$.

$|V_{cd}|$ is best determined from dimuon production in (anti-) neutrino scatterings off nuclei.



Of course, prior knowledge of the PDFs and V_{cs} is required for this analysis.

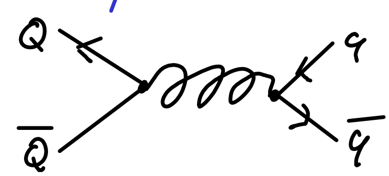
B mesons are extensively studied particles in current high energy physics. There are two important experiments (Belle and BaBar) that produce B mesons from electron positron collisions (the maximal collision energy of the machines is designed to be at the B resonance).

Besides, b-quark pairs are copiously produced at hadron colliders, where they are either studied by dedicated experiments (LHCb) or their production must be understood, as they may be a background to more interesting events (such as Higgs boson decays) that come along with b-quark production. Moreover, the following calculations apply to top-quark production as well and they are another important application of the methods for deeply inelastic scattering in hadron collisions.

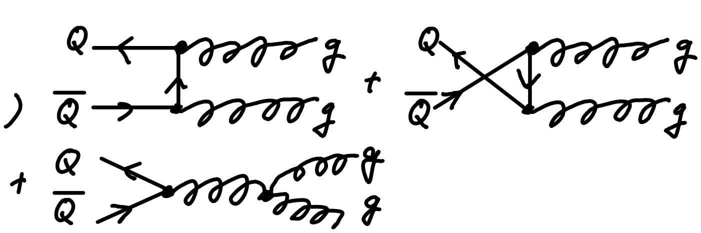
Let Q be a heavy, q a light quark and g a gluon.

The main parton-level production processes are

(a) $q(p_1) + \bar{q}(p_2) \longrightarrow Q(p_3) + \bar{Q}(p_4)$



(b) $g(p_1) + g(p_2) \longrightarrow Q(p_3) + \bar{Q}(p_4)$



Recall that $f_i(x_i)dx_i$ is the probability, that the hadron contains a parton of type i and energy fraction within x_i and x_i+dx_i. We indicate the parton-level Mandelstam parameters and cross-sections with a hat. For the collision of a proton with a (anti-) proton, we then have

$$\sigma(P_1,P_2)=\sum_{ij}\int dx_1\,dx_2\,f_i(x_1)f_j(x_2)\,\hat{\sigma}_{ij}(p_1,p_2)$$

where $p_{1,2}=x_{1,2}P_{1,2}$

Furthermore, notice that (collinear frame)

$$\hat{s}=(p_1+p_2)^2=(E_1+E_2)^2-(E_1-E_2)^2=4E_1E_2$$

such that we may write

$$d\hat{\sigma}_{ij}=\frac{1}{2\hat{s}}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\overline{\sum}|\mathcal{M}_{ij}|^2$$

Here, $\overline{\sum}|\mathcal{M}_{ij}|^2$ is the polarisation summed/averaged invariant matrix elements squared. When we define

$$\tau_1=\frac{2p_1\cdot p_3}{\hat{s}},\qquad \tau_2=\frac{2p_2\cdot p_3}{\hat{s}},\qquad \rho=\frac{4m^2}{\hat{s}},$$

the squared matrix elements are

$$\overline{\sum}|\mathcal{M}_{ij}|^2_{q\bar{q}\to Q\bar{Q}}=g_s^4\,\frac{4}{9}\left(\tau_1^2+\tau_2^2+\frac{\rho}{2}\right)$$

$$\overline{\sum}|\mathcal{M}_{ij}|^2_{gg\to Q\bar{Q}}=g_s^4\left(\frac{1}{6\tau_1\tau_2}-\frac{3}{8}\right)\left(\tau_1^2+\tau_2^2+\rho-\frac{\rho^2}{4\tau_1\tau_2}\right)$$

Since the heavy quark pair is in general boosted with respect to the proton-(anti-) proton collision, it is useful to parametrise the outgoing momenta in a way that can easily be transformed with respect to Lorentz boosts along the collision axis. Choosing this axis to be z, we may write

$$k^\mu = (E, k_x, k_y, k_z) = (m_T \cosh y, k_T \sin\phi, k_T \cos\phi, m_T \sinh y),$$

where $m_T = \sqrt{k_T^2 + m_Q^2}$ is the transverse mass and

$y = \frac{1}{2} \log \frac{E + k_z}{E - k_z}$ the rapidity.

$$\left(\begin{array}{l} \sinh y = \beta\gamma, \\ \cosh y = \gamma, \end{array} \quad -\sinh y + \cosh y = e^{-y} = (1-\beta)\gamma = \sqrt{\frac{1-\beta}{1+\beta}} \right)$$

$$\frac{dk^3}{dy} = E \quad \Rightarrow \quad d^3k = \frac{dy}{E}\, d^2k_T$$

Recall as well that the rapidity is additive under subsequent boosts.

For the present example, we parametrise (hadronic CMS)

$$p_1 = \frac{1}{2}\sqrt{s}\,(x_1, 0, 0, x_1)$$
$$p_2 = \frac{1}{2}\sqrt{s}\,(x_2, 0, 0, -x_2)$$
$$p_3 = (m_T \cosh y_3, p_T, 0, m_T \sinh y_3)$$
$$p_4 = (m_T \cosh y_4, -p_T, 0, m_T \sinh y_4)$$

Energy-momentum conservation gives the relations

$$\frac{1}{2}(x_1 + x_2) = \frac{m_T}{\sqrt{s}}(\cosh y_3 + \cosh y_4) \qquad x_1 = \frac{m_T}{\sqrt{s}}(e^{y_3} + e^{y_4})$$
$$\Rightarrow$$
$$\frac{1}{2}(x_1 - x_2) = \frac{m_T}{\sqrt{s}}(\sinh y_3 + \sinh y_4) \qquad x_2 = \frac{m_T}{\sqrt{s}}(e^{-y_3} + e^{-y_4})$$

$$\hat{s} = (p_3 + p_4)^2 = m_T^2 (2 + 2\cosh y_3 \cosh y_4 - 2 \sinh y_3 \sinh y_4)$$
$$= 2 m_T^2 (1 + \cosh(y_3 - y_4))$$

$$(p_1 + p_2)^2 = \hat{s} = \frac{1}{4} s\, 4 x_1 x_2 = s\, x_1 x_2$$

We make use of the four-dimensional δ-function in order to perform two of the p_T integrals as well as the x_1, x_2 integrations, which give the factor

$$\int dx_1\, dx_2 \;\; \delta\left(\frac{1}{2}\sqrt{s}\,(x_1+x_2) - p_3^0 - p_4^0\right)\, \delta\left(\frac{1}{2}\sqrt{s}\,(x_1-x_2) - p_3^z - p_4^z\right)$$

$$= \frac{1}{2s}$$

$$a = \frac{1}{2}(x_1+x_2) \qquad \left|\frac{\partial(a,b)}{\partial(x_1,x_2)}\right| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{vmatrix}$$

$$b = x_1 - x_2$$

Putting these pieces together, we express the cross section as

$$\frac{d\sigma}{dy_3\, dy_4\, d^2p_T} = \frac{1}{16\pi^2 \hat{s}^2} \sum_{ij} x_1 f_i(x_1)\, x_2 f_j(x_2)\, \overline{\sum} |\mathcal{M}_{ij}|^2$$

$$= \frac{1}{64\pi^2 m_T^4 \left(1+\cosh(y_3-y_4)\right)^2} \sum_{ij} x_1 f_i(x_1)\, x_2 f_j(x_2)\, \overline{\sum} |\mathcal{M}_{ij}|^2$$

In terms of the above variables, the squared matrix elements are

$$\overline{\sum} |\mathcal{M}|^2_{q\bar{q}\to Q\bar{Q}} = \frac{4g_s^4}{9} \frac{1}{1+\cosh(y_3-y_4)} \left(\cosh(y_3-y_4) + \frac{m_Q^2}{m_T^2}\right)$$

$$\overline{\sum} |\mathcal{M}|^2_{gg\to Q\bar{Q}} = \frac{g_s^4}{24} \frac{8\cosh(y_3-y_4)-1}{1+\cosh(y_3-y_4)} \left(\cosh(y_3-y_4) + \frac{2m_Q^2}{m_T^2} - \frac{2m_Q^4}{m_T^4}\right)$$

Notice that for large $y_3 - y_4$

$$\overline{\sum} |\mathcal{M}|^2_{q\bar{q}\to Q\bar{Q}} \sim \text{const.} \qquad \overline{\sum} |\mathcal{M}|^2_{gg\to Q\bar{Q}} \sim e^{|y_3-y_4|}$$

The cross section is therefore exponentially suppressed for large $|y_3-y_4|$. Moreover, heavy quarks that are produced in light quark annihilation are more likely to be

correlated in rapidity than heavy quarks from gluon fusion.
Moreover, we observe that due to the factor $\frac{1}{m_T^4}$, the production of heavy quark pairs of high transversal momentum is suppressed.
To summarise, heavy quark pairs are preferentially produced
- collinear with the collision axis,
- and correlated in rapidity (small $|y_3 - y_4|$). cf. Figure

Note that this explains the peculiar design of the LHCb detector.
Now that the kinematics is worked out, it is in order to justify the treatment of these strong interaction processes in a perturbative manner. The square-momentum transfers in the propagators of the heavy quark production amplitudes are given by

$$
\begin{aligned}
(p_1+p_2)^2 &= 2p_1\cdot p_2 = 2m_T^2\left(1+\cosh(y_3-y_4)\right)\\
(p_1-p_3)^2-m_Q^2 &= -2p_1\cdot p_3 = -m_T^2\left(1+e^{-(y_3-y_4)}\right)\\
(p_2-p_3)^2-m_Q^2 &= -2p_2\cdot p_3 = -m_T^2\left(1+e^{y_3-y_4}\right)
\end{aligned}
$$

Hence, all of these virtualities are bounded from below by the heavy quark scale $m_Q \gg \Lambda_{QCD}$, such that we are indeed in the perturbative domain. In turn, above calculation is not applicable to the production of light quarks.
Notic however that there is a strong scale dependence on the renormalisation scale μ, which should be compensated for by the DGLAP evolution of the PDFs. In practice, it turns out that accurate predictions

for the $t\bar{t}$ productions are available, while the $b\bar{b}$ and $c\bar{c}$ production at hadron colliders suffer from a large theoretical uncertainty.

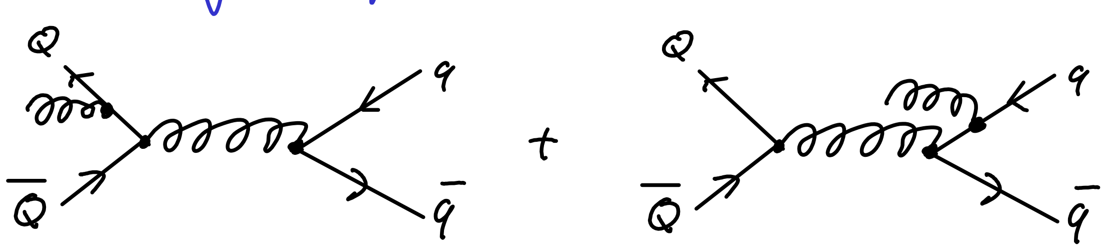
We should mention here also the forward-backward asymmetry A_{FB}, that is currently widely discussed. For a generic proces $f\bar{f} \longrightarrow f'\bar{f}'$, we define $\Delta y = y' - \bar{y}$ (the difference of the rapidities of f' and $\bar{f}'$). Then

$$A_{FB} = \frac{\left.\frac{d\sigma}{dX}\right|_{\Delta y > 0} - \left.\frac{d\sigma}{dX}\right|_{\Delta y < 0}}{\left.\frac{d\sigma}{dX}\right|_{\Delta y > 0} + \left.\frac{d\sigma}{dX}\right|_{\Delta y < 0}}$$

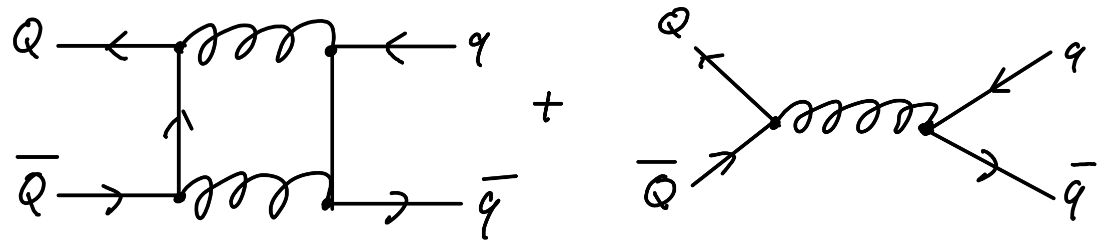
We immediatly observe two points:

- While this quantity can be directly observed at the Tevatron $p\bar{p}$ collider, the pp collider LHC defines no preferred directions for fermions over anti-fermions. Hence, A_{FB} cannot directly be determined at the LHC (there are indirect methods, to extract this quantity from the data, though, but these are comparably imprecise).
- The leading order (LO) QCD processes that we discussed above obviously do not lead to $A_{FB} \neq 0$.

To address the second point we have to go to next-to-leading order. Consider the interferences of the following diagrams:

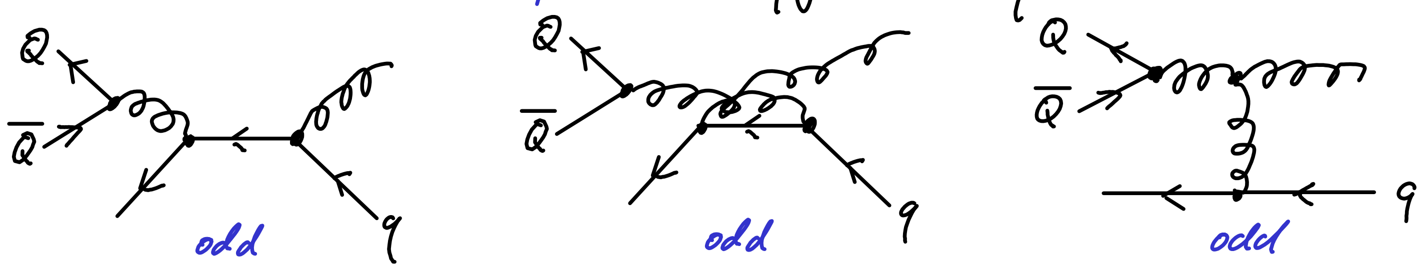


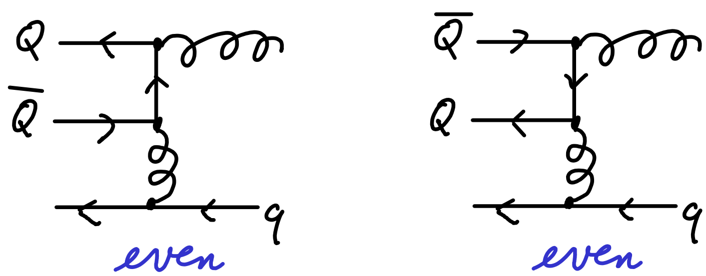
and



The diagrams on the left are even under $Q \leftrightarrow \overline{Q}$, while the diagrams on the right are odd. Their interferences can therefore lead to correlations of the rapidities of Q and q (and of course for the other pairings).

At the same order, the following diagrams contribute to A_{FB} from the process $qg \to Q\overline{Q}q$:





The thing is now, that the NLO theory prediction

$$A_{FB} = \left(4{,}8\,^{+0{,}5}_{-0{,}4}\right)\% \qquad \text{arXiv:0901.0002}$$

deviates from the measurement of the CDF detector at Tevatron

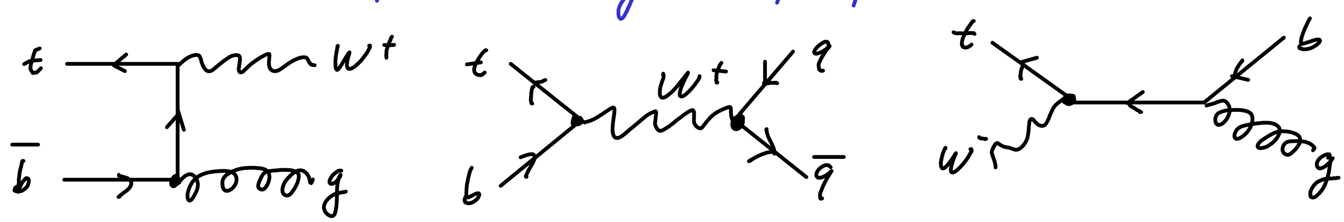
$$A_{FB} = 15{,}0 \pm 5{,}5\%$$

With Tevatron being decommissioned and LHC not being competitive in this measurement, this anomaly persists for the time being and leaves room for BSM explanations.

5.2.2 Top Quark

As discussed above, top quarks are produced predominantly in pairs. The top decays preferrably $t \to W^+ + b$

The top can then be identified in case the W^+ decays into leptons. No deviation of $|V_{ub}|$ from $1 - \mathcal{O}(\lambda^4)$ has been detected. This is also in accordance with observations from single top production



$|V_{td}|$ and $|V_{ts}|$ cannot be effectively measured in tree-level processes. These can be probed in $b \to s + \gamma$ decays (as indicated above) and in mixing of B and $\bar{B}$ mesons (as we will discuss below).

5.2.3 Quarkonia

Before discussing the bound states of the heavy quarks $c\bar{c}$ and $b\bar{b}$, it is useful to review some facts about the e^+e^- bound state positronium. For vanishing angular momentum, there are four possible linear combination (let the first factor represent the positron spin, the second the electron spin, for definiteness):

$$\vec{S} = \vec{S}_{e^+} + \vec{S}_{e^-}$$

$$\left.\begin{aligned} |S=1, S_z=1\rangle &= |\uparrow\rangle|\uparrow\rangle \\ |S=1, S_z=0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right) \\ |S=1, S_z=-1\rangle &= |\downarrow\rangle|\downarrow\rangle \end{aligned}\right\} \; {}^3S_1$$

$$|S=0, S_z=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right) \qquad {}^1S_0$$

notation: ${}^{2S+1}L_J$

As we will need this later in a different context, we verify the eigenvalues

$$\langle\uparrow|\langle\uparrow|\, \vec{S}^2\, |\uparrow\rangle|\uparrow\rangle = \langle\uparrow|\langle\uparrow|\, \vec{S}_{e^+}^2 + \vec{S}_{e^-}^2 + 2\vec{S}_{e^+}\cdot\vec{S}_{e^-}\, |\uparrow\rangle|\uparrow\rangle$$

$$= \frac{3}{4} + \frac{3}{4} + 2\,\frac{1}{2}\,\frac{1}{2} = 2 = S(S+1)$$

$$\frac{1}{2}\left(\langle\uparrow|\langle\downarrow| \pm \langle\downarrow|\langle\uparrow|\right)\, \vec{S}^2\left(|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle\right) =$$

$$= \frac{3}{4} + \frac{3}{4} - 2\,\frac{1}{2}\,\frac{1}{2} \pm \langle\uparrow|\langle\downarrow|\, \vec{S}_{e^+}\cdot\vec{S}_{e^-}\, |\downarrow\rangle|\uparrow\rangle \pm \langle\downarrow|\langle\uparrow|\, \vec{S}_{e^+}\cdot\vec{S}_{e^-}\, |\uparrow\rangle|\downarrow\rangle$$

$$= 1 \pm \frac{1}{2} \pm \frac{1}{2} = S(S+1)$$

The first three states belong to the ortho-positronium triplet and the last one is referred to as the para-positronium singlet.

Since the ortho-positronium is a vector particle, we can infer from our table of fermion bilinears that it is C-odd, while the pseudoscalar para-positronium is C even. Alternatively, we note that the C-conjugation takes the effect on above wave functions that it exchanges particle and anti-particle spinors. Due to the anti-commuting nature of the fermion fields, this yields a factor of -1 and moreover, for the para-positronium, there is an extra factor of -1

from the wave function.
Now, since photons are C-odd and QED is C-conserving, para-positronium will always decay into even numbers of photons, while ortho-positronium decays into odd numbers. The decay into a single photon is not possible, and decays into larger numbers of photons are suppressed by powers of α and by phase-space factors. Thus, the life-time of para-positronium is found to be 0,125 ns, whereas the life-time of ortho-positronium takes the much larger value of 142 ns. This is a typical example for the use of discrete symmetries in the qualitative understanding of particle physics reactions.
For positronium, the reduced mass is $\mu = \frac{m_e^2}{m_e + m_e} = \frac{m_e}{2}$.
In the non-relativistic approximation, the energy levels are given by $E_n = -\frac{R}{n}$, where $n = N + l + 1$ is the principal quantum number following the convention of atomic physics (N: number of nodes of the radial wave function, l: orbital angular momentum) and $R = \frac{1}{2}\mu\alpha^2$ is the Rydberg constant. The fact that $\frac{1}{2}\alpha^2 \ll 1$ justifies the non-relativistic approximation.
Ortho- and para-positronium are S-wave states ($l=0$). For these, the non-relativistic radial wave function is given by

$$\psi(r,\vartheta,\varphi) = \frac{1}{\sqrt{4\pi}} R_0(r), \qquad R_0(r) = 2\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-\frac{r}{a_0}},$$

where $a_0 = \frac{1}{\alpha\mu}$ is the Bohr radius.

The square modulus of this state at the origin is

$$|R_0(0)|^2 = \frac{4}{a_0^3} = 4\alpha^3\mu^3.$$

From this, we obtain the flux of e^+e^- into the annihilation region (the origin)

$$\mathcal{F} = \frac{v}{4\pi}|R(0)|^2,$$ where v is the relative velocity.

On the other hand, the polarised annihilation cross section into two photons at low energies is given by

$$\sigma_{e^+e^-\to\gamma\gamma} = \frac{4\pi}{v}\left(\frac{\alpha}{m_e}\right)^2$$

The decay rate then is

$$\Gamma_{^1S_0\to\gamma\gamma} = \mathcal{F}\,\sigma = \left(\frac{\alpha}{m_e}\right)^2|R(0)|^2 = \frac{1}{2}\alpha^5 m_e.$$

A less simple derivation for the ortho-positronium yields

$$\Gamma_{^3S_1\to\gamma\gamma\gamma} = \frac{\alpha^3}{m_e^2}\,\frac{4(\pi^2-9)}{9\pi}\,|R_0(0)|^2 = \frac{2(\pi^2-9)}{9\pi}\alpha^6 m_e$$

Now back to the hadrons.

In 1974, a narrow resonance was discovered and named the J/ψ. Its quantum numbers $J^{PC} = 1^{--}$ are the same as for the ortho-positronium. The mass and the width are

$$m_{J/\psi} = 3096 \pm 0{,}011 \text{ MeV}$$
$$\Gamma_{J/\psi} = 92{,}9 \pm 2{,}8 \text{ keV}$$

PDG 2012, p. 110

The simple origin of the curious name is that it was discovered simultaneously by two different groups: at SLAC, it was named ψ and at Brookhaven J. Apparently, there is a similarity with the family name of the leader of the Brookhaven group Samuel Ting, which is written in Chinese by the character J. He thus modestly named the particle after himself.

Finding the J/ψ amounted to discovery of the c quark that was predicted within the GIM mechanism, as it was interpreted the as the $c\bar{c}$ bound state. The Υ, which is the $b\bar{b}$ bound state, was then discovered at Fermilab in 1977.

Moreover, quarkonia served as a spectacular confirmation of QCD: while for the light mesons made up out of u, d, s, the theory is in the strong coupling regime, for the quarkonia, QCD is close to the perturbative regime and hence they bear essential similarities with positronium, in particular the narrow width of J/ψ in analogy with the long life time of ortho positronium. When r is the size of the bound state and v the typical velocity, α_s is to be evaluated as

$$v \sim \alpha_s\left(\frac{1}{r^2}\right), \quad r \sim \frac{1}{mv}$$

For charmonium, detailed studies find that $v^2 \simeq 0{,}23\, c^2$, for bottonium $v^2 \simeq 0{,}08\, c^2$. This suggests that quarkonia are described by QCD at the transition between the non-perturbative and the perturbative regimes. This can be mimicked by a phenomenological quark-antiquark potential, that takes at short distances the perturbative, single gluon exchange, Coulomb form

$$V(r) \sim -\frac{4}{3}\frac{\alpha_s\left(\frac{1}{r^2}\right)}{r}$$

and at large distances the non-perturbative

$$V(r) \sim K r,$$

where K is a string tension.

Observed masses of the quarkonia agree well with an improved version of this phenomenological potential.

We should note that for quarkonia, the definition of the principal quantum number follows nuclear physics, where $n = N+1$. In the non-relativistic weak coupling limit, we expect therefore a degeneracy between the 2S and 1P states, which in reality is lifted due to non-Abelian and relativistic corrections.

We can also draw the analogy to the positronium decays. The $c\bar{c}$ state $J^{PC} = 0^{-+}$ is called the η_c (PDG 2012 p. 109), and it is analogous to para-positronium. Denoting the heavy quark charge by Q, the ratios between gluonic and photonic amplitudes are as follows:

$$\frac{|\mathcal{M}(\eta_c \to gg)|}{|\mathcal{M}(\eta_c \to \gamma\gamma)|} = \frac{\alpha_s}{Q^2\alpha}\frac{(\lambda^a)^i{}_j(\lambda^b)^j{}_i}{\delta^i{}_j\,\delta^j{}_i} = \frac{\alpha_s}{Q^2\alpha}\frac{\frac{1}{2}\delta_{ab}}{N}$$

where we have used $\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$ and $N=3$.

Since $\delta_{ab}\delta^{ab} = N^2-1$, we obtain

$$\frac{\Gamma(\eta_c \to gg)}{\Gamma(\eta_c \to gg)} = \left(\frac{\alpha_s}{Q^2\alpha}\right)^2 \frac{\frac{1}{4}(N^2-1)}{N^2} = \frac{2}{9}\frac{\alpha_s^2}{Q^4\alpha^2}$$

From the comparison with the decay rate of para-positronium, we can then infer that

$$\Gamma(\eta_c \to gg) = \frac{2}{3}\left(\frac{\alpha_s}{m_c}\right)^2 |R(0)|^2$$

For the J/ψ, we find the relations

$$\frac{|\mathcal{M}(\eta_c \to ggg)|}{|\mathcal{M}(\eta_c \to \gamma\gamma\gamma)|} = \frac{\alpha_s^{\frac{3}{2}}}{Q^3 \alpha^{\frac{3}{2}}} \frac{[(\lambda_a)^i{}_j (\lambda_b)^j{}_k (\lambda_c)^k{}_i]_{sym}}{\delta^i{}_j \delta^j{}_k \delta^k{}_i}$$

where the subscript sym refers to a symmetrisation of the colour indices a, b, c (bosonic states must be totally symmetric). Symmetrising the a, b indices first leads to

$$\mathrm{tr}[\lambda_a \lambda_b \lambda_c]_{sym} = \frac{1}{2}\mathrm{tr}[\{\lambda_a, \lambda_b\}\lambda_c]$$

The additional symmetrisation of c is then automatically implied by the trace.

In the fundamental representation, we may use the identities

$$\{\lambda_a, \lambda_b\} = \frac{1}{N}\delta^{ab}\mathbb{1} + d^{abc}\lambda_c$$

$$d^{abc}d^{abd} = \frac{N^2-4}{N}\delta^{cd} \Rightarrow d^{abc}d^{abc} = (N^2-1)\frac{N^2-4}{N} = \frac{40}{3}$$

what leads to

$$\frac{1}{2}\mathrm{tr}[\{\lambda_a, \lambda_b\}\lambda_c] = \frac{1}{2}\mathrm{tr}[d^{abd}\lambda_d\lambda_c] = \frac{1}{4}d^{abc}$$

$$\frac{1}{4}\mathrm{tr}[\{\lambda_a, \lambda_b\}\lambda_c]\,\mathrm{tr}[\{\lambda_a, \lambda_b\}\lambda_c] = \frac{1}{16}d^{abc}d^{abc} = \frac{5}{6}$$

such that

$$\frac{|\Gamma(\eta_c \to ggg)|}{|\Gamma(\eta_c \to \gamma\gamma\gamma)|} = \frac{5}{6}\frac{1}{9}\frac{\alpha_s^3}{Q^6\alpha^3} = \frac{5}{54}\frac{\alpha_s^3}{Q^6\alpha^3}$$

Comparing with the decay rate of ortho-positronium, we conclude that

$$\Gamma_{J/\psi \to \gamma\gamma\gamma} = \frac{5}{18}\frac{\alpha_s^3}{m_c^2}\frac{4(\pi^2-9)}{9\pi}|R_0(0)|^2$$

5.3 Light Mesons and Chiral Symmetry Breaking

The formation of the eight bound states $\pi^0, \pi^\pm, K^\pm, K^0, \overline{K^0}$ and η from the light quarks u, d, s can be interpreted as a consequence of the spontaneous breakdown of chiral symmetry, i.e. these particles can be understood as (pseudo-) Goldstone bosons. Moreover, this point of view gives rise to quantitative predictions in the framework of an effective field theory: Chiral Perturbation Theory (χPT).

5.3.1 Chiral Symmetry and Isospin

We first introduce chiral symmetry transformations. Let us therefore consider a global $U(N)$ symmetry with fundamental massless Dirac fermions ψ_a.

$$\mathcal{L} = \overline{\psi_a}\, i\partial\!\!\!/\, \psi_a$$

Since the kinetic term does not mix left- and right-handed fermions, we have the separate transformations

$$\psi \longrightarrow U_L \frac{1-\gamma^5}{2} \psi$$

$$\psi \longrightarrow U_R \frac{1+\gamma^5}{2} \psi$$

where $U_{L/R}$ are unitary matrices. Of course, the Lagrangian is invariant under linear combinations of these transformations as well,

$$\psi \longrightarrow U_V \psi = e^{i\alpha_V \cdot \lambda} \psi$$

$$\psi \longrightarrow U_A \psi = e^{i\alpha_A \cdot \lambda \gamma^5} \psi$$

Note: the λ are understood here to include the $\mathbb{1}$ matrix to allow for the $U(1)$ phases.

Here, V and A stand for "vector" and "axial vector" because of the Noether currents associated with these transformations:

$$j^\mu_{Va} \equiv j^\mu_a = \frac{1}{\alpha_{Va}} \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_b)}}_{=\overline{\psi}_b i\gamma^\mu} \underbrace{\Delta \psi_b}_{= i\alpha_{Va} \lambda^a_{bc} \psi_c} = \overline{\psi}_b \gamma^\mu \lambda^a_{bc} \psi_c$$

$$j^\mu_{Aa} \equiv j^{5\mu}_a = \frac{1}{\alpha_{Aa}} \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_b)}}_{=\overline{\psi}_b i\gamma^\mu} \underbrace{\Delta \psi_b}_{= i\alpha_{Aa} \lambda^a_{bc} \gamma^5 \psi_c} = \overline{\psi}_b \gamma^\mu \gamma^5 \lambda^a_{bc} \psi_c$$

The axial U_A transformation, that essentially transforms left and right chiralities with a different sign, is also called a chiral transformation.

Now, when we add a diagonal mass to the Lagrangian, such that

$$\mathcal{L} = \overline{\psi}_a i\not{\partial} \psi_a - m \overline{\psi}_a \delta_{ab} \psi_b ,$$

the Lagrangian is still invariant under vectorial transformations, while an axial transformation induces the change

$$\bar{\psi} m \psi \to m\, \psi^{\dagger} e^{-i\alpha_A \cdot \lambda_A \gamma^5} \gamma^0 e^{i\alpha_A \cdot \lambda_A \gamma^5} \psi$$

$$= m\, \bar{\psi}\, e^{2i\alpha_A \cdot \lambda_A \gamma^5} \psi$$

We say that the mass term <u>explicitly</u> (in contrast to spontaneously) breaks the chiral symmetry.

When in addition, the mass matrix is non-diagonal, i.e.

$$\mathcal{L} = \bar{\psi}_a i \not{\partial} \psi_a - \bar{\psi}_a m_{ab} \psi_b ,$$

the $SU(N)_V$-symmetry is in general explicitly broken as well, such that the only intact symmetry remains $U(1)_V$.

For many Particle Physics reactions, we can neglect the light quark masses $m_{u,d}$ and sometimes m_s as well. We first consider the Lagrangian involving the two lightest quarks

$$\mathcal{L} = \bar{u}\, i \not{\partial} u + \bar{d}\, i \not{\partial} d - m_u \bar{u} u - m_d \bar{d} d$$

When we arange these quarks into $SU(2)$-doublets

$$Q = \begin{pmatrix} u \\ d \end{pmatrix}$$

and neglect the quark masses, we obtain a Lagrangian

$$\mathcal{L} = \bar{Q}\, i \not{\partial} Q ,$$

which is globally $U(2)_V \times U(2)_A$ symmetric. The quark masses lead to the small but explicit symmetry breaking to $U(1)_V$ (which of course happens to coincide with the local electromagnetic symmetry).

The approximate $SU(2)_V$ symmetry of the quark model is referred to as isospin and it explains why some hadrons consisting of light quarks (i.e. π^0 and $\pi^\pm$ or p and n) are approximately equal in mass.
The pions are bound states of u, d and their antiquarks:

$$|\pi^+\rangle = |u\bar{d}\rangle = |Q\, i\gamma^5 \tfrac{1}{2}(\sigma^1 - i\sigma^2)\bar{Q}\rangle \qquad I_3 = +1$$
$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) = \frac{1}{\sqrt{2}}|Q\, i\gamma^5 \sigma^3 \bar{Q}\rangle \qquad I_3 = 0$$
$$|\pi^-\rangle = |d\bar{u}\rangle = |Q\, i\gamma^5 \tfrac{1}{2}(\sigma^1 + i\sigma^2)\bar{Q}\rangle \qquad I_3 = -1$$

Note the analogy with the ortho-positronium spin-triplet.
Denoting $|\pi^{\pm,0}\rangle = |Q\, e^{\pm,0}\bar{Q}\rangle$, we note that the 3-component of isospin is $I_3\, e = \sigma^3 e - e\,\sigma^3$. Therefore, the pions transform into one another under $U(2)_V$ isospin transformations. In the above notation, $|\pi^{\pm,0}\rangle$ is understood as a spin and flavour wave-function and not as a single particle excitation in the Fock space. Notice that the factor $i\gamma^5$ makes sure that the wave-function is pseudoscalar, as we expect for a spin 0 $q\bar{q}$ bound state with no orbital angular momentum.
The fact that pions as the lowest-lying excitations of QCD are $\bar{q}q$ bound states, which corresponds to an operator that is not invariant under chiral rotations, suggests that already the vacuum does not respect this symmetry. With only the quark fields at hand, we postulate that chiral symmetry is spontaneously broken through the chiral quark condensate

$$\langle 0|\bar{Q}Q|0\rangle = -F_\pi^2 B_0$$

Here, F_π and B_0 are constant, where F_π is the pion decay constant and the meaning of both will be clarified below. The existence of the chiral condensate is confirmed in lattice simulations as well as by the successful phenomenological description of low-energy QCD discussed below. We now observe that the excitations π^0, $\frac{1}{\sqrt{2}}(\pi^+ + \pi^-)$, $\frac{i}{\sqrt{2}}(\pi^+ - \pi^-)$ point in the same direction as the chiral symmetries broken by the chiral condensate. The pions can therefore be identified with the pseudo-Goldstone bosons from the spontaneous breakdown of chiral symmetry. As this is a global symmetry, they are not "eaten" by gauge bosons, unlike the Goldstone components of the Higgs field in the Electroweak theory. Due to the explicit breaking of chiral symmetry by the quark masses, the pions are not exactly massless and therefore only qualify as pseudo-Goldstone bosons.

5.3.2 Chiral Perturbation Theory as an Effective Theory of Low-Energy QCD

Since at low energies, the quarks do not appear as degrees of freedom of QCD and are replaced by pions in this role, we now aim for an effective theory of hadron dynamics at low energies. The idea is to construct a Lagrangian that has the same symmetries as the underlying quark Lagrangian and the same

pattern of chiral symmetry breaking. The couplings in this Lagrangian must then be determined experimentally or by non-perturbative, i.e. lattice, methods.
We now introduce the isospin nucleon doublet

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

and the pion triplet $\vec{\pi} = \{\pi^i\}$ $(i=1,2,3)$, which forms an adjoint field of the isospin $SU(2)$ as $\vec{\pi}\cdot\vec{\tau}$, where $\vec{\tau}$ is the vector of the three Pauli matrices.
By a certain choice of basis in accordance with the discussion above, we can identify

$\pi^0 = \pi^3$ and $\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2)$

A Lagrangian that is invariant under the global isospin symmetry $U = e^{-\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}}$

$$\psi \longrightarrow U\psi$$
$$\vec{\tau}\cdot\vec{\pi} \longrightarrow U\,\vec{\tau}\cdot\vec{\pi}\,U^\dagger$$

is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - M_N)\psi + \frac{1}{2}\left[\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - m_\pi^2\,\vec{\pi}\cdot\vec{\pi}\right] + ig\,\bar{\psi}\vec{\tau}\cdot\vec{\pi}\,\gamma_5\psi - \frac{\lambda}{4}(\vec{\pi}\cdot\vec{\pi})^2$$

Here, $M_N = \mathrm{diag}(m_N, m_N)$ is the nucleon mass matrix, where $m_N \simeq m_p \simeq m_n$.
We have written the pion-nucleon interaction in such a manner that it is parity conserving as it should, since it derives from QCD. Moreover, in order to verify the isospin symmetry, it is useful to note that

$$\vec{\pi}\cdot\vec{\pi} = \frac{1}{2}\,\mathrm{tr}\,(\vec{\tau}\cdot\vec{\pi}\;\vec{\tau}\cdot\vec{\pi})$$

We can use Noether's theorem to identify the isospin currents as

$$\psi \longmapsto e^{-\frac{i}{2}\tau^i\alpha^i}\psi \approx \psi - \frac{i}{2}\tau^i\alpha^i\psi$$

$$\tau^i\pi^i \longmapsto e^{-\frac{i}{2}\tau^j\alpha^j}\,\tau^i\pi^i\, e^{\frac{i}{2}\tau^k\alpha^k}$$

$$\approx \tau^i\pi^i + \frac{i}{2}[\tau^i,\tau^j]\,\alpha^j\pi^i = \tau^i\pi^i - \varepsilon^{ijk}\tau^k\alpha^j\pi^j$$

$$\Rightarrow \pi^i \longmapsto \pi^i - \varepsilon^{ijk}\pi^j\alpha^k \qquad j^\mu = \frac{1}{\alpha}\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)}\Delta X$$

$k \to i$
$i \to j$
$j \to k$

scalar part of current:

$$V^{Si}_\mu = -\frac{1}{\alpha^i}\partial_\mu\pi^j\varepsilon^{jli}\pi^l\alpha^i$$
$$= -\partial_\mu\pi^j\varepsilon^{jli}\pi^l$$

$$V^i_\mu = \bar\psi\gamma^i_\mu\frac{\tau^i}{2}\psi + \varepsilon^{ijk}\pi^j\partial_\mu\pi^k$$

Next, we remove the bare nucleon mass and instead introduce a scalar field σ, such that we obtain the linear sigma model

$$\mathcal{L} = \bar\psi i\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\vec\pi\cdot\partial^\mu\vec\pi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma$$
$$-g\bar\psi(\sigma - i\vec\tau\cdot\vec\pi\gamma_5)\psi + \frac{\mu^2}{2}(\sigma^2+\vec\pi^2) - \frac{\lambda}{4}(\sigma^2+\vec\pi^2)^2$$

For $\mu^2>0$, there is spontaneous symmetry breaking, but first, we consider the symmetries of the Lagrangian. It is useful to introduce the matrix field

$$\Sigma = \sigma + i\vec\tau\cdot\vec\pi$$

such that we may express

$$\sigma^2 + \vec\pi^2 = \frac{1}{2}\mathrm{tr}(\Sigma\Sigma^\dagger)$$

We can then write the Lagrangian as

$$\mathcal{L} = \bar\psi_L i\partial\!\!\!/\psi_L + \bar\psi_R i\partial\!\!\!/\psi_R + \frac{1}{4}\mathrm{tr}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger) + \frac{1}{4}\mu^2\mathrm{tr}(\Sigma^\dagger\Sigma)$$
$$-\frac{\lambda}{16}\left(\mathrm{tr}[\Sigma\Sigma^\dagger]\right)^2 - g(\bar\psi_L\Sigma\psi_R + \bar\psi_R\Sigma^\dagger\psi_L)$$

We see that the chiral fields $\psi_{L,R}$ only couple via Σ.

The Lagrangian has the left and right invariances

$$U_{L,R} = e^{-\frac{i}{2}\vec{\alpha}_{L,R}\cdot\vec{\tau}}, \quad \psi_{L,R} \longmapsto U_{L,R}\psi_{L,R}, \quad \Sigma \longmapsto U_L \Sigma U_R^+$$

Notice that this symmetry transforms between the σ and the $\vec{\pi}$ fields:

$$\sigma \longmapsto \frac{1}{2}\operatorname{tr}(U_L U_R^+)\sigma + \frac{i}{2}\operatorname{tr}(U_L \tau^k U_R^+)\pi^k$$

$$\approx \sigma + \frac{1}{2}(\vec{\alpha}_L - \vec{\alpha}_R)\cdot\vec{\pi}$$

$$\pi^k \longmapsto -\frac{i}{2}\operatorname{tr}(\tau^k U_L U_R^+)\sigma + \frac{1}{2}\operatorname{tr}(\tau^k U_L \tau^l U_R^+)\pi^l$$

$$\approx \pi^k - \frac{1}{2}(\alpha_L^k - \alpha_R^k)\sigma - \frac{i}{4}\operatorname{tr}(\tau^k \alpha_L^i \tau^i \tau^l - \tau^k \tau^l \alpha_R^i \tau^i)\pi^l$$

$$= \pi^k - \frac{1}{2}(\alpha_L^k - \alpha_R^k)\sigma + \frac{1}{2}\varepsilon^{kil}(\alpha_L^i + \alpha_R^i)\pi^l$$

$$= \pi^k - \frac{1}{2}(\alpha_L^k - \alpha_R^k)\sigma - \frac{1}{2}\varepsilon^{klm}(\alpha_L^m + \alpha_R^m)\pi^l$$

We refer to the full symmetry group as $SU(2)_L \times SU(2)_R$. The left and right conserved currents are

$$J_{L\mu}^k = \overline{\psi_L}\gamma_\mu \frac{\tau^k}{2}\psi_L - \frac{i}{8}\operatorname{tr}(\tau^k(\Sigma\partial_\mu\Sigma^+ - \partial_\mu\Sigma\Sigma^+))$$

$$= \overline{\psi_L}\gamma_\mu \frac{\tau^k}{2}\psi_L - \frac{1}{2}(\sigma\partial_\mu\pi^k - \pi^k\partial_\mu\sigma) + \frac{1}{2}\varepsilon^{klm}\pi^l\partial_\mu\pi^m$$

$$J_{R\mu}^k = \overline{\psi_R}\gamma_\mu \frac{\tau^k}{2}\psi_R + \frac{i}{8}\operatorname{tr}(\tau^k(\partial_\mu\Sigma^+\Sigma - \Sigma^+\partial_\mu\Sigma))$$

$$= \overline{\psi_R}\gamma_\mu \frac{\tau^k}{2}\psi_R + \frac{1}{2}(\sigma\partial_\mu\pi^k - \pi^k\partial_\mu\sigma) + \frac{1}{2}\varepsilon^{klm}\pi^l\partial_\mu\pi^m$$

The vector current is identical to the previously considerd isospin current

$$V_\mu^k = J_{L\mu}^k + J_{R\mu}^k = \overline{\psi}\gamma_\mu \frac{\tau^k}{2}\psi + \varepsilon^{klm}\pi^l\partial_\mu\pi^m$$

and there is also an axial vector current

$$A^k_\mu = -j^k_{L\mu} + j^k_{R\mu} = -\bar{\psi}\gamma_\mu\gamma_5\frac{\tau^k}{2}\psi - \pi^k\partial_\mu\sigma + \sigma\partial_\mu\pi^k$$

Now, for $\mu^2 > 0$, the axial symmetry turns out to be spontaneously broken. The minimisation of the scal potential requires that ar

$$\sigma^2 + \vec{\pi}^2 = \frac{\mu^2}{\lambda}.$$

Consider now the particular ground state

$$\langle\sigma\rangle = \sqrt{\frac{\mu^2}{\lambda}} = F_\pi, \quad \langle\vec{\pi}\rangle = \vec{0},$$ where F_π is referred to a as the pion decay constant. s

Other choices are physically indistinguishable, up to field redefinitions. Now we shift the field to

$$\tilde{\sigma} = \sigma - F_\pi$$

such that we obtain the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - gF_\pi)\psi + \frac{1}{2}\left[\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} - 2\mu^2\tilde{\sigma}^2\right] + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}$$
$$- g\bar{\psi}(\tilde{\sigma} - i\vec{\tau}\cdot\vec{\pi}\gamma_5)\psi - \lambda F_\pi\tilde{\sigma}(\tilde{\sigma}^2 + \vec{\pi}^2) - \frac{\lambda}{4}\left[(\tilde{\sigma}^2 + \vec{\pi}^2)^2 - F_\pi^4\right]$$

which manifests the spontaneous breakdown of the $SU(2)_A$ symmetry, while it leaves $SU(2)_V$ intact. We can describe the pattern of symmetry breaking by

$$SU(2)_L \times SU(2)_R \longrightarrow SU(2)_V \quad \text{or} \quad SU(2)_V \times SU(2)_A \longrightarrow SU(2)_V.$$

The three pions $\vec{\pi}$ can be identified as the associated massless Goldstone bosons.

We should now wonder whether the linear sigma model is the unique way of specifying a theory of pions that complies with the symmetries of QCD and their spontaneous breakdown. For this purpose, we introduce the nonlinear exponential parametrisation of the sigma model

$$\Sigma = \sigma + i\vec{\tau}\cdot\vec{\pi} = (v+S)\,U \qquad U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi}}$$

such that $\vec{\pi}' = \vec{\pi} + \ldots$ and $S = \tilde{\sigma} + \ldots$ where the higher order terms are suppressed by powers of the field excitations divided by F_π. We can therefore note the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu S)^2 - 2\mu^2 S^2\right] + \frac{(F_\pi + S)^2}{4}\,\mathrm{tr}\left(\partial_\mu U \partial^\mu U^+\right) - \lambda F_\pi S^3$$
$$- \frac{\lambda}{4} S^4 + \bar{\psi} i \partial\!\!\!/ \psi + g(F_\pi + S)\left(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^+ \psi_L\right)$$

which agrees with the one previously considered up to the mentioned higher order corrections. It is interesting to note that there are no potential terms for the pions left, leaving only derivative interactions. We therefore expect that this model has very good convergence properties for soft momentum exchange. To investigate this further, and whether the linear and the nonlinear representation lead to the same predictions, we consider pion-pion scattering $\pi^+\pi^0 \longrightarrow \pi^+\pi^0$.

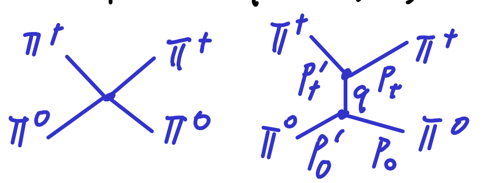
The interaction term in the linear representation is

$$\mathcal{L}_I = -\frac{\lambda}{4}\left(\vec{\pi}^2\right)^2 - \lambda F_\pi \tilde{\sigma}\,\vec{\pi}^2 .$$

When setting $q = (p_+' - p_+) = p_0 - p_0'$ and $m_{\tilde\sigma}^2 = 2\lambda F_\pi^2 = 2\mu^2$, we obtain

$$\mathcal{M}_{\pi^+\pi^0 \to \pi^+\pi^0} = -2i\lambda + (-2i\lambda v)^2 \frac{i}{q^2 - m_{\tilde\sigma}^2}$$

exercise: determine Feynman rules and factors



$$= -2i\lambda\left[1 + \frac{2\lambda F_\pi^2}{q^2 - 2\lambda F_\pi^2}\right] = i\frac{q^2}{F_\pi^2} + \mathcal{O}\left(\lambda\left(\frac{q^2}{\lambda F_\pi^2}\right)^2\right)$$

Now in the exponential representation,

$$\mathcal{L}_I = \frac{(F_\pi + S)^2}{4} \, tr\left(\partial_\mu U \partial^\mu U^+\right)$$ exercise

$$\approx \frac{F_\pi^2}{4} \, tr\Big[\Big(\partial_\mu \Big(i \frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi} - \frac{1}{2}\Big(\frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi}\Big)^2 - \frac{i}{6}\Big(\frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi}\Big)^3 \Big)\Big)$$

$$* \Big(\partial^\mu \Big(-i \frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi} - \frac{1}{2}\Big(\frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi}\Big)^2 + \frac{i}{6}\Big(\frac{\vec{\tau}\cdot\vec{\pi}'}{F_\pi}\Big)^3 \Big)\Big)\Big]$$

$$\approx \frac{1}{2} \partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi} + \frac{1}{6F_\pi^2}\left[(\vec{\pi}'\cdot\partial_\mu \vec{\pi}')^2 - \vec{\pi}'^2 (\partial_\mu \vec{\pi}'\cdot\partial^\mu \vec{\pi}') \right]$$

and one finds

$$\mathcal{M}_{\pi^+\pi^0 \to \pi^+\pi^0} = i \frac{(p_+' - p_+)^2}{F_\pi^2} .$$

This is an example of the Haag theorem, that states when two fields are related nonlinearly through $\varphi = \chi F(\chi)$ with $F(0)=1$, then the same observables result from either using $\mathcal{L}(\varphi)$ or $\mathcal{L}(\chi F(\chi))$.
For the present purposes, we conclude that the nonlinear representation is more suitable, since the calculation of the scattering amplitudes does not rely on the cancellation of large contributions and it makes no reference to the exchange of the heavy field S or σ.
The effective Lagrangian will also include higher order terms, i.e.

$$\mathcal{L} = \frac{F_\pi^2}{4} \, tr\left(\partial_\mu U \partial^\mu U^+\right) + \alpha_1 \left[tr\left(\partial_\mu U \partial^\mu U^+\right)\right]^2$$

$$+ \alpha_2 \, tr\left(\partial_\mu U \partial_\nu U^+\right) tr\left(\partial^\mu U \partial^\nu U^+\right) + \dots$$

Just as for F_π, the coefficient of these higher order terms must be determined experimentally. However, for small momentum exchange, these terms are suppressed. Note that there are no derivative-free terms, since $\mathrm{tr}(UU^+)=2$ is constant.

To make the effective Lagrangian more realistic, we must introduce a pion mass. Since the vanishing mass is protected by the Goldstone theorem, we must slightly break the axial symmetry by introducing a term

$$\mathcal{L}_{breaking} = a\sigma = \frac{a}{4}\,\mathrm{tr}(\Sigma + \Sigma^+)$$

Note that this preserves the vectorial symmetry, such that it can be used to model the explicit breaking of chiral symmetry through the quark masses.

The minimum of the potential to first order in a is

$$\langle\sigma\rangle = v = \sqrt{\frac{\mu^2}{\lambda}} + \frac{a}{2\mu^2}$$

The pion mass is induced as

$$\mathcal{L}_{breaking} = \frac{a}{4}(F_\pi + S)\,\mathrm{tr}(U+U^+) = \frac{a}{4}(F_\pi+S)\left(4 - \left(\frac{\vec{\tau}\cdot\vec{\pi}}{F_\pi}\right)^2 + \ldots\right)$$

$$= a(F_\pi+S) - \frac{a}{2F_\pi}\vec{\pi}\cdot\vec{\pi} + \ldots = a(F_\pi+S) - \frac{m_\pi^2}{2}\vec{\pi}\cdot\vec{\pi} + \ldots$$

where $m_\pi^2 = \frac{a}{F_\pi}$

Another advantage of the nonlinear (exponential) representation of the sigma model is that the symmetry currents can easily be read off from the Lagrangian as the Noether currents when neglecting m_π^2:

$$\mathcal{L} = \frac{F_\pi^2}{4}\,\mathrm{tr}(\partial_\mu U \partial^\mu U^+) + \frac{F_\pi^2 m_\pi^2}{4}\,\mathrm{tr}(U+U^+)$$

$$U_{L,R} = e^{-\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}_{L,R}}$$

$$U \to U_L U U_R^+$$

$$j^\mu = \frac{1}{\alpha}\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\Delta X$$

$$L^{k\mu} = \frac{1}{\alpha_L^k}\frac{F_\pi^2}{4}\,\mathrm{tr}\left[-\frac{i}{2}\tau^k\alpha_L^k U\partial^\mu U^\dagger + \frac{i}{2}(\partial^\mu U)U^\dagger\tau^k\alpha_L^k\right]$$

$$= -\frac{i}{2}\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\tau^k\left(U\partial^\mu U^\dagger - (\partial^\mu U)U^\dagger\right)\right]$$

$$= -i\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\tau^k U\partial^\mu U^\dagger\right]$$

$$R^{k\mu} = \frac{1}{\alpha_R^k}\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\frac{i}{2}U\tau^k\alpha_R^k\partial^\mu U^\dagger - \frac{i}{2}(\partial^\mu U)\tau^k\alpha_R^k U^\dagger\right]$$

$$= -\frac{i}{2}\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\tau^k\left(U^\dagger\partial^\mu U - (\partial^\mu U^\dagger)U\right)\right]$$

$$= -i\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\tau^k U^\dagger\partial^\mu U\right]$$

$\Longrightarrow$

$$V_\mu^k = -i\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\tau^k\left(U^\dagger\partial_\mu U + U\partial_\mu U^\dagger\right)\right]$$

$$A_\mu^k = -i\frac{F_\pi^2}{4}\,\mathrm{tr}\left[\tau^k\left(U^\dagger\partial_\mu U - U\partial_\mu U^\dagger\right)\right]$$

The addition of the explicit symmetry breaking implies that the axial current is only partially conserved, what is referred to as the Partial Conservation of the Axial Current (PCAC). To see this explicitly, note the equation of motion for $\vec{\pi}$

$$\partial^\mu(U^\dagger\partial_\mu U) + \frac{m_\pi^2}{2}(U - U^\dagger) = 0$$

such that we obtain $\qquad U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{F_\pi}}$

$$\langle 0|\partial^\mu A_\mu^k(x)|\pi^j(\vec{p})\rangle = -i\frac{F_\pi^2}{4}\frac{m_\pi^2}{2}\langle 0|-4i\,\mathrm{tr}\left(\tau^k\frac{\vec{\tau}\cdot\vec{\pi}(x)}{F_\pi}\right)|\pi^j(\vec{p})\rangle$$

$$= -F_\pi m_\pi^2\langle 0|\pi^k(x)|\pi^j(\vec{p})\rangle = -F_\pi m_\pi^2\,\delta^{kj}\,e^{-ip\cdot x}$$

It is also useful to note the matrix element

$$\langle 0 | A_\mu^k(x) | \pi^j(\vec{p}) \rangle = -i \frac{F_\pi^2}{4} \langle 0 | \mathrm{tr}\left(2\tau^k (-ip_\mu) i \frac{\vec{\tau}\cdot\vec{\pi}(x)}{F_\pi}\right) | \pi^j(\vec{p}) \rangle$$

$$= -i F_\pi p_\mu \delta^{kj} e^{-ip\cdot x}$$

These relations can be used to prove the soft pion theorem,

$$\lim_{q^\mu \to 0} \langle \pi^k(\vec{q}) \beta | \mathcal{O} | \alpha \rangle = -\frac{i}{F_\pi} \langle \beta | [Q_5^k, \mathcal{O}] | \alpha \rangle ,$$

where $\mathcal{O}$ is a local operator and $Q_5^k = \int d^3x\, A_0^k(x)$.
To show this, consider the matrix element for $\alpha \to \beta + \pi^k(q)$ and manipulate it by using the LSZ reduction formula. Moreover, we infer from the PCAC relations that we may replace

$$\pi^k = -\frac{1}{F_\pi m_\pi^2} \partial^\mu A_\mu^k ,$$

such that we obtain

$$\langle \pi^k(q) \beta | \mathcal{O}(0) | \alpha \rangle = i \int d^4x\, e^{iq\cdot x} (\partial_\mu \partial^\mu + m_\pi^2) \langle \beta | T[\pi^k(x) \mathcal{O}(0)] | \alpha \rangle$$

$$= i \int d^4x\, e^{iq\cdot x} (-q^2 + m_\pi^2) \langle \beta | T[\pi^k(x) \mathcal{O}(0)] | \alpha \rangle$$

$$= -i \frac{m_\pi^2 - q^2}{F_\pi m_\pi^2} \int d^4x\, e^{iq\cdot x} \langle \beta | T[\partial^\mu A_\mu^k(x) \mathcal{O}(0)] | \alpha \rangle$$

We commute the derivative past the time ordering by

$$\partial^\mu \langle \beta | T[A_\mu^k(x) \mathcal{O}(0)] | \alpha \rangle = \langle \beta | T[\partial^\mu A_\mu^k(x) \mathcal{O}(0)] | \alpha \rangle$$

$$+ \delta(x^0) \langle \beta | [A_0^k(x), \mathcal{O}(0)] | \alpha \rangle$$

such that

$$\langle \pi^k(q)\beta|\mathcal{O}(0)|\alpha\rangle = i\,\frac{m_\pi^2-q^2}{F_\pi m_\pi^2}\int d^4x\, e^{iq\cdot x}$$

$$* \left[\langle\beta|[A_0^k(x),\mathcal{O}(0)]|\alpha\rangle\,\delta(x^0) + iq^\mu\langle\beta|T[A_\mu^k(x),\mathcal{O}(0)]|\alpha\rangle\right]$$

Now assume that the matrix element does not vary much between the on-shell value $q^2=m_\pi^2$ and the limit $q^\mu\to 0$. We then obtain

$$\lim_{q^\mu\to 0}\langle\pi^k(q)\beta|\mathcal{O}(0)|\alpha\rangle = \frac{i}{F_\pi}\langle\beta|[Q_5^k,\mathcal{O}(0)]|\alpha\rangle + \lim_{q^\mu\to 0} iq^\mu R_\mu^k$$

where

$$Q_k^5 = \int d^3x\, A_0^k(x), \quad R_\mu^k = \frac{i}{F_\pi}\int d^4x\, e^{iq\cdot x}\langle\beta|T[A_\mu^k(x),\mathcal{O}(0)|\alpha\rangle$$

The remainder vanishes unless there is a singularity in R_μ^k for $q_\mu\to 0$, which may occur when there are intermediate states that are mass-degenerate with α or β.

When we express the charge operators in terms of the quark fields

$$Q^k = \int d^3x\, \bar\psi\gamma^0\frac{\lambda^k}{2}\psi, \qquad Q_5^k = \int d^3x\, \bar\psi\gamma^0\gamma^5\frac{\lambda^k}{2}\psi$$

we obtain the current algebra

$$[Q^i, V_\mu^j] = if^{ijk}V_\mu^k, \qquad [Q_5^i, V_\mu^j] = if^{ijk}A_\mu^k,$$

$$[Q^i, A_\mu^j] = if^{ijk}A_\mu^k, \qquad [Q_5^i, A_\mu^j] = if^{ijk}V_\mu^k.$$

Here, λ are the generators of the flavour symmetry, i.e Pauli matrices for two flavours (u,d) and Gell-Mann matrices for three flavours (u,d,s).

Due to the spontaneous breakdown of chiral symmetry, the vacuum is nontrivial, as expressed e.g. by the relation $\langle 0|\bar{Q}Q|0\rangle = -F_\pi^2 B_0$. Comparing the symmetry currents of the QCD Lagrangian with those of the low-energy effective Lagrangian, we can identify B_0 as a parameter in the latter Lagrangian.

In functional quantisation, we compute the expectation value of a current by coupling it to a source in the Lagrangian. Taking the functional derivative of the generating functional with respect to the source yields the desired expectation value.

For QCD with three flavours, we modify the Lagrangian by adding the sources

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \bar{\psi} i \not{D} \psi - \bar{\psi}\gamma_\mu \frac{1-\gamma^5}{2} l^\mu \psi - \bar{\psi}\gamma_\mu \frac{1+\gamma^5}{2} r^\mu \psi$$

$$- \bar{\psi}_L (s+ip)\psi_R - \bar{\psi}_R (s-ip)\psi_L$$

where

$$l_\mu = l_\mu^0 + l_\mu^a \lambda^a, \quad r_\mu = r_\mu^0 + r_\mu^a \lambda^a, \quad s = s^0 + s^a \lambda^a, \quad p = p^0 + p^a \lambda^a.$$

Note that the Gell-Mann matrices λ are understood to act here on flavour space (u,d,s). The QCD Lagrangian is recovered for $l_\mu = r_\mu = p = 0$ and $s = m$, with m being the 3×3 quark mass matrix.

For example, the left-handed current follows as

$$J^k_{L\mu}(x) = -\frac{\partial \mathcal{L}}{\partial l_k^\mu(x)} = \bar{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi$$

the scalar density as $\bar{\psi}(x)\psi(x) = -\dfrac{\partial \mathcal{L}}{\partial s^0(x)}$

and the pseudoscalar density as $\bar{\psi}(x) i\gamma^5 \psi(x) = -\frac{\partial \mathcal{L}}{\partial p^0(x)}$

Expectation values follow from the generating functional, for example

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle = i \frac{\delta \log Z}{\delta s^0(x)}\Bigg|_{\substack{l=r=p=0 \\ s=m}}$$

In order to construct the low-energy effective Lagrangian, it is useful to define the external sources such that these transform like gauge fields. Denoting the local flavour transformations by $L(x)$ and $R(x)$ (note that these are no actual gauge symmetries: we only introduce the sources for variational purposes and ultimately fix $l=r=p=0$, $s=m$), we impose:

$$\psi_L \to L(x)\psi_L \qquad \psi_R \to R(x)\psi_R$$

$$l_\mu \to L(x) l_\mu L^\dagger(x) \qquad r_\mu \to R(x) r_\mu R^\dagger(x)$$

$$(s+ip) \to L(x)(s+ip)R^\dagger(x)$$

The same local invariance is recovered in the effective Lagrangian when we use the covariant derivative

$$D_\mu U = \partial_\mu U + i l_\mu U - i U r_\mu$$

such that

$$U \to L(x) U R^\dagger(x) \qquad D_\mu U \to L(x) D_\mu U R^\dagger(x)$$

and

$$\mathcal{L} = \frac{F_\pi^2}{4} \operatorname{tr}(D_\mu U D^\mu U^\dagger) + \frac{F_\pi^2}{4} \operatorname{tr}(\chi U^\dagger + U\chi^\dagger)$$

where $\chi = 2B_0(s+ip)$.

By taking derivatives, we again obtain currents from the effective Lagrangian, for example

$$j^k_{L\mu}(x) = -\frac{\partial \mathcal{L}}{\partial l^\mu_k(x)} = -\frac{F_\pi^2}{4}\frac{\partial}{\partial l^\mu_k} \operatorname{tr}\left(i l^\mu_\ell \lambda_\ell U \partial_\mu U^\dagger - (\partial_\mu U) i l^\mu_\ell \lambda_\ell U^\dagger\right)$$

$$= -i\frac{F_\pi^2}{2} \operatorname{tr}\left(\lambda_k U \partial_\mu U^\dagger\right)$$

5.3.5 Pions, Kaons and Etas

We have now constructed an effective Lagrangian that describes the pseudoscalar bound states of low energy QCD. Besides the parameters of the fundamental theory (quark masses), F_π and B_0 are parameters that need to be determined from experiment or the lattice. In principle, they derive from the scale Λ_{QCD} and are of similar order.

In order to obtain the pion masses, we compare the variation of the QCD Lagrangian and the effective Lagrangian with respect to the scalar density:

$$\langle 0|\bar\psi(x)\psi(x)|0\rangle = i\frac{\delta \log Z}{\delta s^0(x)} = \dots - F_\pi^2 B_0 \simeq (250\ MeV)^3$$

$$F_\pi \simeq 93\ MeV$$

Expanding the chiral symmetry breaking term as

$$\frac{F_\pi^2}{4}\operatorname{tr}(\chi U^\dagger + U\chi^\dagger) \qquad U = e^{i\frac{\vec\tau\cdot\vec\pi}{F_\pi}}$$

$$\simeq \frac{F_\pi^2}{4} 2B_0 \operatorname{tr}\Big[m\Big(\mathbb{1} - i\frac{\vec\tau\cdot\vec\pi}{F_\pi} - \frac{1}{2}\frac{\vec\pi^2}{F_\pi^2}\Big) + \Big(\mathbb{1} + i\frac{\vec\tau\cdot\vec\pi}{F_\pi} - \frac{1}{2}\frac{\vec\pi^2}{F_\pi^2}\Big)m\Big]$$

we find that $\frac{1}{2}m_\pi^2\vec\pi^2 = -\frac{1}{2}\operatorname{tr}[m]B_0$

(Note that the pion kinetic term in the effective Lagrangian is negative.) This is the celebrated Gell-Mann-Oakes-Renner (GMOR) relation

$$m_\pi^2 = -B_0 \operatorname{tr}[m] = -B_0 (m_u + m_d)$$

We can therefore relate the pion mass to the expectation value of the quark condensate and the pion decay constant. The expectation value of the condensate can be "observed" in lattice simulations.

Next, we consider what F_π does to deserve its name. $(99{,}98770 \pm 0{,}00004)\%$ of the charged pions decay into $\mu^+ \nu_\mu$ or $\mu^- \bar{\nu}_\mu$. Note that only a fraction $1{,}23 * 10^{-4}$ goes into $e^+ \nu_e$ or $e^- \bar{\nu}_e$. These decays are mediated by the weak Hamiltonian density

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V_{ud} \bar{\psi}_d \gamma_\lambda (1-\gamma_5) \psi_u \left[\bar{\psi}_{\nu_e} \gamma^\lambda (1-\gamma_5) \psi_e + \bar{\psi}_{\nu_\mu} \gamma^\lambda (1-\gamma_5) \psi_\mu \right]$$

Using

$$\langle 0 | A_\mu^j (0) | \pi^k (\vec{p}) \rangle = -i F_\pi p_\mu \delta^{jk}$$

$$\hookrightarrow \pi^+ = \frac{1}{\sqrt{2}} (\pi^1 + i \pi^2)$$

We find that the decay $\pi^+ \longrightarrow \mu^+ \nu_\mu$ has the amplitude

$$\mathcal{M}_{\pi^+ \to \mu^+ \nu_\mu} = \frac{G_F}{\sqrt{2}} V_{ud} \sqrt{2} F_\pi p_\lambda \bar{u}_{\nu_\mu} \gamma^\lambda (1-\gamma^5) v_\mu$$

$$= -G_F V_{ud} F_\pi m_\mu \bar{u}_{\nu_\mu} (1-\gamma^5) v_\mu$$

The second equality is a consequence of the Dirac equation. Since the expression for the decay into a positron is analogous, we can explain the strong relative suppression. It is a consequence of helicity suppression. Weak interactions only produce left-handed fermions and right-handed antifermions in the massless limit. Along the decay axis, the total angular momentum is then one, which is forbidden for the decay of a spin-0 particle. Therefore, there must be a helicity flip, which explains the occurrence of a factor of the lepton mass.

Performing the phase-space integral, one finds the decay rate

$$\Gamma_{\pi^+\to\mu^+\nu_\mu} = \frac{G_F^2}{4\pi} F_\pi^2 m_\mu^2 m_\pi |V_{ud}|^2 \left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2,$$

and accordingly for decays into positrons.

Note that this is in agreement with the experimental observation (that we quoted above)

$$\frac{\Gamma_{\pi^+\to e^+\nu_e}}{\Gamma_{\pi^+\to\mu^+\nu_\mu}} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2-m_e^2}{m_\pi^2-m_\mu^2}\right)^2 = 1{,}3 * 10^{-4}.$$

The small discrepancy is resolved at next order in perturbation theory.

Now, we extend the analysis to include strange quarks, i.e. we consider

$$U = e^{i\frac{\lambda^a\pi^a}{F}}$$

and the effective Lagrangian

$$\mathcal{L} = \frac{F^2}{4}\operatorname{tr}\left[D_\mu U D^\mu U^+\right] + \frac{F^2 B_0}{2}\operatorname{tr}\left[(s+ip)U + U^+(s-ip)\right]$$

When the symmetry of this Lagrangian corresponds to QCD without external sources, we have

$$s = m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad p = 0, \quad D_\mu U = \partial_\mu U.$$

The extra particles can be associated with the additional generators of SU(3) as

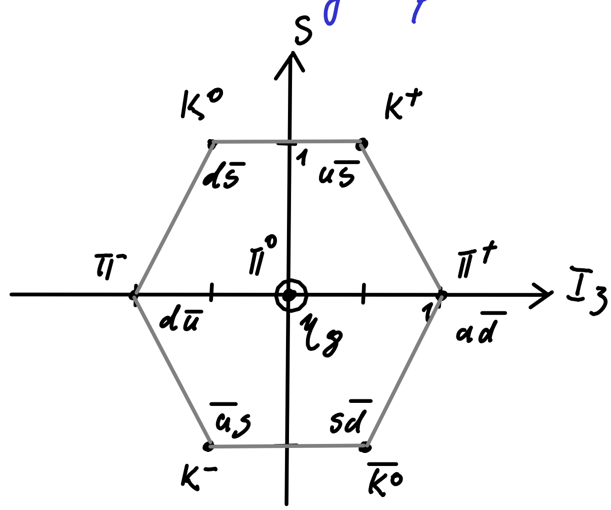
$$\frac{1}{\sqrt{2}}\sum_{a=1}^{8}\lambda^a\pi^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

and the GMOR relations are

$$m_\pi^2 = B_0(m_u+m_d), \qquad m_{K^\pm}^2 = B_0(m_s+m_u),$$

$$m_{K^0}^2 = B_0(m_s+m_d), \qquad m_{\eta_8}^2 = \frac{1}{3}B_0(4m_s+m_u+m_d).$$

These particles can be ordered geometrically in a meson octet, where the horizontal axis is the isospin I_3 and the vertical axis the strangeness S, which is defined such that the strange quark has $S=-1$.



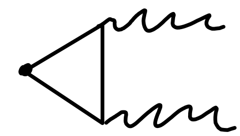
Just as the chiral effective Lagrangian, we may interpret the Standard Model Lagrangian as an effective theory. An important difference is that chiral perturbation theory predicts its own breakdown for large momenta (cf. pion-pion scattering), because the interaction terms are nonrenormalisable, while the Standard Model could be valid up to the scale of vacuum instability. Recall also our reasoning that the SM without a Higgs boson would break down at TeV-scale energies due to the increase in the cross section for WW scattering.

5.3.5 Axial Anomaly, Instantons, η' Problem, Strong CP Problem and Such

The neutral pion branches to 98,8% into two photons. The process is mediated by the axial anomaly. Recall that for a single chiral fermion, it is given by

$$\partial_\mu A^\mu = -\frac{Q^2 g^2}{32\pi^2}\,\varepsilon^{\alpha\beta\gamma\delta} F^a_{\alpha\beta} F^a_{\gamma\delta}$$

and represented by the triangle diagram



Here, Q is the gauge charge of the particular chiral fermion. Pions therefore couple to photons via

$$\partial_\mu A^\mu_3 = -\underset{\text{QCD}}{3}\,\frac{\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 e^2}{16\pi^2}\,\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} = -\frac{e^2}{16\pi^2}\,\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta},$$

where 3 indicates the generator τ^3 of isospin associated with the neutral pion.

Instead of calculating the triangle diagrams, the non-conservation of axial currents can also be derived from functional methods [K. Fujikawa, PRL 42, 1195 (1979); Phys. Rev D21, 2848 (1980)].

For a chiral transformation

$$U(x) = e^{i\gamma^5 \alpha(x) t}, \quad \psi(x) \longrightarrow U(x)\,\psi(x)$$

where t is a hermitian matrix in flavour space, we obtain

$$\mathcal{L} \longrightarrow \mathcal{L}(x) + \alpha(x)\mathcal{A}(x)$$

for a chirally invariant Lagrangian $\mathcal{L}$. (In the presence of fermion masses, that violate chiral symmetry, these pick up phases as well.)

The anomaly function is given by

$$\mathcal{A}(x) = -\frac{1}{16\pi^2}\,\varepsilon_{\mu\nu\varrho\sigma} F_a^{\mu\nu}(x) F_b^{\mu\nu}(x)\,\mathrm{tr}\{t_a t_b t\}$$

where $t_{a,b}$ are the gauge representation matrices for the fermions. For the pion field with $t = \tau^3$, we have (cf. above)

$$\mathcal{A}(x) = -\frac{e^2}{16\pi^2}\,\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}\,.$$

For a small α, the pion field transforms

$$\pi^0 \longrightarrow \pi^0 + \alpha F_\pi$$

Since we demand that the effective Lagrangian has the same symmetry properties as the QCD Lagrangian, we must add the term

$$\frac{\pi^0(x)\,\mathcal{A}(x)}{F_\pi} = -\frac{e^2}{16\pi^2}\,\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}\,\frac{\pi^0}{F_\pi}\,.$$

This is an effective vertex for $\pi^0 \longrightarrow \gamma\gamma$ and we immediately obtain

$$i\mathcal{M}_{\pi^0\to\gamma\gamma} = i\,\frac{e^2}{4\pi^2 F_\pi}\,\varepsilon_\nu^* \varepsilon_\lambda^*\, \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta$$

Performing the phase space integral, one obtains for the decay rate

$$\Gamma_{\pi^0 \to \gamma\gamma} = \frac{e^2 m_\pi^3}{2^{10}\pi^5 F_\pi^2} = 1{,}11 * 10^{16} \frac{1}{s}$$

in agreement with the experimental value $\Gamma_{\pi^0 \to \gamma\gamma} = (1{,}19 \pm 0{,}08) * 10^{16} \frac{1}{s}$. Note that the theory value scales with the factor $\left(\frac{N_c}{3}\right)^2$, where N_c is the number of colours. The success of this calculation was therefore another early confirmation of QCD.

A problem in early QCD was however, that the tree-level Lagrangian exhibits besides the $SU(3)_A$ symmetry, that is spontaneously broken by the chiral quark condensate and leads to the octet mesons also a $U(1)_A$ symmetry. One might conclude that there should be an associated pseudo-Goldstone boson η_1. However, the particle η' that has the according quantum numbers has the mass 958 MeV - much above what would be predicted from the GMOR relation.

Now the $U(1)_A$ current is anomalous under QCD. This alone does however not resolve the η'-problem by adding explicit symmetry breaking. Introducing the dual field strength tensor $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\varrho\sigma} F^{\varrho\sigma}$, we note that we can express

$$\frac{1}{4} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \partial_\mu K^\mu$$

where $K^\mu = \varepsilon^{\mu\nu\alpha\beta}\left(\frac{1}{2} A_\nu \partial_\alpha A_\beta - \frac{ig}{3} A_\nu A_\alpha A_\beta\right)$

The anomaly function is therefore a total derivative, and provided the fields decay fast enough (faster than $\frac{1}{|\vec{r}|}$), the integral vanishes. In fact, the chiral anomaly of electromagnetism does not prevent the π^0 from being a pseudo-Goldstone boson, even though A_3^μ is anomalous.

Now in QCD, there exist extended field configurations, so called instantons, that are saddle points of the Minkowskian action. They can be found as classical solutions in Euclidean space, where they are solitons. In Minkowski space, instantons are classically forbidden configurations, but they occur in quantum theory as tunneling transitions. Specifically, Euclidean instantons are given by the solutions to the Euclidean Yang-Mills equations $D_\mu F_{\mu\nu}=0$

$$A_\mu = -\frac{i}{g}\frac{x^2}{x^2+\lambda^2}\,(\partial_\mu \Omega)\Omega^{-1}$$

where λ is an arbitrary scale, $x^2=\sum_{i=1}^{4} x_i^2$ and

$$\Omega = \frac{x_4 \pm \sigma_i x_i}{\sqrt{x^2}}.$$

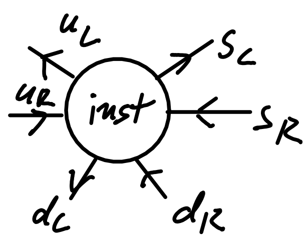
The σ_i act on an $SU(2)$ subgroup of $SU(N)$. Since Ω is a unitary matrix, A_μ reduces to a pure gauge for $x^2 \longrightarrow \infty$. However, since $A_\mu \sim \frac{1}{\sqrt{x^2}}$, the integral over the instanton does not need to vanish. In fact, it can be shown that

$$\frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \pm 1$$

There are additional solutions such that the integral can take any integer value n, which is called

the winding number. Hence, quantum effects explicitly break $U(1)_A$ in QCD and the η' is not a Goldstone boson.
We note that the observed mesons exhibit a small singlet-octet mixing such that the mass eigenstates are $\eta \approx \eta_8$ and $\eta' \approx \eta_1$.
The chiral anomaly implies that instantons are the source for chiral currents. This has been explicitly demonstrated by 't Hooft [PRL 37, 8 (1976); PRD 14, 3432 (1976); Phys. Rept. 142, 357 (1986)], who has derived an effective operator

$$\kappa e^{i\vartheta} \det \overline{\psi_L}(x)\psi_R(x) + \kappa e^{-i\vartheta} \det \overline{\psi_R}(x)\psi_L(x)$$



The determinant is over flavour space and ϑ results from a possible CP-odd Lagrangian term

$$\vartheta F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

The 't Hooft operator explicitly breaks $U(1)_A$. An according term in the low-energy effective Lagrangian should have the form

$$\propto -e^{i\frac{\eta_1}{F_\pi}} - e^{-i\frac{\eta}{F_\pi}} = -2\cos\frac{\eta_1}{F_\pi},$$

which leads to a mass for $\eta' \approx \eta_1$.
Now return to the CP-violating term $\vartheta F\tilde{F}$. Since it has the form of the anomaly function, it is

generated when rephasing the quark fields in the CKM matrix up to the one remaining phase, even if $\vartheta = 0$ for some reason before this rephasing.
An experimental signature for a non-zero ϑ would be an electric dipole moment (EDM) of the neutron

$$d_n \sim \vartheta * 10^{-15}\, e\, cm$$

whereas the current experimental upper bounds are

$$d_n \lesssim 3 * 10^{-26}\, e\, cm,$$

requiring ϑ to be of order 10^{-10} or smaller, even though we should expect it to be of order one. This is the strong CP problem.
We finally note that ϑ is sometimes interpreted as a feature of the QCD vacuum (ϑ-vacuum). For this purpose, we characterise the states with different winding number by $|n\rangle$. A gauge transformation Ω takes $|n\rangle \to |n \pm 1\rangle$. The true vacuum should only pick up a phase. We can construct a state with this property as

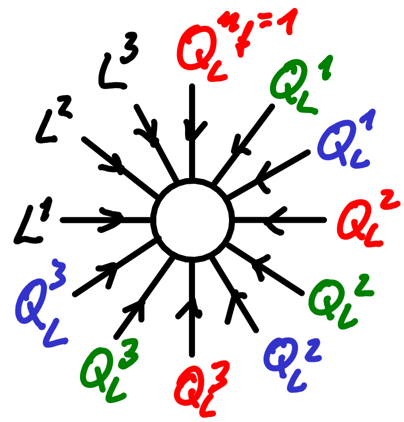
$$|vac\rangle_\vartheta = \sum_{n=-\infty}^{\infty} e^{in\vartheta} |n\rangle$$ and therefor Ω takes

$$|vac\rangle_\vartheta \to e^{\mp i\vartheta} |vac\rangle_\vartheta .$$

Sometimes, this is compared to the ground state in a double-well potential, which is due to tunneling transitions a superposition of the two naive ground states

Instanton transitions occur also for $SU(2)_L$. At zero temperature, such transitions are strongly suppressed due to the breakdown of the Electroweak symmetry. At high temperatures, in the Early Universe, Electroweak symmetry is restored and moreover, sphaleron transitions can be induced by classical statistical (thermal) fluctuations.

The effective sphaleron operator acts of course only on left-handed fermions and it violates $B+L$ (baryon plus lepton number) by six units.



Note that B and L are classical symmetries of the SM Lagrangian that are anomalously violated at the quantum level.

The weak sphaleron is likely to be of pivotal importance in cosmology, as it can transfer a lepton asymmetry that may result from the decays of sterile neutrinos (Leptogenesis) into the observed baryon asymmetry.

In case of a first order Electroweak phase transition (coexistence of symmetric and broken phase), a $B+L$ asymmetry can be produced at the phase boundary that is captured and frozen in the expanding bubbles of broken symmetry, where the sphaleron is quenched.

5.4 CP Violation and Mixing of Neutral Mesons

5.4.1 Asymmetric Decays of Sterile Neutrinos

CP-violation in meson systems is convoluted with hadronic effects. In order to introduce CP-violating effects, we therefore study first a perturbative setup that is of possible cosmological relevance as it may account for the observed baryon asymmetry of the Universe.

The Leptogenesis scenario is implemented by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\overline{N_i}(i\partial\!\!\!/ - M_i)N_i + \overline{l_a}\,i\partial\!\!\!/\, l_a + (\partial^\mu\phi^\dagger)(\partial_\mu\phi)$$
$$- Y_{ia}^*\,\overline{l_a}\,\tilde{\phi}\,\frac{1+\gamma^5}{2}N_i - Y_{ia}\,\overline{N_i}\,\tilde{\phi}^\dagger\,\frac{1-\gamma^5}{2}l_a$$

where l_a are the SM lepton doublets, ϕ the SM Higgs doublet and N_i are extra sterile Majorana neutrinos, often referred to as right-handed neutrinos.

In order to form $SU(2)_L$ invariants, we define $\tilde{\phi} = (\varepsilon\phi)^t$, where ε is the antisymmetric $SU(2)_L$ invariant tensor.

The leading-order matrix element for sterile neutrino decay is

$$i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\phi}^\dagger} = Y_{ia}^*\;\overline{u}(\vec{k},r)\,\frac{1+\gamma^5}{2}\,u(\vec{p},s)$$

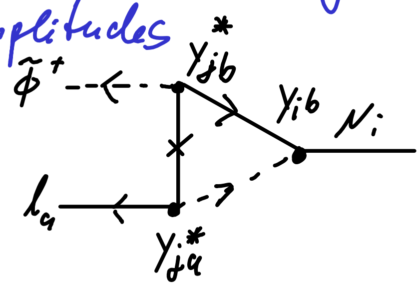
We work in the rest frame of N_i, where $\vec{p}=0$ and $|\vec{k}| = \frac{M_i}{2}$. Next, we obtain

$$\overline{\sum_{\substack{pol,\\ SU(2)_L}}} \left| i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\phi}^+}\right|^2 = \frac{1}{2}\, 2\, |Y_{ia}|^2\, \mathrm{tr}\left[\not{k}\, \frac{1+\gamma^5}{2}(\not{p}-M_i)\right]$$
$$= 2|Y_{ia}|^2\, k\cdot p = |Y_{ia}|^2 M_i^2$$

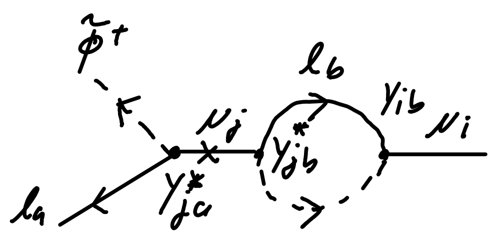
and

$$\Gamma^{LO}_{N_i\to l_a\tilde{\phi}^+} = \frac{1}{2M_i}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{M_i^2}(2\pi)^4\delta^4(p-k-k')|Y_{ia}|^2 M_i^2$$
$$= \frac{|Y_{ia}|^2}{2M_i}\int\frac{d^3k}{(2\pi)^3}\, 2\pi\, \delta(M_i - 2|\vec{k}|)$$
$$= \frac{|Y_{ia}|^2}{4M_i}\int_0^\infty \frac{\vec{k}^2\, d|\vec{k}|}{2\pi^2}\, 2\pi\, \delta\left(\frac{M_i}{2}-|\vec{k}|\right) = \frac{|Y_{ia}|^2}{16\pi} M_i$$

Similarly, we obtain $\Gamma^{LO}_{N_i\to \bar{l}_a\hat{\phi}} = \frac{|Y_{ia}|^2}{16\pi} M_i$. At LO, no asymmetry is generated in decays.
At next-to-leading order, there are two additional amplitudes



$i\mathcal{M}^{vert}_{N_i\to l_a\tilde{\phi}^+}$



$i\mathcal{M}^{wv}_{N_i\to l_a\tilde{\phi}^+}$

The decay rates that include these corrections are given by

$$\Gamma_{N_i\to l\tilde{\phi}^+} = \frac{1}{16\pi M_i}\overline{\sum_{\substack{pol,\\ SU(2)_L}}}\left| i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\phi}^+} + i\mathcal{M}^{vert}_{N_i\to l_a\tilde{\phi}^+} + i\mathcal{M}^{wv}_{N_i\to l_a\tilde{\phi}^+}\right|^2$$

The Feynman diagrams for the decay into anti-leptons are the same, up to complex conjugation of the Yukawa couplings.

The loop amplitudes naturally decompose into absorptive and dispersive contributions. The absorptive contribution is from the cut where both loop particles are on shell and is a purely imaginary term to be multiplied with the couplings. The dispersive part is purely real up to multiplication by the couplings.
We can therefore write

$$i\mathcal{M}^{vert}_{N_i\to l_a\tilde{\varphi}^+} + i\mathcal{M}^{wv}_{N_i\to l_a\tilde{\varphi}^+} = i\mathcal{M}^{abs}_{N_i\to l_a\tilde{\varphi}^+} + i\mathcal{M}^{disp}_{N_i\to l_a\tilde{\varphi}^+}$$

We attach according superscripts for the amplitudes including antileptons as well as to the individual contributions vert and wv.

Now, $i\mathcal{M}^{dis}_{N_i\to l_a\tilde{\varphi}^+} = \left(i\mathcal{M}^{dis}_{N_i\to \bar{l}_a\tilde{\varphi}}\right)^*$

$$i\mathcal{M}^{abs}_{N_i\to l_a\varphi^+} = -\left(i\mathcal{M}^{abs}_{N_i\to \bar{l}_a\tilde{\varphi}}\right)^*$$

This implies that in $\Gamma_{N_i\to l_a\tilde{\varphi}^+}$, there is an interference term

$$i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\varphi}^+}\left(i\mathcal{M}^{abs}_{N_i\to l_a\tilde{\varphi}^+}\right)^* + \left(i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\varphi}^+}\right)^* i\mathcal{M}^{abs}_{N_i\to l_a\tilde{\varphi}^+}$$

whereas in $\Gamma_{N_i\to \bar{l}_a\tilde{\varphi}}$, there is the term

$$-i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\varphi}^+}\left(i\mathcal{M}^{abs}_{N_i\to l_a\tilde{\varphi}^+}\right)^* - \left(i\mathcal{M}^{LO}_{N_i\to l_a\tilde{\varphi}^+}\right)^* i\mathcal{M}^{abs}_{N_i\to l_a\varphi^+}$$

Note that when all coupling constants are real, it follows that $i\mathcal{M}^{abs}_{N_i\to l_a\tilde{\varphi}^+} = -\left(i\mathcal{M}^{abs}_{N_i\to l_a\tilde{\varphi}^+}\right)^*$ and $i\mathcal{M}^{LO}_{N_i\to l^a\tilde{\varphi}^+} = \left(i\mathcal{M}^{LO}_{N_i\to l^a\tilde{\varphi}^+}\right)^*$, such that there is no decay asymmetry.
We therefore need to go after the absorptive parts of the

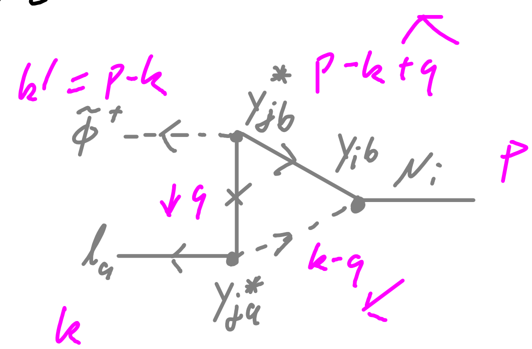
one-loop amplitudes.

For the vertex contribution, we obtain

$$\overline{\sum}\, i\mathcal{M}^{LO*}_{N_i\to l_a\tilde{\phi}^+}\, i\mathcal{M}^{vert,abs}_{N_i\to l_a\tilde{\phi}^+} = -\sum_{b,j} Y_{ia}\, Y^+_{aj}\, Y^*_{jb}\, Y^t_{bi}\, i\,\mathrm{Im}\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{(k-q)^2+i\varepsilon}$$

$$\mathrm{tr}\Big[\, \not{k}\, \frac{1+\gamma_5}{2}\, \frac{i(\not{q}+M_j)}{q^2-M_j^2+i\varepsilon}\, \frac{1+\gamma_5}{2}\, \frac{i(\not{p}-\not{k}+\not{q})}{(p-k+q)^2+i\varepsilon}\, \frac{1-\gamma_5}{2}\, (\not{p}-M_i)\Big]$$

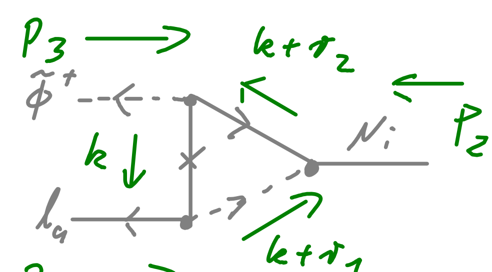
$$= i\sum_{b,j} Y_{ia}\, Y^+_{aj}\, Y^*_{jb}\, Y^t_{bi}\; \mathrm{Re}\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{(k-q)^2+i\varepsilon}$$



$$* \, 2\, \frac{M_i M_j\;\; k\cdot(p-k+q)}{[q^2-M_j^2+i\varepsilon][(p-k+q)^2+i\varepsilon]}$$

The evaluation of the loop integral is straightforward. Since we are only interested in the imaginary part, one good method is to use the optical theorem. Here, we make use of the integral tables of Passarino & Veltman (PV).

$$J_V = \mathrm{Re}\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{(k-q)^2+i\varepsilon}\,\frac{k\cdot(p-k+q)}{[q^2-M_j^2+i\varepsilon][(p-k+q)^2+i\varepsilon]}$$



$$= \mathrm{Re}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+r_1)^2+i\varepsilon}\,\frac{r_1\cdot(k+r_2)}{[k^2-M_j^2+i\varepsilon][(k+r_2)^2+i\varepsilon]}$$

$r_1 = p_1$

$r_2 = p_1+p_2$

$q = k$

$p-k+q = k+r_2$

$k = r_1$

$k-q = k+r_1$

$$= \mathrm{Re}\,\Gamma(3)\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \Big\{ r_1\cdot r_2\, I_3(r_1, r_2, \underset{m_0}{M_j}, \underset{m_1}{0}, \underset{m_2}{0}) + r_{1\mu}\, I_3^{\mu}(r_1, r_2, \underset{m_0}{M_j}, \underset{m_1}{0}, \underset{m_2}{0}) \Big\}$$

$$r_1^2 = p_1^2 = 0 \qquad r_2^2 = (p_1+p_2)^2 = M_i^2 + 2p_1\cdot p_2 = 0$$

$$p_3^2 = (p_1+p_2^2) = 0 \qquad r_1\cdot r_2 = -p_1\cdot p_3 = -\frac{1}{2}M_i^2 \quad (p_1+p_3)^2 = 2p_1\cdot p_3 = M_i^2$$

$$I_3 = \frac{i}{16\pi^2}\frac{-1}{2C} \qquad I_3^\mu = \frac{i}{16\pi^2}\frac{\underline{P}^\mu}{2C} \qquad \underline{P}^\mu = x_1 r_1^\mu + x_2 r_2^\mu$$

$$C = x_1^2 r_1^2 + x_2^2 r_2^2 + 2x_1x_2\, r_1\cdot r_2 + x_1 m_1^2 + x_2 m_2^2 + (1-x_1-x_2)m_0^2$$

$$- x_1 r_1^2 - x_2 r_2^2$$

$$= -x_1x_2 M_i^2 + (1-x_1-x_2)M_j^2$$

$$J_{V_1} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{-x_1x_2M_i^2 + (1-x_1-x_2)M_j^2 + i\varepsilon} = \int_0^1 dx_1 \frac{\log\left(-\frac{M_j^2}{x_1M_i^2} - i\varepsilon\right)}{M_j^2 - M_i^2 x_1}$$

$$i\,\mathrm{Im}\, J_{V_1} = -i\pi \int_0^1 dx \frac{1}{M_j^2 + M_i^2 x} = -i\pi \frac{1}{M_i^2}\log\left(1+\frac{M_i^2}{M_j^2}\right)$$

$$r_{1\mu}\cdot \underline{P}^\mu = -\frac{1}{2}x_2 M_i^2$$

$$J_{V_2} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{-\frac{1}{2}x_2}{-x_1x_2M_i^2 + (1-x_1-x_2)M_j^2}$$

$$i\,\mathrm{Im}\, J_{V_2} = i\pi \frac{1}{2M_i^2}\left[1 - \frac{M_j^2}{M_i^2}\log\left(1+\frac{M_i^2}{M_j^2}\right)\right]$$

$$i\,\mathrm{Im}\left(\frac{1}{2}M_i^2 J_{V_1} + M_i^2 J_{V_2}\right) = \frac{i\pi}{2}\left[1 - \left(1+\frac{M_j^2}{M_i^2}\right)\log\left(1+\frac{M_i^2}{M_j^2}\right)\right]$$

$$\mathcal{J}_V = -\frac{1}{32\pi}\left[1 - \left(1+\frac{M_j^2}{M_i^2}\right)\log\left(1+\frac{M_i^2}{M_j^2}\right)\right]$$

Putting these results together yields

$$\overline{\sum} i\mathcal{M}^{LO*}_{N_i\to l_a\tilde{\varphi}^+} i\mathcal{M}^{virt,abs}_{N_i\to l_a\tilde{\varphi}^+} = (YY^+)_{ij}(Y^*Y^t)_{ji}\; 2i\mathcal{J}_V\, M_iM_j$$

and

$$\overline{\sum} i\mathcal{M}^{LO*}_{N_i\to \bar{l}_a\tilde{\varphi}} i\mathcal{M}^{virt,abs}_{N_i\to \bar{l}_a\tilde{\varphi}} = \left[(YY^+)_{ij}(Y^*Y^t)_{ji}\right]^*\, 2i\mathcal{J}_V\, M_iM_j$$

When we define the decay asymmetry as

$$\varepsilon^i = \frac{\Gamma_{N_i \to \ell^a \tilde{\phi}^\dagger} - \Gamma_{N_i \to \bar{\ell}^a \tilde{\phi}}}{\Gamma_{N_i \to \ell^a \tilde{\phi}^\dagger} + \Gamma_{N_i \to \bar{\ell}^a \tilde{\phi}}} = \varepsilon^i_{vert} + \varepsilon^i_{wv}$$

we find that

$$\varepsilon^i_{vert} = \frac{2\overline{\sum} i\mathcal{M}^{LO*}_{N_i \to \ell_a \tilde{\phi}^\dagger} i\mathcal{M}^{vert,abs}_{N_i \to \ell_a \tilde{\phi}^\dagger} - 2\overline{\sum} i\mathcal{M}^{LO*}_{N_i \to \bar{\ell}_a \tilde{\phi}} i\mathcal{M}^{vert,abs}_{N_i \to \bar{\ell}_a \tilde{\phi}}}{2\overline{\sum} \left| i\mathcal{M}^{LO}_{N_i \to \ell^a \tilde{\phi}^\dagger} \right|^2}$$

$$= \frac{-8 g_V \; M_i M_j}{2 Y_{ia} Y^*_{ia} M_i^2} \operatorname{Im}\left[(YY^\dagger)_{ij}\right]^2$$

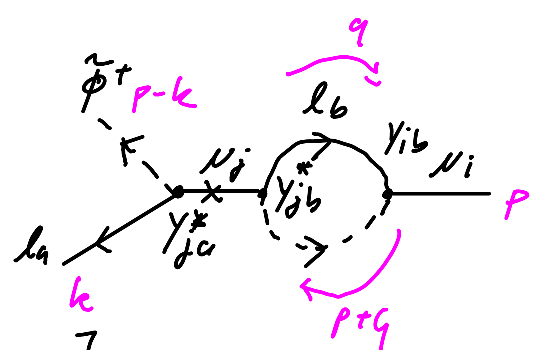
$$= \frac{1}{8\pi} \frac{\operatorname{Im}\left[(YY^\dagger)_{ij}\right]^2}{(YY^\dagger)_{ii}} \frac{M_j}{M_i} \left[1 - \left(1 + \frac{M_j^2}{M_i^2}\right) \log\left(1 + \frac{M_i^2}{M_j^2}\right)\right]$$

For completeness, we also derive the asymmetry from the wave-function contribution.

SU(2) doublet in the loop

$$\overline{\sum} i\mathcal{M}^{LO*}_{N_i \to \ell_a \tilde{\phi}^\dagger} i\mathcal{M}^{wv,abs}_{N_i \to \ell_a \tilde{\phi}^\dagger} = -\sum_{b,j} Y_{ia} Y^\dagger_{aj} Y^*_{jb} Y^t_{bi} \; 2$$

$$* \, i \operatorname{Im} \operatorname{tr}\left[\not{k} \frac{1+\gamma^5}{2} \frac{i(\not{p} + M_j)}{p^2 - M_j^2} \frac{1-\gamma^5}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i\not{q}}{q^2 + i\varepsilon} \frac{i}{(p+q)^2 + i\varepsilon} \frac{1+\gamma^5}{2} (\not{p} - M_i) \right]$$



$$= -\sum_{b,j} Y_{ia} Y^\dagger_{aj} Y^*_{jb} Y^t_{bi} \; M_i M_j$$

$$* \, 4 i \operatorname{Im}\left[\frac{i}{M_i^2 - M_j^2} \int \frac{d^4q}{(2\pi)^4} \frac{i k\cdot q}{q^2 + i\varepsilon} \frac{i}{(p+q)^2 + i\varepsilon} \right]$$

$$= i \sum_{b,j} Y_{ia} Y^\dagger_{aj} Y^*_{jb} Y^t_{bi} \frac{2 M_i M_j}{M_i^2 - M_j^2} \operatorname{Re} \int_0^1 dx \frac{i}{16\pi^2} \left(-\Delta_\varepsilon + \log c\right) \times k\cdot p$$

where $C = -M_i^2 x(1-x)$ and Δ_ε is a real term that contains the UV divergence. To extract the non-vanishing contribution from on shell leptons and Higgs bosons, we only need to evaluate the contribution from the logarithm

$$\mathrm{Re} \int_0^1 dx \, \frac{i}{16\pi^2} \times \log\left(-M_i^2 x(1-x) + i\varepsilon\right) k\cdot p = -\frac{M_i^2}{64\pi}$$

It follows that

$$\overline{\sum} \, i\mathcal{M}^{LO^*}_{N_i \to l_a \tilde{\varphi}} + i\mathcal{M}^{wv,\,abs}_{N_i \to l_a \tilde{\varphi}^\dagger} = -\sum_j \frac{i}{16\pi}\left[(YY^\dagger)_{ij}\right]^2 \frac{M_i M_j}{M_i^2 - M_j^2}$$

and

$$\varepsilon^i_{wv} = \frac{1}{8\pi} \frac{M_i M_j}{M_i^2 - M_j^2} \frac{\mathrm{Im}\left[(YY^\dagger)_{ij}\right]^2}{(YY^\dagger)_{ii}}$$

Eventually, the lepton asymmetry from N_i decays at high temperatures may be converted into a baryon asymmetry through sphaleron processes.

For our present purposes, we may recapitulate to this end that CP-violating processes occur due to interferences of amplitudes (at leading order typically an interference of a tree and a loop process) where the loop diagram can be cut in two pieces in such a manner that the cut particles are on shell (which becomes evident when considering above results in the light of the optical theorem).

5.4.2 Mixing in Neutral Meson Systems

There are four neutral mesons
$K^0 = |d\bar{s}\rangle$, $D^0 = |c\bar{u}\rangle$, $B^0 = |d\bar{b}\rangle$, $B^0_s = |s\bar{b}\rangle$
that can mix with their anti-particles, i.e. their charge eigenstates not need to be their mass eigenstates. Note that the top quark is too short-lived in order to effectively form and to observe bound states.
When the mass eigenstates are no C eigenstates, CP-violation must be present (the above C eigenstates are C-odd and P-odd, hence CP-even). Moreover, the mixing is mediated by FCNCs only, which makes it a sensitive probe of the SM and its extensions.
Suppose the meson is a coherent superposition of the states P^0 and $\overline{P^0}$. From the observation of the decay products in a detector we can tell whether it was a P^0 or $\overline{P^0}$ that has decayed, i.e. the coherence is lost (cf. Schrödinger's cat).
Such a situation can be described as an open system, i.e. by a non-hermitian Hamiltonian. For that purpose, we describe the meson as

$$|\psi(t)\rangle = a(t)|P^0\rangle + b(t)|\overline{P^0}\rangle$$

and take a Hamilton operator of the form

$$H = M - \frac{i}{2}\Gamma$$

where M and Γ are hermitian 2×2 matrices.

Note that these naturally arise from the computation of the meson self-energies and that Γ arises from on-shell intermediate states in the loop and that it is therefore related through the optical theorem to the meson decay rate.

We remark that if we extend the Hilbert space beyond the meson system to include the decay products, the detector and the Universe around, we should obtain again a closed system with a higher dimensional Hamilton operator that is Hermitian.

Now we label the mass eigenstates "light" and "heavy" with $m_H > m_L$ as P_L, P_H. For the K^0, one typically labels the states by their life time "long" and "short" as K_L and K_S, where it turns out that K_L is the heavier of the two states.

Now we determine these eigenstates in terms of $\mathcal{H}$. From the CPT theorem, we observe that $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$. Next, we parametrise the mass eigenstates as

$$|P_{L,H}\rangle = p\,|P^0\rangle \pm q\,|\overline{P^0}\rangle, \qquad |p|^2 + |q|^2 = 1$$

The solution to the eigenvalue problem

$$H\,|P_{L,H}\rangle = \left(M_{L,H} - \frac{i}{2}\Gamma_{L,H}\right)|P_{L,H}\rangle$$

is

$$\left(\frac{q}{p}\right)^2 = \frac{M_{12}^{*2} - \frac{i}{2}\Gamma_{12}^{*}}{M_{12}^{2} - \frac{i}{2}\Gamma_{12}}$$

(i.e. we can write $M - \frac{i}{2}\Gamma = \begin{pmatrix} A & p^2 \\ q^2 & A \end{pmatrix}$, $A \in \mathbb{R}$ $p, q \in \mathbb{C}$)

In the absence of CP-violating phases in the underlying theory, M_{12} and Γ_{12} are relatively real, such that

$$\left|\frac{p}{q}\right| = 1 \quad \text{and} \quad \langle P_H | P_L \rangle = |p|^2 - |q|^2 = 0.$$

In order to determine the eigenvalues, since the mass splitting is small, we define

$$\Delta m = M_H - M_L, \quad \Delta\Gamma = \Gamma_H - \Gamma_L,$$

$$m = \frac{M_H + M_L}{2}, \quad \Gamma = \frac{\Gamma_H + \Gamma_L}{2},$$

$$x = \frac{\Delta m}{\Gamma}, \quad y = \frac{\Delta\Gamma}{2\Gamma},$$

$$\vartheta = \arg(M_{12}\Gamma_{12}^*).$$

The eigenvalue equations then yield

$$(\Delta m)^2 - \frac{1}{4}(\Delta\Gamma)^2 = 4|M_{12}|^2 - |\Gamma_{12}|^2, \quad \Delta m \Delta\Gamma = 4\,\mathrm{Re}[M_{12}\Gamma_{12}^*]$$

In the CP-conserving case, one further obtains

$$\Delta m^2 - \frac{4|M_{12}|^2|\Gamma_{12}|^2}{\Delta m^2} = 4|M_{12}|^2 - |\Gamma_{12}|^2$$

$$\Delta m^2 = \frac{4|M_{12}|^2 - |\Gamma_{12}|^2}{2} \pm \sqrt{\frac{(4|M_{12}|^2 - |\Gamma_{12}|^2)^2}{4} + 4|M_{12}|^2|\Gamma_{12}|^2}$$

$$= 4|M_{12}|^2 \implies \Delta m = 2|M_{12}|$$

$$\Delta\Gamma = 2|\Gamma_{12}|$$

Next, we consider the time evolution and concentrate on the CP-conserving case first.
In this limit, we may choose $p = q = \frac{1}{\sqrt{2}}$ real, such that

$$|P_{L(H)}\rangle = \frac{1}{\sqrt{2}}\left(|P^0\rangle \pm |\bar{P}^0\rangle\right)$$

Suppose we produce in some reaction at time $t=0$ the flavour eigenstate $|P^0\rangle = \frac{1}{\sqrt{2}}(|P_L\rangle + |P_H\rangle)$. With $\Delta E = \gamma \Delta m$ (γ being the Lorentz factor of the meson in the laboratory frame), we obtain

$$|P^0(t)\rangle = e^{-iHt}|P^0(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i(E-\frac{\Delta E}{2})t}|P_L\rangle + e^{-i(E+\frac{\Delta E}{2})t}|P_H\rangle\right)$$

$$= \frac{1}{\sqrt{2}} e^{-iEt}\left(e^{i\frac{\Delta E}{2}t}|P_L\rangle + e^{-i\frac{\Delta E}{2}}|P_H\rangle\right)$$

$$= \frac{1}{2} e^{-iEt}\left[\left(e^{i\frac{\Delta E}{2}t} + e^{-i\frac{\Delta E}{2}t}\right)|P^0\rangle + \left(e^{i\frac{\Delta E}{2}t} - e^{-i\frac{\Delta E}{2}t}\right)|\bar{P}^0\rangle\right]$$

$$= e^{-iEt}\left[\cos\left(\frac{\Delta E t}{2}\right)|P^0\rangle + i\sin\left(\frac{\Delta E t}{2}\right)|\bar{P}^0\rangle\right]$$

The probability to measure a specific flavour therefore is oscillatory in time

$$\mathcal{P}(P^0 \to P^0)[t] = |\langle P^0(t)|P^0\rangle|^2 = \cos^2\left(\frac{\Delta E t}{2}\right) = \frac{1+\cos(\Delta E t)}{2}$$

$$\mathcal{P}(P^0 \to \bar{P}^0)[t] = |\langle P^0(t)|\bar{P}^0\rangle|^2 = \sin^2\left(\frac{\Delta E t}{2}\right) = \frac{1-\cos(\Delta E t)}{2}$$

We can now comment on the phenomenological implication of the parameter $x = \frac{\Delta m}{\Gamma}$. In the PDG booklet, we can see that nowadays, Δm has been determined for all neutral meson systems with different precision.

When $x \ll 1$, the meson has typically no time to oscillate before it decays. Before measuring Δm, first upper bounds on this parameter have been determined.

When $x \gg 1$, the meson performs many oscillations before it decays, which also makes it hard to determine Δm. In this case, the experimental observation effectively averages over many oscillations, i.e. $\mathcal{P}(P^0 \longrightarrow P^0) \approx \mathcal{P}(P^0 \longrightarrow \overline{P^0}) \approx \frac{1}{2}$. Experiments have therefore first been able to determine a lower bound on Δm before measuring it.

We list the orders of magnitude for the x and $y = \dfrac{\Delta \Gamma}{2\Gamma}$ parameters:

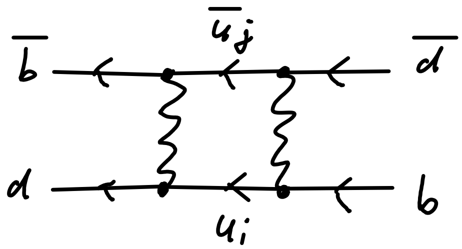
K^0	$x \sim 1$	$y \sim 1 \longrightarrow$	$\mathcal{O}(1)$ difference in life-time of mass eigenstates
D^0	$x \sim 10^{-2}$	$y \sim 10^{-2} \longrightarrow$	y experimentally accessible in decay rates of approximate $D^0, \overline{D}^0$ states
B^0	$x \sim 1$	$y \lesssim 10^{-2} \longrightarrow$	effect of y is hard to measure before decay
B_s^0	$x \sim 10$	$y \sim 10^{-1} \longrightarrow$	averaging effect over oscillations, but the life-time difference of the mass eigenstates can be observed (many oscillations before decay)

We now outline the calculation of the mixing parameters M_{12} and Γ_{12}, and first focus on M_{12}. Within the Lagrangian or the Hamiltonian, there are mass square terms of the form

$$\frac{1}{2} M^2 = \frac{1}{2}\left[\begin{pmatrix} m_B & 0 \\ 0 & m_B \end{pmatrix} + \begin{pmatrix} 0 & M_{12} \\ M_{12} & 0 \end{pmatrix}\right]^2 \approx \frac{1}{2}\left[\begin{pmatrix} m_B^2 & 0 \\ 0 & m_B^2 \end{pmatrix} + 2 m_B \begin{pmatrix} 0 & M_{12} \\ M_{12} & 0 \end{pmatrix}\right]$$

such that $M_{12} \approx \frac{1}{2} \frac{\mathcal{M}^2_{12}}{m_B} = \frac{1}{2} \frac{\langle B|\mathcal{O}|\bar{B}\rangle}{m_B}$

The operator $\mathcal{O}$ annihilates a $\bar{B}$ and creates a B.
It should therefore be of the form
$\mathcal{O} \sim (\bar{b} d)(\bar{b} d)$
A typical contribution arises from box diagrams:



From these, we find that

$$M_{12} \sim \frac{g^4}{m_W^2} \underbrace{\langle B|(\bar{b}_L \gamma_\mu d_L)(\bar{b}_L \gamma^\mu d_L)|\bar{B}\rangle}_{\text{hadronic matrix element}} \sum_{i,j} V_{id}^* V_{ib} V_{jd}^* V_{jb} \underbrace{F(x_i, x_j)}_{\text{form factor}}$$

where $x_i = \frac{m_i^2}{m_W^2}$.

The important point about the form factor is that due to the CKM unitarity (GIM mechanism), the m_i-independent terms vanish, such that the loop is dominated by the top quark and we may set $i=j=t$.

A similar calculation determines the mixing of the K^0-mesons. Since the mixing between the second and third generation is CKM suppressed, the loop turns out to be c-quark dominated. Therefore, there is an $\frac{m_c^2}{m_t^2}$ GIM suppression. This qualitatively explains why for both, K^0 and B^0 it can be that $x = \mathcal{O}(1)$, such that

Both systems are well suited for experimental studies. The parameter Γ_{12} can be determined from the same diagrams with on-shell intermediate quarks. For these however QCD corrections are more important and consequently, there are larger theoretical uncertainties. Finally, the hadronic matrix element is best determined from lattice QCD.

We therefore conclude that $\Delta m_B \propto |V_{tb} V_{td}|^2$ and therefore offers a good method of determining $|V_{td}|$, cf. PDG 11.2.7.

Now we turn to the aspect of CP violation in meson systems. Let us suppose that the amplitude of a process $X \longrightarrow f$ is given by two contributions of the form

$$A_{X\to f} = |a_1|\, e^{i(\delta_1+\phi_1)} + |a_2|\, e^{i(\delta_2+\phi_2)}$$

Suppose further that ϕ_i results from complex couplings and δ_i from cuts through intermediate on shell particle. We have encountered an example of this general form in the decays of sterile neutrinos.

For the CP conjugate process $\overline{X} \longrightarrow \overline{f}$, it then follows that

$$A_{\overline{X}\to\overline{f}} = |a_1|\, e^{i(\delta-\phi_1)} + |a_2|\, e^{i(\delta_2-\phi_2)}$$

CP violation occurs then in the combination

$$|A_{X\to f}|^2 - |A_{\overline{X}\to\overline{f}}|^2 = |a_1||a_2| \Big[e^{i(\delta_1+\phi_1-\delta_2-\phi_2)} + e^{-i(\delta_1+\phi_1-\delta_2-\phi_2)}$$

$$- e^{i(\delta_1-\phi_1-\delta_2+\phi_2)} + e^{-i(\delta_1-\phi_1-\delta_2+\phi_2)}\Big]$$

$$= 2|a_1||a_2|\left[\cos(\delta_1+\phi_1-\delta_2-\phi_2) - \cos(\delta_1-\phi_1-\delta_2+\phi_2)\right]$$

$$= 4|a_1||a_2| \sin(\delta_1-\delta_2)\sin(\phi_1-\phi_2)$$

while

$$\left|1+ae^{-i\varphi}\right|^2 = 1+a^2+2a\cos\varphi$$

$$\varepsilon^{CP} = \frac{|A_{X\to f}|^2 - |A_{\overline{X}\to\overline{f}}|^2}{|A_{X\to f}|^2 + |A_{\overline{X}\to\overline{f}}|^2}$$

$$= \frac{\left|1+\frac{|a_2|}{|a_1|}e^{-i(\delta_1-\delta_2+\phi_1-\phi_2)}\right|^2 - \left|1+\frac{|a_2|}{|a_1|}e^{i(\delta_1-\delta_2-\phi_1+\phi_2)}\right|^2}{\left|1+\frac{|a_2|}{|a_1|}e^{-i(\delta_1-\delta_2+\phi_1-\phi_2)}\right|^2 + \left|1+\frac{|a_2|}{|a_1|}e^{i(\delta_1-\delta_2-\phi_1+\phi_2)}\right|^2}$$

$$= \frac{2\frac{|a_2|}{|a_1|}\left[\cos(\delta_1+\phi_1-\delta_2-\phi_2) - \cos(\delta_1-\phi_1-\delta_2+\phi_2)\right]}{2+2\left(\frac{|a_2|}{|a_1|}\right)^2 + 2\frac{|a_2|}{|a_1|}\left[\cos(\delta_1+\phi_1-\delta_2-\phi_2)+\cos(\delta_1-\phi_1-\delta_2+\phi_2)\right]}$$

$$= \frac{2\frac{|a_2|}{|a_1|}\left[\cos(\delta_1+\phi_1-\delta_2-\phi_2) - \cos(\delta_1-\phi_1-\delta_2+\phi_2)\right]}{2+2\left(\frac{|a_2|}{|a_1|}\right)^2 + 2\frac{|a_2|}{|a_1|}\left[\cos(\delta_1+\phi_1-\delta_2-\phi_2)+\cos(\delta_1-\phi_1-\delta_2+\phi_2)\right]}$$

$$= \frac{2\frac{|a_2|}{|a_1|}\sin(\delta_1-\delta_2)\sin(\phi_1-\phi_2)}{2+2\left(\frac{|a_2|}{|a_1|}\right)^2 + 2\frac{|a_2|}{|a_1|}\cos(\delta_1-\delta_2)\cos(\phi_1-\phi_2)}$$

When e.g. $|a_1| \gg |a_2|$ (as typical for the interference of a large tree and a suppressed loop amplitude), we can approximate

$$\varepsilon^{CP} \approx \frac{|a_2|}{|a_1|}\sin(\delta_1-\delta_2)\sin(\phi_1-\phi_2).$$

In flavour physics, the phases ϕ_i are sometimes referred to as "weak phases", whereas the phases δ_i typically result from cuts over strongly interacting particles. Phases of these type appear also in other contexts, e.g. the decays of sterile neutrinos. In general, this categorisation is therefore not related to weak and strong interactions.

SM CP violation has so far been observed in K_L, B^0 and $B^\pm$ decays. Flavour physicists distinguish between CP violation from decays ("direct CP violation") and from mixing ("indirect CP violation").

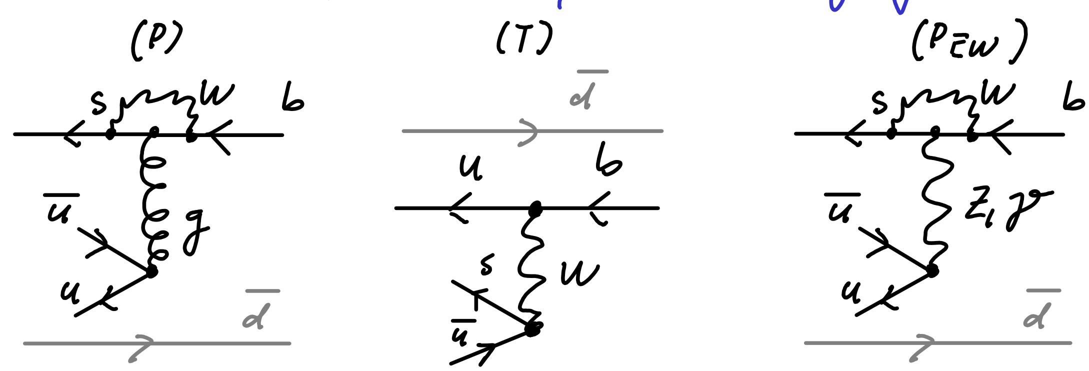
We discuss a few relevant examples.

$B^0 \longrightarrow K^+\pi^-$ ($\bar{B}^0 \to K^-\pi^+$) $\varepsilon^{CP} = -0{,}097 \pm 0{,}12$

This is an example for CP-violation from decays, namely, at the quark level, of the b quark.

Recall that $B^0 = |d\bar{b}\rangle$, $K^+ = |u\bar{s}\rangle$ and $\pi^- = |d\bar{u}\rangle$.

The relevant amplitudes for $\bar{B}^0$ decay are (we indicate the spectator quark in grey)



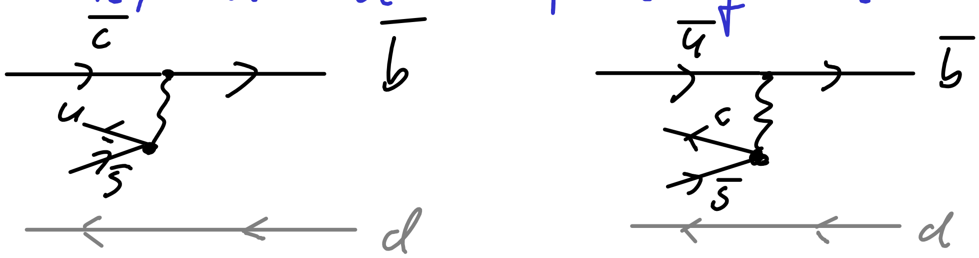
The tree level diagram (T) is highly CKM suppressed.

The one loop diagrams apparently look like penguins.

The strong phase of (P) cannot be calculated using present techniques, but we know that the CP violating interference of (P) or (P_{EW}) is with (T) relies on the weak phase $\arg[V_{ts}^* V_{tb} V_{ub} V_{us}^*]$.

$B \to D/\bar{D} K$

Another example for CP violation from decays.
Here, we have interference of the tree diagrams



There can be interference of states with an intermediate D^0 or $\bar{D}^0$, provided these decay to a final state f_D that is accessible from both D^0 & $\bar{D}^0$.
We are sensitive here to the weak phase

$\arg[V_{cb} V_{us}^* V_{ub}^* V_{cs}] \approx \delta = \gamma$

Experimentally, it is possible to determine the hadronic rates $B \to D K X$ and $B \to \bar{D} K X$, where X stands for additional particles in the final state.
Besides, one can measure $D/\bar{D} \to f_D$ for the different final states f_D. In the process $B \to K X f_D$, the amplitudes $B \to D/\bar{D} K X$ and $D/\bar{D} \to f_D$ factorize.
When we have n different X and k different D, we have $n*k$ measurements of $B \to K X f_D$ that depend on $n+k$ hadronic decay rates. For large enough n, k, this is sufficient to determine the theoretically unknown strong phases as well as the weak phase

γ, cf. also PDG 11.3.4.

We next move to CP violation from mixing. For this purpose, we generalise the oscillation formulae to include CP violation.
For definiteness, we focus on the B^0 system. Let f denote a final state that is accessible to both, B^0 and $\overline{B^0}$. We recall that we introduced

$$|B^0_{L,H}\rangle = p\,|B^0\rangle \pm q\,|\overline{B^0}\rangle$$

and define the amplitudes

$$A_f = A(B^0 \longrightarrow f) \qquad \overline{A_f} = A(\overline{B^0} \longrightarrow f)$$

as well as the ratio

$$\lambda_f = \frac{q\,\overline{A_f}}{p\,A_f}$$

The flavour eigenstates are

$$|B^0\rangle = \frac{1}{2p}\left(|B^0_L\rangle + |B^0_H\rangle\right)$$

$$|\overline{B^0}\rangle = \frac{1}{2q}\left(|B^0_L\rangle - |B^0_H\rangle\right)$$

such that for the time evolution, we obtain

$$\begin{aligned}|B^0(t)\rangle &= \frac{1}{2p}\Big[e^{-im_L t - \frac{1}{2}\Gamma_L t}\left(p\,|B^0\rangle + q\,|\overline{B^0}\rangle\right) \\ &\qquad + e^{-im_H t - \frac{1}{2}\Gamma_H t}\left(p\,|B^0\rangle - q\,|\overline{B^0}\rangle\right)\Big] \\ &= g_+(t)\,|B^0\rangle - \frac{q}{p}\,g_-(t)\,|\overline{B^0}\rangle\end{aligned}$$

where

$$g_\pm(t) = \frac{1}{2}\left[e^{-im_H t - \frac{1}{2}\Gamma_H t} \pm e^{-im_L t - \frac{1}{2}\Gamma_L t}\right]$$

and

$$|\overline{B}^0(t)\rangle = \frac{1}{2q}\Big[e^{-im_L t - \frac{1}{2}\Gamma_L t}(p|B^0\rangle + q|\overline{B}^0\rangle)$$
$$- e^{-im_H t - \frac{1}{2}\Gamma_H t}(p|B^0\rangle - q|\overline{B}^0\rangle)\Big]$$
$$= g_+(t)|\overline{B}^0\rangle - \frac{p}{q} g_-(t)|B^0\rangle$$

Defining $\tau = \Gamma t$, we obtain for the decay rates (more precisely, the probability divided by dt for initial states B^0 or $\overline{B}^0$ to decay into f between t and $t+dt$) (recall $x = \frac{\Delta m}{\Gamma}$, $y = \frac{\Delta\Gamma}{2\Gamma}$)

$$\Gamma(B^0\to f)[t] = \left|\langle B^0|B^0(t)\rangle A_f + \langle\overline{B}^0|B^0(t)\rangle \overline{A}_f\right|^2$$
$$= |A_f|^2 e^{-\tau}\Big\{(\cosh y\tau + \cos x\tau) + |\lambda_f|^2(\cosh y\tau - \cos x\tau)$$
$$-2\,\mathrm{Re}\left[\lambda_f(\sinh y\tau + i\sin x\tau)\right]\Big\}$$

$$\Gamma(\overline{B}^0\to f)[t] = \left|\langle B^0|\overline{B}^0(t)\rangle A_f + \langle\overline{B}^0|\overline{B}^0(t)\rangle \overline{A}_f\right|^2$$
$$|\overline{A}_f|^2 e^{-\tau}\Big\{(\cosh y\tau + \cos x\tau) + |\lambda_f|^{-2}(\cosh y\tau - \cos x\tau)$$
$$-2\,\mathrm{Re}\left[\lambda_f^{-1}(\sinh y\tau + i\sin x\tau)\right]\Big\}$$

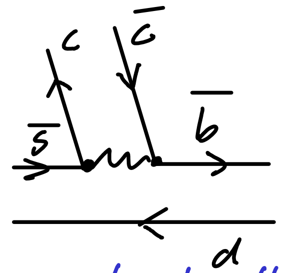
The last terms in the curly brackets result from interferences between the decays of B^0 and $\overline{B}^0$ states. For the B^0 system, $\Delta\Gamma \approx 0$, $\left|\frac{q}{p}\right| \approx 1$ and $|\overline{A}_f| \approx |A_f|$. Consequently, this interference term dominates the time dependent asymmetry, which we define

$$\mathcal{E}_f(t) = \frac{\Gamma(\overline{B}^0\to f)[t] - \Gamma(B^0\to f)[t]}{\Gamma(\overline{B}^0\to f)[t] + \Gamma(B^0\to f)[t]} \approx \mathrm{Im}[\lambda_f]\sin(x\tau)$$

$$= \sin(\arg[\lambda_f])\sin(\Delta m t)$$

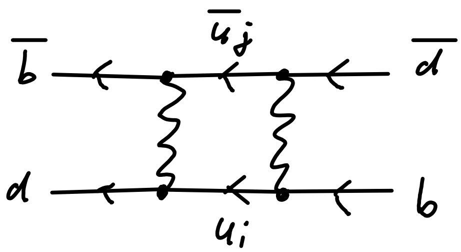
This approximation is suitable for treating the processes

$B^0/\overline{B^0} \longrightarrow J/\psi\, K_S$, that are mediated by e.g.



While we see that a B^0 decay results in a K^0 and $\overline{B^0}$ in a $\overline{K^0}$, we will discuss shortly that the kaons oscillate rapidly into a state K_S^0. Therefore, we can treat these as a common final state f.

Now recall that the B^0 mixing is dominated by



$$M_{12} \sim \frac{g^4}{m_W^2} \langle B | (\bar{b}_L \gamma_\mu d_L)(\bar{b}_L \gamma^\mu d_L) | \bar{B} \rangle$$
$$* \sum_{ij} V_{id}^* V_{ib} V_{jd}^* V_{jb} \; F(x_i, x_j)$$

Due to the GIM mechanism, the t quarks dominate the loop such that we obtain for the relevant phase

$$\lambda_f \approx \frac{q}{p} \frac{\bar{A}_f}{A_f} \approx -\arg(M_{12}) \frac{\bar{A}_f}{A_f} = \frac{V_{tb}^* V_{td}}{V_{tb} \; V_{td}^*} \frac{\bar{A}_f}{A_f}$$

In addition, the $\bar{b} \longrightarrow \bar{c} c \bar{s}$ transition adds a phase $\arg[V_{cb}^* V_{cs}]$ to A_f and oppositely to $\bar{A}_f$.

Now, since $|\lambda_f| \approx 1$

$$\lambda_f = \arg\left[\frac{(V_{cb} V_{cs}^*)^2}{(V_{tb} V_{td}^*)^2} \right] = 2 \arg\left[-\frac{V_{cs} V_{cb}^*}{V_{td} V_{tb}^*} \right]$$

$$\approx 2 \arg\left[-\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*} \right] + \mathcal{O}(\lambda^5)$$

At the bottom line, there is a rather direct determination of the CKM angle β, that circumvents the use of hadronic matrix elements.

Next, we describe the $K^0/\overline{K^0}$ system. For that purpose, it is useful to consider the CP violating effects in the decays of mass eigenstates. In particular, since the life-time of the two eigenstates is very different, we can effectively produce the state K_L by simply waiting until most of the K_S have decayed.

In the absence of CP violation, $|p|=|q|$. It is useful to expand around this limit in terms of a parameter $\overline{\varepsilon}$ such that $\frac{p}{q} = \frac{1+\overline{\varepsilon}}{1-\overline{\varepsilon}}$. For $\overline{\varepsilon}=0$, mass and CP eigenstates coincide and

$$|K^0_\pm\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle \mp |\overline{K^0}\rangle\right], \quad CP|K^0_\pm\rangle = \pm|K^0_\pm\rangle,$$

$$|K_S\rangle \approx |K^0_+\rangle, \qquad |K_L\rangle \approx |K^0_-\rangle$$

Now, including CP violating effects, we have

$$|K_{S}^{L}\rangle = \frac{1}{\sqrt{1+|\overline{\varepsilon}|^2}}\left[|K^0_\mp\rangle + \overline{\varepsilon}\,|K^0_\pm\rangle\right]$$

$$\overline{\varepsilon} = \frac{p-q}{p+q} \approx \frac{i}{2}\,\frac{\operatorname{Im} M_{12} - \frac{i}{2}\operatorname{Im}\Gamma_{12}}{\operatorname{Re} M_{12} - \frac{i}{2}\operatorname{Re}\Gamma_{12}} \approx \frac{1}{2}\,\frac{M_{12}-M_{21} - \frac{i}{2}(\Gamma_{12}-\Gamma_{21})}{M_L - M_S - \frac{i}{2}(\Gamma_L-\Gamma_S)}$$

A rather immediate consequence of the mixing is the asymmetry in semileptonic decays of K_L, with the particular amplitudes

$$A_{K_L\to\pi^- e^+\nu_e} = \frac{1+\overline{\varepsilon}}{\sqrt{2}}\,A_{K^0\to\pi^- e^+\nu_e}$$

$$A_{K_L\to\pi^+ e^-\overline{\nu_e}} = \frac{1-\overline{\varepsilon}}{\sqrt{2}}\,A_{K^0\to\pi^+ e^-\overline{\nu_e}}$$

Since $A_{K^0\to\pi^- e^+\nu_e} = A_{K^0\to\pi^+ e^-\overline{\nu_e}}$, we obtain

$$\frac{\Gamma_{K_L \to \pi^- e^+ \nu_e}}{\Gamma_{K_L \to \pi^+ e^- \bar{\nu}_e}} \simeq \frac{1 + 2 \operatorname{Re} \bar{\varepsilon}}{1 - 2 \operatorname{Re} \bar{\varepsilon}} \simeq 1 + 4 \operatorname{Re} \bar{\varepsilon}$$

and it is found experimentally that $\operatorname{Re} \bar{\varepsilon} \simeq 1{,}6 * 10^{-3}$.
More dramatic is the effect of CP violation in hadronic decays.
When decaying in two pions, these should be in an S-wave ($l=0$) state, with

$$CP|\pi^0\pi^0\rangle = (-1)^2(-1)^{l}|\pi^0\pi^0\rangle = +|\pi^0\pi^0\rangle$$

hence, the decay in two pions is possible for $|K^0_+\rangle$, but not for $|K^0_-\rangle$. This is the main reason for the rather different life times ($9 * 10^{-11}$ s and $5 * 10^{-8}$ s).
For charged pion pairs in the final state, the same conclusion follows from $CP|\pi^\pm\rangle = -|\pi^\mp\rangle$.
Now, the first experimental evidence for CP violation has been the discovery that K_L decays with a small branching ratio into the CP even two-pion states:

$$\frac{\Gamma_{K_L \to \pi^0\pi^0}}{\Gamma_{K_L}} = (1{,}967 \pm 0{,}010) * 10^{-3}, \quad \frac{\Gamma_{K_L \to \pi^+\pi^-}}{\Gamma_{K_L}} = (8{,}64 \pm 0{,}06) * 10^{-3}$$

More on the phenomenology can be found in L.L. Chau, Phys Repts. <u>95</u>, 1 (1983).
The amplitudes under consideration depend on the total isospin of the final two pions, that may add up to either two or zero.
Identifying isospin with angular momentum as

$$\langle j_1 m_1 \, j_2 m_2 | \equiv \langle I^{(1)} I_3^{(1)} I^{(2)} I_3^{(2)} |$$
$$\langle j m \, j_1 j_2 | \equiv \langle I \, I_3 \; I^{(1)} I^{(2)} |$$

and noting the Clebsch-Gordan coefficients

$$\langle 1\,0\,1\,0 | 2\,0\;1\,1 \rangle = \sqrt{\frac{2}{3}}$$
$$\langle 1\,0\,1\,0 | 0\,0\;1\,1 \rangle = -\sqrt{\frac{1}{3}}$$
$$\langle 1\,1\,1\,-1 | 2\,0\;1\,1 \rangle = \sqrt{\frac{1}{6}}$$
$$\langle 1\,1\,-1\,1 | 2\,0\;1\,1 \rangle = \sqrt{\frac{1}{6}}$$
$$\langle 1\,1\,1\,-1 | 0\,0\;1\,1 \rangle = \sqrt{\frac{1}{3}}$$
$$\langle 1\,1\,-1\,1 | 0\,0\;1\,1 \rangle = \sqrt{\frac{1}{3}}$$

we infer that

$$\langle \pi^0 \pi^0 | = -\sqrt{\frac{1}{3}} \langle (\pi\pi)_{I=0} | + \sqrt{\frac{2}{3}} \langle (\pi\pi)_{I=2} |$$

$$\langle \pi^+ \pi^- | = \sqrt{\frac{2}{3}} \langle (\pi\pi)_{I=0} | + \sqrt{\frac{1}{3}} \langle (\pi\pi)_{I=2} |$$

We can hence parametrise the hadronic matrix elements as

weak interaction Hamiltonian

$$\langle (\pi\pi)_{I=0} | H_W | K^0 \rangle = a_0 \, e^{i\delta_0}, \quad \langle (\pi\pi)_{I=2} | H_W | K^0 \rangle = a_2 \, e^{i\delta_2},$$

$$\langle (\pi\pi)_{I=0} | H_W | \bar{K}^0 \rangle = a_0^* \, e^{i\delta_0}, \quad \langle (\pi\pi)_{I=2} | H_W | \bar{K}^0 \rangle = a_2^* \, e^{i\delta_2},$$

where a_0, a_2 are weak interaction phases and δ_0, δ_2 are strong phases. Theoretical calculation face large uncertainties due to QCD.

For the mixing states, we define

$$a_{0\,L}^{\,S} = \langle (\pi\pi)_{I=0} | H_W | K_L^S \rangle = \left[(1+\bar{\varepsilon}) a_0 \mp (1-\bar{\varepsilon}) a_0^* \right] e^{i\delta_0}$$

$$a_{2\,L}^{\,S} = \langle (\pi\pi)_{I=2} | H_W | K_L^S \rangle = \left[(1+\bar{\varepsilon}) a_2 \mp (1-\bar{\varepsilon}) a_2^* \right] e^{i\delta_2}$$

Combining with the isospin decomposition, we find

$$A_{00\,L}^{\;S} = \langle \pi^0\pi^0 | H_W | K_L^S \rangle = -\frac{1}{\sqrt{3}}\, a_{0\,L}^{\;S} + \frac{\sqrt{2}}{\sqrt{3}}\, a_{2\,L}^{\;S}$$

$$A_{+-\,L}^{\;S} = \langle \pi^+\pi^- | H_W | K_L^S \rangle = \frac{\sqrt{2}}{\sqrt{3}}\, a_{0\,L}^{\;S} + \frac{1}{\sqrt{3}}\, a_{2\,L}^{\;S}$$

The experimental results for CP violation are often quoted in terms of η_{00} and η_{+-}, defined as

$$\eta_{00} = \frac{A_{00L}}{A_{00S}} = \frac{a_{0L} - \sqrt{2}\, a_{0S}}{a_{0S} - \sqrt{2}\, a_{2S}}, \qquad \eta_{+-} = \frac{A_{+-L}}{A_{+-S}} = \frac{a_{0L} + \frac{1}{\sqrt{2}}\, a_{0S}}{a_{0S} + \frac{1}{\sqrt{2}}\, a_{0L}}$$

Now, when we define

$$\varepsilon = \frac{a_{0L}}{a_{0S}}, \qquad \varepsilon_2 = \frac{a_{2L}}{a_{2S}}, \qquad \omega = \frac{a_{2S}}{a_{0S}}, \qquad \varepsilon' = \frac{1}{\sqrt{2}}\, \omega \left(\frac{\varepsilon_2}{\omega} - \varepsilon \right),$$

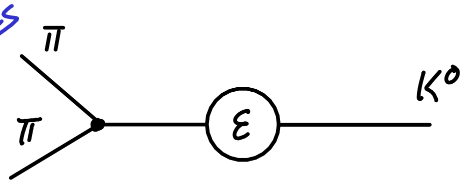
we find

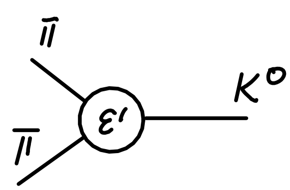
$$\eta_{00} = \frac{\varepsilon - \sqrt{2}\, \varepsilon_2}{1 - \sqrt{2}\, \omega} = \varepsilon - \frac{2\varepsilon'}{1 - \sqrt{2}\, \omega} \approx \varepsilon - 2\varepsilon'$$

$$\eta_{+-} = \frac{\varepsilon + \frac{1}{\sqrt{2}}\, \varepsilon_2}{1 + \frac{1}{\sqrt{2}}\, \omega} = \varepsilon + \frac{\varepsilon'}{1 + \frac{1}{\sqrt{2}}\, \omega} \approx \varepsilon + \varepsilon'$$

The last approximation relies on the experimental fact that $\left| \frac{a_2}{a_0} \right| \approx \frac{1}{20}$.

To some approximation, the ε-type CP violation is therefore insensitive to the final state, whereas the ε'-type is. Diagrammatically, this is sometimes shown as





and we note the resemblance with the different

perturbative contributions to the decays of sterile neutrinos.
We finally quote the experimental values reported by PDG

$|\eta_{00}| = (2{,}220 \pm 0{,}011) * 10^{-3}$ $\qquad$ $|\varepsilon| = (2{,}228 \pm 0{,}011) * 10^{-3}$

$|\eta_{+-}| = (2{,}232 \pm 0{,}011) * 10^{-3}$ $\qquad$ $Re\left[\frac{\varepsilon'}{\varepsilon}\right] = (1{,}66 \pm 0{,}23) * 10^{-3}$

The fact that $\varepsilon >> \varepsilon'$ may be attributed to the resonant mixing of the mass eigenstates K^0_L and K^0_S, cf. also the discussion on the decays of right-handed neutrinos.

5.5 Concluding Remarks

The success of the SM is impressively shown on the global fits of the unitarity triangle. We have discussed here how to determine a few elements of the CKM matrix or combinations of these. Interplay between theory and experiment is crucial and we have gained some insights into the theoretical methods. The fact that the unitarity triangle closes within the range of uncertainty tells us that there is currently no indication of flavour violating processes besides the CKM mechanism. Beyond the Standard Model physics must therefore either be flavour insensitive (and this must stand after radiative corrections) or it may only occur at the multi-TeV scale, which is not what one would theoretically desire in view of the hierarchy problem, or more precisely the desire to stabilise the Higgs boson mass at 125 GeV without introducing excessive parametric tuning. It would be nice if these lectures also included theory and phenomenology of neutrinos. Mixing & oscillations are of pivotal experience as well. After studying the neutral meson systems, one should therefore be prepared to also learn about that topic.