Theoretical Particle Physics

The topic of this lecture is the application of Relativistic Quantum Theory [Quantum Field Theory (QFT)] in order to describe Particle Physics phenomena. Clearly, an overlap with a lecture on QFT is unavoidable. The basic concepts of QFT are here neither derived in a complete nor pedagogical manner. However, we attempt to make clear what basic concepts such as Green functions, the S-matrix, cross-sections etc. are, how these are computed and how they are related. The (present) outline of the lecture is as follows:

1. <u>Prerequisites</u> (a review of / introduction to QFT)
2. <u>Quantum Electrodynamics</u> (QED). Elementary scattering processes & renormalisation
3. <u>Quantum Chromodynamics</u> (QCD). Non-Abelian Gauge Theories, running coupling & asymtotic freedom, deep inelastic scattering – partons & quarks
4. <u>Electroweak Interactions.</u> Spontaneous breakdown of gauge symmetry, Electroweak unification, the Higgs boson

Possible topics of a follow-up course in the next term include hadron physics, flavour physics, heavy quarks & weak interactions, neutrino physics & physics beyond the Standard Model.

1. Prerequisites

We go here through some basics that are better covered within QFT I. The course catalogue does not list that lecture as a requirement for the present course. The following crash course is not pedagogical. For those not familiar with QFT I, it should give directions for self-study, for the others, it is meant as a warmup & repetition (which is an excellent pedagogical concept, by the way).

Framework: Relativistic Quantum Field Theory

$\longrightarrow$ discuss now
1.1 Relativity
1.2 Relativistic Field Theory
1.3 Relativistic Quantum Fields
1.4 Scattering Processes

1.1 Relativity

Contravariant vector : $x^\mu = (x^0, \vec{x})$ $\quad (\mu = 0, 1, 2, 3)$

Covariant vector : $x_\mu = (x^0, -\vec{x})$

Indices are raised and lowered using the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A Lorentz transformation Λ leaves $g_{\mu\nu}$ invariant:

$$g_{\mu\nu} \longmapsto \Lambda^\rho{}_\mu \, g_{\rho\sigma} \, \Lambda^\sigma{}_\nu = [\Lambda^T g \Lambda]_{\mu\nu} = g_{\mu\nu}$$

$\Rightarrow$ The Λ form the group $SO(3,1)$

Indices are lowerd & raised with $g_{\mu\nu}$ and $g^{\mu\nu}$:

$$x_\mu = g_{\mu\nu} x^\nu$$
$$x^\mu = g^{\mu\nu} x_\nu$$

$g^{\mu\nu}$ is the same matrix as $g_{\mu\nu}$, such that $g^{\mu\varrho} g_{\varrho\nu} = \delta^\mu_\nu$.

The invariance of $g_{\mu\nu}$ implies that, e.g. for a boost in 1-direction

$$\Lambda = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

φ: rapidity $\qquad \beta = \frac{v}{c}$

$$\sinh\varphi = -\beta\gamma \qquad \cosh\varphi = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

For a rotation by an angle φ around the 3-axis:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Contravariant vectors transform as:

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Covariant vectors as

$$x_\mu \longmapsto x'_\mu = \Lambda_\mu{}^\nu x_\nu = x_\nu (\Lambda^{-1})^\nu{}_\mu$$
$$\hookrightarrow \Lambda_\mu{}^\nu = g_{\mu\varrho} \Lambda^\varrho{}_\sigma g^{\sigma\nu}$$

The identity $\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu$ is because

$$x_\mu p^\mu = x'_\mu p'^\mu = x_\varrho (\Lambda^{-1})^\varrho{}_\mu \Lambda^\mu{}_\sigma p^\sigma = x_\varrho \delta^\varrho{}_\sigma p^\sigma = x_\varrho p^\sigma$$

Due to the invariance of $g_{\mu\nu}$, Lorentz-scalars can be formed by contracting indices, e.g.:

$$x_\mu p^\mu = x^\mu g_{\mu\nu} p^\nu \equiv x \cdot p = x' \cdot p' = \Lambda^\mu{}_\rho x^\rho g_{\mu\nu} \Lambda^\nu{}_\sigma p^\sigma$$
$$= x^\rho [\Lambda^T g \Lambda]_{\rho\sigma} p^\sigma$$

Derivatives w.r.t. x^μ are covariant: $\frac{\partial}{\partial x^\mu} = \partial_\mu$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu \Rightarrow \frac{\partial x^\nu}{\partial x'^\mu} = [\Lambda^{-1}]^\nu{}_\mu = \Lambda_\mu{}^\nu$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu$$

Derivatives w.r.t. x_μ are contravariant: $\frac{\partial}{\partial x_\mu} = \partial^\mu$

Poincaré Group:

Lorentz transformations & space-time translations

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$\rightarrow$ a 10 parameter group (3 boosts, 3 rotations, 4 space-time translations).

Lie-Algebra of the Poincaré Group

The elements of a Lie Group can be parametrised by a set of parameters, such that each group element is continuously connected with the identity. For the Poincaré transformation, this construction is as follows:

$$\Lambda^\mu{}_\nu = e^{-\frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma}]^\mu{}_\nu}$$

$$[J^{\rho\sigma}]^\mu{}_\nu = i\left[g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu\right]$$

$$a^\mu = e^{-i\varepsilon_\rho P^\rho} x^\mu \qquad \text{where } P^\mu = i\partial^\mu$$

The parameters $\omega_{\rho\sigma}$, ε_ρ can be identified as:

Boost φ in i-direction: $\omega_{0i} = -\omega_{i0} = \varphi$

rotation φ around k-axis: $\omega_{ij} = -\omega_{ji} = \varepsilon_{ijk}\,\varphi$ &
translation: $a^\mu = \varepsilon^\mu$

For infinitesimal parameters ε_a, $\omega_{\varrho\sigma}$, study the commutators of the Poincaré-transformation. For this purpose, it is useful to define:

$$\vec{P} = \begin{pmatrix} P^1 \\ P^2 \\ P^3 \end{pmatrix}, \quad H = P^0, \quad \vec{J} = \begin{pmatrix} J^{23} \\ J^{31} \\ J^{12} \end{pmatrix}, \quad \vec{K} = \begin{pmatrix} J^{10} \\ J^{20} \\ J^{30} \end{pmatrix}$$

The commutators are called the

<u>Lie-Algebra of the Poincaré-Group</u>

$[J^i, J^j] = i\,\varepsilon_{ijk}\,J^k$ angular momentum algebra
$[J^i, K^j] = i\,\varepsilon_{ijk}\,K^k$
$[K^i, K^j] = -i\,\varepsilon_{ijk}\,J^k$
$[P^i, J^j] = i\,\varepsilon_{ijk}\,P^k$ angular momentum/momentum commutator
$[P^i, K^j] = -i H\,\delta_{ij}$
$[J^i, H] = [P^i, H] = [H, H] = 0$ angular momentum, momentum & energy conservation
$[K^i, H] = i P^i$

A set of operators that satisfies the same group multiplication table is called a <u>representation</u> of the Poincaré group, e.g.

$$D(\Lambda, a)\, D(\Lambda', a') = D(\Lambda\Lambda', \Lambda a' + a)$$

For the time being, we restrict our considerations to the Lorentz group. Immediately, we recognise the following representations:

$D(\Lambda) \equiv 1$: scalar representation

$D(\Lambda) = \Lambda^{\mu}{}_{\nu}$: vector representation $V' = D(\Lambda)V \quad V'^{\mu} = \Lambda^{\mu}{}_{\nu}$

$D(\Lambda) = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} \Lambda_{\varrho_1}{}^{\sigma_1} \cdots \Lambda_{\varrho_m}{}^{\sigma_m}$: tensor representation

$$T'^{\mu_1 \cdots \mu_n}{}_{\varrho_1 \cdots \varrho_m} = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} \Lambda_{\varrho_1}{}^{\sigma_1} \cdots \Lambda_{\varrho_m}{}^{\sigma_m} T^{\nu_1 \cdots \nu_n}{}_{\sigma_1 \cdots \sigma_m}$$

An important property of different representations of the same Lie-group is that these satisfy the same Lie-algebra. Besides the trivial representation, vectors and tensor products of vectors, the additional important category are spinor representations. Recall from Quantum Mechanics, that the description of spin-$\frac{1}{2}$ particles followed from the observation that the Pauli-matrices satisfy the angular momentum algebra with total angular momentum $l = \frac{1}{2}$:

$$J^i \longrightarrow \frac{1}{2}\sigma^i$$

$$\vec{J}^2 = l(l+1)\mathbb{1} = \frac{1}{4}\vec{\sigma}^2 = \frac{3}{4}\mathbb{1} \Rightarrow l = \frac{1}{2}$$

The relativistic generalisation is found with the change of basis

$$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}), \qquad \vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$$

such that the Lorentz-Algebra takes the form:

$$[A^i, A^j] = i\varepsilon^{ijk}A^k$$

$$[B^i, B^j] = i\varepsilon^{ijk}B^k$$

$$[A^i, B^j] = 0$$

We denote the eigenvalues of $\vec{A}^2, \vec{B}^2$ by $A(A+1)\mathbb{1} = \vec{A}^2$ and $B(B+1)\mathbb{1} = \vec{B}^2$ and the representations by (B, A).

For definiteness, $(\frac{1}{2},0)$ is called the left-handed and $(0,\frac{1}{2})$ the right handed spinor representation. This is motivated by the fact that the operator

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$ performs a mirror reflection of the spatial coordinates. For the generators of the Lorentz Group, this implies $(P^{-1}=P)$

$$\begin{aligned} \vec{J} &\to P\vec{J}P = \vec{J} \\ \vec{K} &\to P\vec{K}P = -\vec{K} \end{aligned} \quad \text{such that} \quad \begin{aligned} \vec{A} &\to P\vec{A}P = \vec{B} \\ \vec{B} &\to P\vec{B}P = \vec{A} \end{aligned}$$

While QED and QCD (quantum electro- and chromodynamics) are chirally invariant, chirality is of pivotal importance in the theory of weak interactions.

Left and right handed spinors are two-component objects. Now, we explicitly construct the representation matrices. In the right-handed case, we have

$$J^{i0} = K^i = -i(A^i - B^i) = -\frac{i}{2}\sigma^i$$
$$J^{ij} = \varepsilon^{ijk}J^k = \varepsilon^{ijk}(A^k + B^k) = \frac{1}{2}\varepsilon^{ijk}\sigma^k$$

It is thus useful to define

$$\sigma^\mu = (\mathbb{1}, \sigma^i), \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$$
$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$$

Therefore, for the right-handed spinor

$J^{\mu\nu} = \bar{\sigma}^{\mu\nu}$ and

$$D_{(0,\frac{1}{2})}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}$$

Analogously, in the left-handed case,

$$J^{i0} = i\frac{\sigma^i}{2}, \quad J^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma^k \Rightarrow J^{\mu\nu} = \sigma^{\mu\nu},$$

$$D_{(\frac{1}{2},0)}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}$$

Apparently,

$$D^+_{(\frac{1}{2},0)} = D^{-1}_{(0,\frac{1}{2})}$$

In analogy with the Lorentz indices, it is useful to introduce upper and lower spinorial indices that may take the values 1,2. Undotted indices are left-handed, dotted ones are right-handed. On the Pauli-matrices, these appear as follows:

$$\sigma^{\mu\nu}{}_\alpha{}^\beta, \quad \bar\sigma^{\mu\nu\,\dot\alpha}{}_{\dot\beta}, \quad \sigma^\mu{}_{\alpha\dot\beta}, \quad \bar\sigma^{\mu\dot\alpha\beta}$$

The indices are contracted according to ${}^\alpha{}_\alpha$ and ${}_{\dot\alpha}{}^{\dot\alpha}$.
We define $\varepsilon^{12} = -\varepsilon^{21} = \varepsilon_{21} = -\varepsilon_{12} = 1$ where the indices may be either dotted or undotted. This tensor is invariant:

$$\varepsilon^{\alpha\gamma} D_{(\frac{1}{2},0)\alpha}{}^\delta D_{(\frac{1}{2},0)\gamma}{}^\beta = \varepsilon^{\delta\beta}$$

and accordingly for lower and dotted indices.
That follows from the important property:

$$D_{(\frac{1}{2},0)\alpha}{}^\beta = \varepsilon_{\alpha\gamma}\left[D^*_{(0,\frac{1}{2})}\right]^\gamma{}_\delta\, \varepsilon^{\delta\beta}$$

Just as for $g_{\mu\nu}$ for Minkowski indices, we can therefore employ ε for raising & lowering spinor indices.
Since left- and right handed spinors are related by complex conjugation, we adopt the convention of

considering all spinors as left-handed. I.e. ξ_α transforms with $D_{(\frac{1}{2},0)}$ and $(\xi^\dagger)_{\dot\beta}\,\varepsilon^{\dot\beta\dot\alpha}$ with $D_{(0,\frac{1}{2})}$.

Without going into further details, we notice that we can construct all representations of the Lorentz-group from <u>Tensor products</u>, for example

$$\underbrace{(\tfrac{1}{2},0)\otimes(0,\tfrac{1}{2})}_{\text{spinors, } 2\times 2 \text{ dof.}} = \underbrace{(\tfrac{1}{2},\tfrac{1}{2})}_{\text{vector } V^\mu,\ 4 \text{ d.of.}} \longrightarrow \text{cf. } \sigma^\mu{}_{\alpha\dot\beta}\ \&\ \bar\sigma^{\mu\dot\alpha\beta}$$

or

direct sum/product

$$(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2}) = \underbrace{(0,0)}_{\text{scalar}}\oplus\underbrace{(0,1)}_{F^{\mu\nu}} \longrightarrow \text{cf. } \sigma^{\mu\nu}{}_\alpha{}^\beta\ \&\ \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}$$

etc.

Since QED & QCD are chirally invariant we construct the <u>Dirac representation</u> as the <u>direct sum</u> of a left and a right-handed spinor: $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$
The representation matrices take the block-diagonal form:

$$D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})} = \begin{pmatrix} D_{(\frac{1}{2},0)} & 0 \\ 0 & D_{(0,\frac{1}{2})} \end{pmatrix}$$

It is useful to introduce the <u>Dirac-matrices</u>

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

and to define

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}$$

such that

$$D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$$

Above representation of the Dirac-matrices is called the chiral (or Weyl) representation. Other representations can be obtained by similarity transformations
$\gamma^\mu \longmapsto S\gamma^\mu S^{-1}$.

The Dirac-matrices satisfy a so-called <u>Clifford-Algebra</u>

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\mathbb{1}\,g^{\mu\nu}.$$

1.2 Relativistic Field Theory

So far, we have discussed the transformations of scalars, vectors, tensors & spinors. A field has the transformation property:

$$\phi'^A(x') = D^{AB}(\Lambda)\,\phi^B(x) =: \left[U^{-1}(\Lambda,a)\,\phi(x)\,U(\Lambda,a)\right]^A$$

$$x' = \Lambda x + a$$

where U is a unitary operator (note that D is in general not unitary). A is any combination of Lorentz and spinor indices.

Field equations that are consistent with special relativity must transform as a field themselves, i.e. $T^A(x) = T'^A(x') = 0$. Equations with that property are called Lorentz-covariant.

The Lagrangian formalism is a powerful and systematic method of obtaining field equations. It is based on the Lagrangian $\mathcal{L}(x)$, that transforms as a Lorentz scalar field. In particular, it depends on fields as $\mathcal{L}(x) = \mathcal{L}(\phi_1^A(x), \phi_2^B(x), \dots)$

The action is

$$S[\phi_1^A, \phi_2^B, \dots] = \int d^4x\, \mathcal{L}(x),$$

which is a Lorentz-invariant functional.

Stationarity of the action under variations of ϕ^A requires

$$0 = \delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^A}\,\delta\phi^A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\,\delta(\partial_\mu \phi^A)\right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\right)\delta\phi^A = 0$$

For general variations, this yields the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} = 0$$

These are covariant, since they transform as $\frac{\partial}{\partial \phi^A}$, and hence are consistent with Special Relativity.

Now, we restrict ourselves to equations that at most contain second-order derivatives. Moreover, we would like to obtain linear equations that allow for analytical solutions. Therefore, the Lagrangian should contain terms that are at most quadratic in ϕ^A. This corresponds to a free or a Gaussian theory. Higher order terms are called interaction terms and are treated perturbatively in many applications.

The most important examples for relativistic field equations are:

Klein-Gordon Equation

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow (\partial^2 + m^2)\phi = 0$$

Dirac Equation

Define the Dirac spinor $\psi = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}$

and $\overline{\psi} = \psi^\dagger \gamma^0 = (\xi^\dagger_{\dot{\beta}} \;\; \chi^\beta) \begin{pmatrix} 0 & \delta^{\dot{\beta}}{}_{\dot{\alpha}} \\ \delta_\beta{}^\alpha & 0 \end{pmatrix} = (\chi^\alpha \;\; \xi^\dagger_{\dot{\alpha}})$

Lorentz invariants up to 2nd order in $\psi, \bar{\psi}$:

$$\bar{\psi}\psi = \chi^{\alpha}\xi_{\alpha} + \xi^{+}_{\dot{\alpha}}\chi^{+\dot{\alpha}}$$

$$\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi = \xi^{+}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu}\xi_{\alpha} + \chi^{\beta}\sigma^{\mu}{}_{\beta\dot{\beta}}\partial_{\mu}\chi^{+\dot{\beta}}$$

Use the notation: $\not{V} = V_{\mu}\gamma^{\mu} = V^{\mu}\gamma_{\mu}$

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

$$\Rightarrow (i\not{\partial} - m)\psi = 0$$

Majorana Equation

$$\mathcal{L} = \xi^{+}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} i\partial_{\mu}\xi_{\alpha} - \frac{1}{2}m\,\xi^{\alpha}\xi_{\alpha} - \frac{1}{2}m\,\xi^{+}_{\dot{\alpha}}\xi^{+\dot{\alpha}}$$

$$\Rightarrow i\partial_{\mu}\bar{\sigma}^{\mu}\xi - m(\underbrace{\varepsilon\xi}_{\varepsilon^{\alpha\beta}\xi_{\beta}})^{+} = 0$$

Notice that the same physical field can be described with Dirac-spinors when imposing the condition $\xi = \chi$.

Proca-Equation

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^{2}A^{\rho}A_{\rho}$$

$$\frac{\partial}{\partial A^{\sigma}}F^{\mu\nu}F_{\mu\nu} = 0$$

$$\frac{\partial}{\partial^{\rho}(A^{\sigma})}F^{\mu\nu}F_{\mu\nu} = 2(F_{\rho\sigma} - F_{\sigma\rho}) = 4F_{\rho\sigma}$$

$$\Rightarrow \quad \partial^\rho F_{\rho\sigma} - m^2 A_\sigma = 0$$

$$\partial^2 A_\sigma - \partial_\sigma \partial^\rho A_\rho - m^2 A_\sigma = 0$$

Notice that for $m=0$, this equation as well as the Lagrangian are <u>gauge invariant</u>, in the sense that the transformation

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu \alpha(x)$$

has no effect.

1.3 Relativistic Quantum Fields

There are two main approaches toward quantisation of relativistic fields:

- the canonical based on the Hamiltonian formalism
- the functional based on the path integral

In the canonical approch, the canonical momentum density is given by

$$\pi^A(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}_A(x)},$$

the Hamiltonian density by

$$\mathcal{H}(x) = \pi^A(x)\,\dot{\phi}_A(x) - \mathcal{L}(x)$$

and the Hamiltonian by

$$H(t) = \int d^3x\, \mathcal{H}(\vec{x},t)$$

Quantisation is performed by promoting the fields to operators and requiring the (anti-)commutation relations:

$$[\cdot,\cdot]_- = [\cdot,\cdot] \qquad [\cdot,\cdot]_+ = \{\cdot,\cdot\}$$

$$[\phi_A(\vec{x},t),\pi^B(\vec{y},t)]_\mp = i\delta^3(\vec{x}-\vec{y})\,\delta_A^B$$

$$[\phi_A(\vec{x},t),\phi^B(\vec{y},t)]_\mp = [\pi_A(\vec{x},t),\pi^B(\vec{y},t)] = 0$$

When imposing relativistic causality, the spin-statistics theorem states that fields with integer number spins are bosonic (use $[\cdot,\cdot]$), fields with half-integer spin fermionic (use $\{\cdot,\cdot\}$).

With the help of creation & annihilation operators, we can construct field operators that comply with the (anti-) commutation rules.

As we aim to perform calculations in the interaction picture, it is useful to write down field operators that evolve with the free (Gaussian) Hamiltonian H_0:

$$\phi^A(x) = e^{iH_0 t}\phi^A(\vec{x}, t=0)\, e^{-iH_0 t}$$

In the following, we note the field operators for the Klein-Gordon, Dirac and massless Vector field:

Klein-Gordon field operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\left(a(\vec{p})e^{-ip\cdot x} + a^+(\vec{p})e^{ip\cdot x}\right)\Big|_{p^0=\omega(\vec{p})}$$

$$\omega(\vec{p}) = \sqrt{\vec{p}^{\,2}+m^2}$$

$$\pi(x) = \frac{\partial}{\partial t}\phi(x)$$

$$[a(\vec{p}), a^+(\vec{p}\,')] = (2\pi)^3\delta^3(\vec{p}-\vec{p}\,')$$

$$[a(\vec{p}), a(\vec{p}\,')] = [a^+(\vec{p}), a^+(\vec{p}\,')] = 0$$

$\sqrt{2\omega(\vec{p})}\, a^+(\vec{p})|0\rangle = |\vec{p}\rangle$ creates a particle

Notice also the normalisation

$$\langle\vec{p}\,'|\vec{p}\rangle = \sqrt{4\omega(\vec{p}\,')\omega(\vec{p})}\,\langle 0|a(\vec{p}\,')a^+(\vec{p})|0\rangle$$

$$= \sqrt{4\omega(\vec{p}\,')\omega(\vec{p})}\,\langle 0|\, a^+(\vec{p})a(\vec{p}\,') + [a(\vec{p}\,'), a^+(\vec{p})]|0\rangle$$

$$= 2\omega(\vec{p})\,\delta^3(\vec{p}-\vec{p}\,')$$

$\langle 0|0\rangle = 1$

Dirac field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,\omega(\vec{p})}} \sum_{s} \left\{ a(\vec{p},s)\, u(\vec{p},s)\, e^{-ip\cdot x} + b^+(\vec{p},s)\, v(\vec{p},s)\, e^{ip\cdot x} \right\}$$

$s \to$ spin

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,\omega(\vec{p})}} \sum_{s} \left\{ a^+(\vec{p},s)\, \overline{u}(\vec{p},s)\, e^{ip\cdot x} + b(\vec{p},s)\, \overline{v}(\vec{p},s)\, e^{-ip\cdot x} \right\}$$

The basis spinors can be expressed by boosting solutions found for $\vec{p}=0$ by a rapidity $\vec{\varphi}$ that satisfies

$$\frac{\vec{\varphi}}{|\vec{\varphi}|} = \frac{\vec{p}}{|\vec{p}|} \quad \text{and} \quad e^{\pm|\vec{\varphi}|} = \frac{\omega(\vec{p}) \pm |\vec{p}|}{m}.$$

Then

$$u(\vec{p},s) = \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\, \xi_s \\ e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\, \xi_s \end{pmatrix} \qquad v(\vec{p},s) = \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\, \eta_s \\ -e^{\frac{1}{2}\vec{\varphi}\cdot\vec{\sigma}}\, \eta_s \end{pmatrix}$$

Where $\xi_{+\frac{1}{2}} = \eta_{-\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_{-\frac{1}{2}} = \eta_{+\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$u(\vec{p},s)$ and $v(\vec{p},s)$ are by construction solutions to the Dirac equation:

$$\left.\begin{aligned} (i\not{\partial} - m)\, e^{-ip\cdot x}\, u(\vec{p},s) &= (\not{p} - m)\, u(\vec{p},s) = 0 \\ (i\not{\partial} - m)\, e^{ip\cdot x}\, v(\vec{p},s) &= (-\not{p} - m)\, v(\vec{p},s) = 0 \end{aligned}\right\} \text{ where } p^0 = \sqrt{\vec{p}^{\,2} + m^2}$$

The anticommutation relations are:

$$\left.\begin{aligned} \{a(\vec{p},s),\, a^+(\vec{p}\,',s')\} &= (2\pi)^3\, \delta^3(\vec{p}-\vec{p}\,')\, \delta_{ss'} \\ \{b(\vec{p},s),\, b^+(\vec{p}\,',s')\} &= (2\pi)^3\, \delta^3(\vec{p}-\vec{p}\,')\, \delta_{ss'} \end{aligned}\right\} \text{ all remaining anticommutators vanish}$$

$\sqrt{2\omega(\vec{p})}\; a^+(\vec{p},s)|0\rangle = |\vec{p}_+, s\rangle$ creates a particle

$\sqrt{2\omega(\vec{p})}\; b^+(\vec{p},s)|0\rangle = |\vec{p}_-, s\rangle$ creates an anti-particle

Useful identities for the basis spinors are:

$$u^\dagger(\vec{p},s)\,u(\vec{p},s') = v^\dagger(\vec{p},s)\,v(\vec{p},s') = 2\,\omega(\vec{p})\,\delta_{ss'}$$
$$\bar{u}(\vec{p},s)\,u(\vec{p},s') = -\bar{v}(\vec{p},s)\,v(\vec{p},s') = 2m\,\delta_{ss'}$$
$$\sum_s u(\vec{p},s)\bar{u}(\vec{p},s) = \not{p}+m \qquad \sum_s v(\vec{p},s)\bar{v}(\vec{p},s) = \not{p}-m$$

Massless Vector field operator

The better method for quantising gauge theories is the functional method. Nonetheless, it is useful to write down the field operator in covariant ($\partial^\mu A_\mu = 0$, Lorentz gauge) Feynman gauge.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{\frac{1}{2}(\partial_\mu A^\mu)^2}_{\text{gauge fixing term}} \overset{\text{P.I.}}{=} \frac{1}{2}A^\mu g_{\mu\nu}\partial^2 A^\nu$$

Field operator:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{k}|}}\left[a(\vec{k},\lambda)\,\varepsilon_\mu^\lambda(k)\,e^{-ik\cdot x} + a^\dagger(\vec{k},\lambda)\,\varepsilon_\mu^\lambda(k)\,e^{ik\cdot x}\right]_{k^0=|\vec{k}|}$$

Commutation relations:

$$[a(\vec{k},\lambda),\, a^\dagger(\vec{k}',\lambda')] = -(2\pi)^3\,\delta^3(\vec{k}-\vec{k}')\,g_{\lambda\lambda'}$$

It turns out that unphysical states are not propagated, provided we choose the polarisation vectors of the external particles to be of the form $\varepsilon^\mu = (0,\vec{\varepsilon})$ and $\vec{k}\cdot\vec{\varepsilon} = 0$, i.e. they describe transversely polarised photons.
Functional quantisation becomes unavoidable for non-Abelian gauge theories, at which point we discuss this method in detail.

Time-ordered expectation values in the canonical approach

The goal is to calculate expectation values of certain operators for interacting theories. We discuss this for scalar fields, the generalisation should be clear.
Above operators are interaction picture operators and constitute basic building blocks in perturbation theory.
For better distinction, we denote the Heisenberg operators by ϕ_H. On the example of a two-point (Green) function, we see how we formally obtain general expectation values:

$$\overset{\to \text{no time-ordering}}{iG(x,y)} = \langle \Omega | \phi_H(x) \phi_H(y) | \Omega \rangle$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | \left(T e^{-i\int_{x^0}^{T} d\tau H_I(\tau)} \right) \phi(x^0) \left(T e^{-i\int_{y^0}^{x^0} d\tau H_I(\tau)} \right) \phi(y^0) \left(T e^{-i\int_{-T}^{y^0} d\tau H_I(\tau)} \right) | 0 \rangle}{\langle 0 | T e^{-i\int_{-T}^{T} d\tau H_I(\tau)} | 0 \rangle}$$

Here, T stands for time ordering, $|\Omega\rangle$ is the vacuum of the full Hamiltonian $H = H_0 + H_I$, where H_0 is the free Hamiltonian.
The ε-prescription and the normalisation are due to the fact that in general $|\Omega\rangle \neq |0\rangle$ ($|0\rangle$ being the ground state of H_0). (Cf. Peskin & Schroeder.)
In these lectures, we are mostly concerned with scattering processes. As we will recall, scattering amplitudes can be obtained from time-ordered expectation values, for which the above expression simplifies:

$$iG^T(x,y) =: iG(x,y) =$$

$$\langle\Omega|T\,\phi_H(x)\phi_H(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\,\phi(x)\,\phi(y)\,e^{-i\int_{-T}^{T} d\tau\, H_I(\tau)}|0\rangle}{\langle 0|T\,e^{-i\int_{-T}^{T} d\tau\, H_I(\tau)}|0\rangle}$$

We assume that the terms in H_I are proportional to small coupling constants. Perturbation theory then amounts to an expansion of the exponentials. Wick's Theorem is an efficient rule for evaluating time-ordered products of the interaction picture operators:

(The normal ordered product : : is defined such that creation operators appear on the left, annihilation operators on the right.

$$T[\phi(x_1)\phi(x_2)\cdots\phi(x_n)] \underset{\substack{\uparrow \\ n\,\text{even}}}{=} :\phi(x_1)\phi(x_2)\cdots\phi(x_n):$$

$$+\sum_{\text{perm}} \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle :\phi(x_3)\cdots\phi(x_n):$$

$$+\sum_{\text{Perm}} \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle\langle 0|T[\phi(x_3)\phi(x_4)]|0\rangle :\phi(x_5)\cdots\phi(x_n):$$

$$+\ \cdots$$

$$+\sum_{\text{perm}} \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle\cdots\langle 0|T[\phi(x_{n-1})\phi(x_n)]|0\rangle$$

When taking the vacuum expectation value, only the terms in the last line survive. Recall that the field operators here are understood to be in the interaction picture.

A useful quantity to know is therefore the Feynman propagator

$$i\Delta^F(x_1,x_2) = \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$$

Fixing a normalisation, it is a solution to the equation

$$(-\partial^2_{x_1} - m^2)\, i\Delta^F(x_1,x_2) = i\delta^4(x_1-x_2)$$

A possible solution is

$$i\Delta^F(x_1,x_2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)} \frac{i}{p^2-m^2+i\varepsilon}$$

The ε-prescription corresponds to the desired choice of time-ordered boundary conditions. $\longrightarrow$ Aufgabe: zeige dies

To illustrate these techniques, we consider the theory

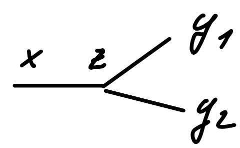
$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \frac{1}{2}m_\chi^2\chi^2$$
$$- \frac{1}{2}g\phi\chi^2$$

and the three-point function

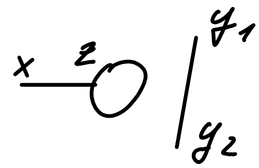
$$iG_{\phi\chi\chi}(x,y_1,y_2) = \langle\Omega|T\,\phi_H(x)\chi_H(y_1)\chi_H(y_2)|\Omega\rangle = iG_{\phi\chi\chi}(x,y_1,y_2) + iG_{\phi\chi\chi}(x,y_1,y_2) + \dots$$

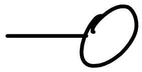
$$iG^{LO}_{\phi\chi\chi}(x,y_1,y_2) = -i\frac{g}{2}\int d^4z\,\langle 0|T\,\phi(x)\phi(z)\chi(y_1)\chi(z)\chi(y_2)\chi(z)|0\rangle$$

$$= -ig\int d^4z\Big[i\Delta^F_\phi(x-z)\,i\Delta^F_\chi(z-y_1)\,i\Delta^F_\chi(z-y_2)$$



$$+\,i\Delta^F_\phi(x-z)\,i\Delta^F_\chi(0)\,i\Delta^F_\chi(y_1-y_2)\Big]$$



The second term is a disconnected diagram with a loop. In particular, the "tadpole" is an unwanted contribution that one cancels by adding a formal counterterm $\otimes$, such that  +  $= 0$. We will come back to renormalisation & counterterms later.

For the time being, we call the first term the "tree" contribution and express it in terms of momentum space propagators as

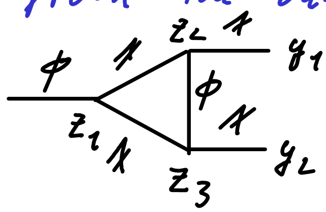
$$iG^{\text{tree}}_{\phi\chi\chi}(x,y_1,y_2) = \int d^4z \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-z)} \int \frac{d^4q_1}{(2\pi)^4} e^{-iq_1\cdot(z-y_1)} \int \frac{d^4q_2}{(2\pi)^4} e^{-iq_2(z-y_2)}$$

$$\underbrace{(-ig)\, i\Delta^F_\phi(p)\, i\Delta^F_\chi(q_1)\, i\Delta^F_\chi(q_2)}_{=: iG^{\text{tree}}_{\phi\chi\chi}(p,q_1,q_2)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} (2\pi)^4 \delta^4(p-q_1-q_2)\, e^{-i(p\cdot x - q_1\cdot y_1 - q_2\cdot y_2)}\, iG^{\text{tree}}_{\phi\chi\chi}(p,q_1,q_2)$$

Evidently, the expression $iG^{\text{tree}}_{\phi\chi\chi}(p,q_1,q_2)$ can be computed in a rather simple manner.

Now for $iG^{NLO}_{\phi\chi\chi}$. Let us disregard the unwanted disconnected contributions from the outset and only consider the 1-loop term



$$iG^{NLO}_{\phi\chi\chi}(x,y_1,y_2) = \frac{1}{3!}\left(-i\frac{g}{2}\right)^3 \int d^4z_1 \int d^4z_2 \int d^4z_3$$

$$\langle 0|T\phi(x)\chi(y_1)\chi(y_2)\phi(z_1)\chi(z_1)\chi(z_1)\phi(z_2)\chi(z_2)\chi(z_2)\phi(z_3)\chi(z_3)\chi(z_3)|0\rangle$$

$\longrightarrow$

$$iG^{1\text{-loop}}_{\phi\chi\chi}(x,y_1,y_2) = \frac{1}{3!}\left(-i\frac{g}{2}\right)^3 \int d^4z_1 \int d^4z_2 \int d^4z_3$$

$$* \Big(i\Delta^F_\phi(x-z_1)\, i\Delta^F_\chi(z_2-z_1)\, i\Delta^F_\chi(z_3-z_1)\, i\Delta^F_\phi(z_3-z_2)\, i\Delta^F_\chi(z_2-y_1)\, i\Delta^F_\chi(z_3-y_2)$$

$$+ \text{ perm. of } (z_1, z_2, z_3) \text{ and } \chi\text{-propagators } [3! \cdot 2^3] \Big)$$

$$= ig^3 \int d^4z_1 \int d^4z_2 \int d^4z_3 \int \frac{d^4p}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

$$e^{-ip\cdot(x-z_1)}\, e^{-iq_1(z_2-y_1)}\, e^{-iq_2(z_3-y_2)}\, e^{-ik_1(z_2-z_1)}\, e^{-ik_2(z_3-z_1)}\, e^{-ik_3(z_3-z_2)}$$

$$i\Delta^F_\phi(p)\, i\Delta^F_\chi(q_1)\, i\Delta^F_\chi(q_2)\, i\Delta^F_\chi(k_1)\, i\Delta^F_\chi(k_2)\, i\Delta^F_\phi(k_3)$$

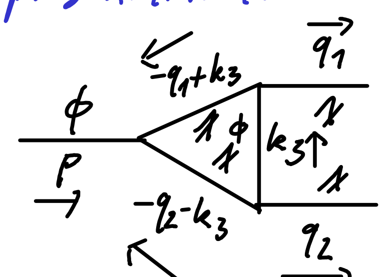
Integrations:
$d^4z_1 \to -p - k_1 - k_2 = 0$
$d^4z_2 \to q_1 + k_1 - k_3 = 0$
$d^4z_3 \to q_2 + k_2 + k_3 = 0$
$\rightsquigarrow k_1 = k_3 - q_1$
$k_2 = -k_3 - q_2$
$\rightsquigarrow -p + q_1 + q_2 = 0$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4}\, e^{-(ip\cdot x - q_1\cdot y_1 - q_2\cdot y_2)}\, (2\pi)^4 \delta^4(p - q_1 - q_2)$$

$$* \underbrace{\Big(ig^3 \int \frac{d^4k_3}{(2\pi)^3}\, i\Delta^F_\phi(p)\, i\Delta^F_\chi(q_1)\, i\Delta^F_\chi(q_2)\, i\Delta^F_\chi(-q_1+k_3)\, i\Delta^F_\chi(-q_2-k_3)\, i\Delta^F_\phi(k_3) \Big)}_{=: iG^{1\text{-loop}}_{\phi\chi\chi}(p, q_1, q_2)}$$

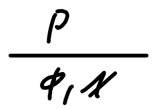
diagrammatic representation:

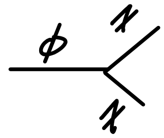
$$iG^{1\text{-loop}}_{\phi\chi\chi}(p, q_1, q_2) =$$



Again, the momentum-space representation of the one-loop correction to the three point function has a simple expression. We recognise propagators that connect vertices and momentum conservation in the vertices. By the choice of normalisation in the interaction Lagrangian term, the combinatorial factors cancel (in this example). Finally, a loop corresponds to one integral over four momentum.

The diagrammatic representations are called Feynman diagrams. Most commonly, we are interested in calculating n-point functions in momentum space. Instead of reiterating the tedious calculations from above, we can draw a Feynman diagram and calculate the n-point functions using the Feynman rules:

- propagator:  $\longrightarrow \dfrac{i}{p^2 - m^2_{\phi,\chi} + i\varepsilon}$

- vertex:  $\longrightarrow -ig$, impose 4-momentum conservation

- loop integrals: $\displaystyle\int \frac{d^4k}{(2\pi)^4}$

- divide by symmetry factors of the diagram (in case the combinatorics does not work out as simple as above).

Now, we discuss the second important quantisation scheme, the <u>functional</u> or the <u>path integral</u> approach. For calculational purposes, we add a current $J(x)$ to the Hamiltonian and consequently, to the Lagrangian as well:

$$\mathcal{H}(x) \longrightarrow \mathcal{H}(x) - J(x)\,\phi(x)$$
$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + J(x)\,\phi(x)$$

The <u>generating functional</u> is the vacuum-to-vacuum amplitude

$$Z[J] = \lim_{T\to\infty} \langle \Omega |\, T\, e^{-i\int_{-T}^{T} dt\, H(t)} \,|\Omega\rangle$$

This has a path integral representation:

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \; e^{i\int d^4x \{\mathcal{L}(x) + J(x)\phi(x)\}}$$

The normalisation $\mathcal{N}$ is chosen such that $Z[J(x)\equiv 0] = 1$.

Time-ordered n-point Green functions are computed as

$$\langle\Omega| T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\Bigg|_{J(x)=0}$$

$$= \mathcal{N}\int \mathcal{D}\phi \; \phi(x_1)\cdots\phi(x_n)\, e^{i\int d^4x \{\mathcal{L}(x) + J(x)\phi(x)\}}\Bigg|_{J(x)=0}$$

The generating functional is the Quantum Field Theoretical analogue of the partition function in statistical physics and hence has an intuitive meaning. For practical purposes, we reexpress the action as follows:

Consider first the free part and shift the field:

$$(-\partial_x^2 - m^2)\, i\Delta^F(x,y) = i\delta^4(x-y)$$

$$\phi'(x) = \phi(x) - i\int d^4y \; i\Delta^F(x,y)\, J(y)$$

$$\int d^4x\, \mathcal{L}_0 = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + J\phi\right]$$

$$= \int d^4x \left[\frac{1}{2}\phi(-\partial^2 - m^2)\phi + J\phi\right]$$

$$= \int d^4x \Big\{ \frac{1}{2}\phi'(x)(-\partial_x^2 - m^2)\phi'(x) + \cancel{J(x)\phi'(x)}$$

$$+ \cancel{i\int d^4y \; \underbrace{i\Delta^F(x,y)\, J(y)\, (-\partial_x^2 - m^2)}_{i\delta^4(x-y)}\, \phi'(x)}$$

$$- \frac{1}{2}\int d^4y \; i\Delta^F(x,y)\, J(y)\, \underbrace{(-\partial_x^2 - m^2)\int d^4z\; i\Delta^F(x,z)}_{i\delta^4(x-z)}\, J(z)$$

$$+ i J(x)\int d^4y \; i\Delta^F(x,y)\, J(y)\Big\}$$

$$= \int d^4x \, \frac{1}{2} \phi'(x) (-\partial_x^2 - m^2) \phi'(x) + \frac{i}{2} \int d^4x \int d^4y \, J(x) \, i\Delta^F(x,y) J(y)$$

→ The free action is now quadratic in ϕ'!

For illustrative purposes, now consider the interaction term

$$\mathcal{L}_{int} \supset -\frac{\lambda}{4!}, \quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}.$$

In terms of the shifted field, we express

$$\frac{Z[J]}{\mathcal{N}} = \int \mathcal{D}\phi' \, e^{i\int d^4x \left[\frac{1}{2}\phi'(-\partial^2 - m^2)\phi' - \frac{\lambda}{4!}\phi^4 \right] - \frac{1}{2}\int d^4x \, d^4y \, J(x) \, i\Delta^F(x,y) \, J(y)}$$

↖ no prime here!

Next, observe that

$$[i\phi(y)]^n \, e^{i\int d^4x \, (\mathcal{L}_0 + J(x)\phi(x))} = \frac{\delta^n}{\delta J^n(y)} \, e^{i\int d^4x \, (\mathcal{L}_0 + J(x)\phi(x))}$$

This can of course be applied to the individual terms in an exponential series. Dropping the prime in the path integral, we can therefore express

$$\frac{Z[J]}{\mathcal{N}} = e^{-i\int d^4x \frac{\lambda}{4!} \frac{\delta^4}{\delta J^4(x)}} \int \mathcal{D}\phi \, e^{i\int d^4x \left[\frac{1}{2}\phi(-\partial^2 - m^2)\phi + J(x)\phi(x) \right]}$$

$$= e^{-i\int d^4x \frac{\lambda}{4!} \frac{\delta^4}{\delta J^4(x)}} \underbrace{\int \mathcal{D}\phi \, e^{i\int d^4x \frac{1}{2}\phi(-\partial^2 - m^2)\phi - \frac{1}{2}\int d^4x \, d^4y \, J(x) \, i\Delta^F(x,y) J(y)}}_{=: \mathcal{N}^{-1} Z_0[J]}$$

Example: two-point function to NLO

In order to distinguish it from the tree-level (free) propagator Δ^F, we call the time-ordered Green function of the interacting theory G^F. It can be obtained by computing perturbative corrections to Δ^F. It should be clear how above manipulations can be adapted from $\frac{\lambda}{4!}\phi^4$ theory to the previously considered $\frac{g}{2}\phi\chi^2$.

$$i G^F_\phi(x_1,x_2) = \langle \Omega | T\, \phi_H(x_1)\, \phi_H(x_2) | \Omega \rangle$$

$$= -\frac{\delta^2}{\delta J_\phi(x_1)\,\delta J_\phi(x_2)}\, e^{-i\int d^4x\, \frac{g}{2} \frac{\delta}{\delta J_\phi(x)} \frac{\delta^2}{\delta J_\chi(x)}}\; Z_0[J_{\phi,\chi}]\Big|_{J_{\phi,\chi}=0}$$

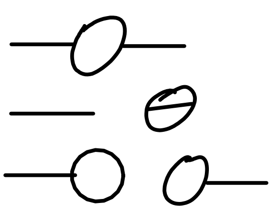
$$\simeq -\frac{\delta^2}{\delta J_\phi(x_1)\,\delta J_\phi(x_2)} \Big(1 - i\int d^4x\, \frac{g}{2} \frac{\delta}{\delta J_\phi(x)} \frac{\delta^2}{\delta J_\chi^2(x)}$$

$$- \frac{1}{2}\frac{g^2}{4} \int d^4x\, \frac{\delta}{\delta J_\phi(x)} \frac{\delta^2}{\delta J_\chi^2(x)} \int d^4y\, \frac{\delta}{\delta J_\phi(y)} \frac{\delta^2}{\delta J_\chi^2(y)} \Big)$$

$$* \mathcal{N}'\, e^{-\frac{1}{2}\int d^4x\, d^4y\, J_\phi(x)\, i\Delta^F_\phi(x,y)\, J_\phi(y) - \frac{1}{2}\int d^4x\, d^4y\, J_\chi(x)\, i\Delta^F_\chi(x,y)\, J_\chi(y)}\Big|_{J_{\phi,\chi}=0}$$

the part that does not depend on J just gives a factor

$$= \mathcal{N}'\, i\Delta^F_\phi(x_1,x_2) + \mathcal{N}' \frac{g^2}{8} \frac{\delta^2}{\delta J_\phi(x_1)\, J_\phi(x_2)} \int d^4x\, d^4y\, \frac{\delta}{\delta J_\phi(x)} \frac{\delta^2}{\delta J_\chi^2(x)} \frac{\delta}{\delta J_\phi(y)} \frac{\delta^2}{\delta J_\chi^2(y)}$$

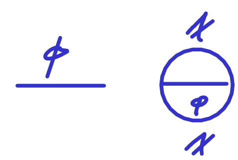


$$* \frac{1}{2!}\left(-\frac{1}{2}\int d^4v_1\, d^4v_2\, J_\phi(v_1)\, i\Delta^F_\phi(v_1,v_2)\, J_\phi(v_2) \right)^2$$

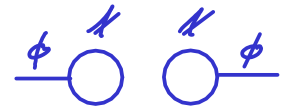
$$* \frac{1}{2!}\left(-\frac{1}{2}\int d^4w_1\, d^4w_2\, J_\chi(w_1)\, i\Delta^F_\chi(w_1,w_2)\, J_\phi(w_2) \right)^2$$

$$= \mathcal{N}'\, i\Delta^F_\phi(x_1,x_2)$$

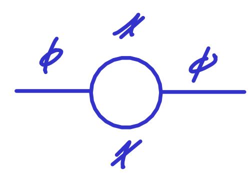
$$- \frac{g^2}{4} \mathcal{N}'\, i\Delta^F_\phi(x_1,x_2) \int d^4x\, d^4y\, i\Delta^F_\phi(x,y) \left(i\Delta^F_\chi(x,y) \right)^2$$



$$- g^2 \mathcal{N}' \int d^4x\, i\Delta^F_\phi(x_1,x)\, i\Delta^F_\chi(x,x) \int d^4y\, i\Delta^F_\phi(x_2,y)\, i\Delta^F_\chi(y,y)$$

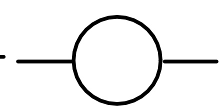


$$- \frac{g^2}{2} \int d^4x\, d^4y\, i\Delta^F_\phi(x_1,x) \left(i\Delta^F_\chi(x,y) \right)^2 i\Delta^F_\phi(y,x_2)$$



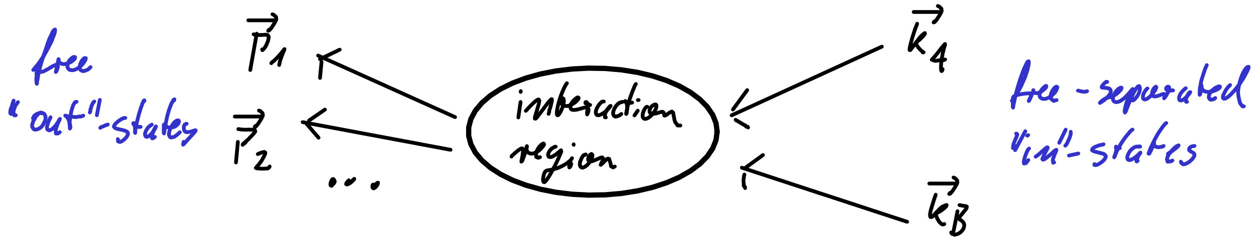
The diagrammatic representations of each term are indicated. Instead of a derivation from the canonical or the functional formalism, of course, one may use Feynman rules.
The phenomenological-minded should however appreciate the origin of the Feynman rules, while the theoretical-minded should be able to derive such rules by either of above methods.

In order to comment on the particular terms, recall that $\mathcal{N}$ is chosen such that $Z[J=0]=1$. Therefore, $\mathcal{N}^{-1}$ is the sum of all vacuum graphs, such that the second term (tree level propagator times a vacuum graph) is cancelled.
The third term is a disconnected diagram that is not cancelled (recall similar disconnected contributions in the 3-point function). The fourth and last term is what we are interested in an is called a self-energy correction.
Without proof, we remark that $\log Z[J_{\phi,x}]$ is a generating functional of connected diagrams, i.e.

$$\frac{\delta^2}{\delta J_\phi(x_1)\,\delta J_\phi(x_2)} \log Z[J_{\phi,x}] =$$ $+ \frac{1}{2}$  $+ \ldots$

1.4 Scattering Processes

Reactions of highly energetic particles in colliders, in astrophysical or cosmological contexts can often be described as scatterings



The probability of a scattering event is described by a unitary operator, the <u>S-matrix</u>:

$$_{out}\langle \vec{p}_1 \vec{p}_2 \cdots | \vec{k}_A \vec{k}_B \rangle_{in} = \langle \underbrace{\vec{p}_1 \vec{p}_2 \cdots}_{f} | S | \underbrace{\vec{k}_A \vec{k}_B}_{i} \rangle$$

i.e. the probability that the initial states i scatter to the final states f is given by

$$P_{fi} = S^{+}_{if} S_{fi}$$

The unitarity guarantees probability conservation in the form

$$\sum_f S^{+}_{if} S_{fk} = \delta_{ik}$$

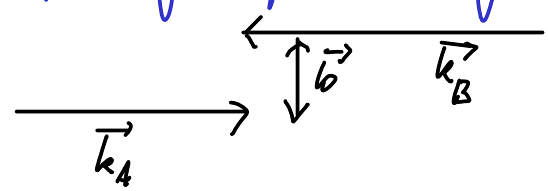
In order to connect this to observables such as cross sections and decay rates, it is useful to define the <u>T-matrix</u>

$$S = \mathbb{1} + i\, T$$

and the <u>invariant matrix-element</u>

$$\langle \vec{p}_1 \vec{p}_2 \cdots | iT | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \, \delta^{(4)}\Big(k_A + k_B - \sum_i p_i\Big) \, i\mathcal{M}(\vec{k}_A, \vec{k}_B \to \vec{p}_1 \vec{p}_2 \dots)$$

Notice that we also encountered a momentum-conserving δ-function when computing n-point functions.
It is useful (and it reflects the situation in collider experiments) to work in the <u>collinear frame</u>, where $\vec{k}_A \parallel \vec{k}_B$.
Nonetheless, the collisions are in general not head-on, but they are <u>spatially</u> separated by the <u>impact parameter</u> $\vec{b}$:



An intuitive definition of the <u>cross section</u> is

$$\sigma = \int \underset{\substack{\uparrow \\ \text{impact parameter}}}{d^2b} \; \underset{\substack{\uparrow \\ \text{probability of a scattering of particles at impact parameter } \vec{b}}}{P(\vec{b})}$$

, i.e. it is the effective area either of the particles "sees" in the particular interaction.

This can be shown to be equivalent to the more common definition of σ as the ratio of the transition rate over the flux. Next, we need to find an expression for $P(\vec{b})$.
For that purpose, we note that the in states of the particles A and B are not momentum eigenstates, but rather wave-packets centered around certain momenta $\vec{p}$.

$$|\phi\rangle = \int \underbrace{\frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{k})}}}_{\text{rel. invariant integration measure}} \, \psi(\vec{k}) \, |\vec{k}\rangle$$

ψ ↳ function, that describes the momentum-distribution of the wave-packet, centered around $\vec{p}$

The proper normalisation follows as

$$\langle\phi|\phi\rangle = \int \frac{d^3k\, d^3k'}{(2\pi)^6} \frac{\Psi^*(\vec{k}')\,\Psi(\vec{k})}{2\sqrt{\omega(\vec{k})\,\omega(\vec{k}')}} \langle 0|\sqrt{2\omega(\vec{k}')}\, a(\vec{k}')\sqrt{2\omega(\vec{k})}\, a^\dagger(\vec{k})|0\rangle$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^6} \Psi^*(\vec{k}')\,\Psi(\vec{k})\,(2\pi)^3\delta^3(\vec{k}-\vec{k}') = \int \frac{d^3k}{(2\pi)^3}|\Psi(\vec{k})|^2 \overset{!}{=} 1$$

The probability that we are after is:

$$P = \left|{}_{out}\langle \vec{p}_1\,\vec{p}_2 \cdots|\phi_A\,\phi_B\rangle_{in}\right|^2$$

Making use of translation invariance, we write

$$|\phi_A\,\phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3}\frac{d^3k_B}{(2\pi)^3}\frac{\Psi_A(\vec{k}_A)\,\Psi_B(\vec{k}_B)\, e^{-i\vec{b}\cdot\vec{k}_B}}{2\sqrt{\omega(\vec{k}_A)\,\omega(\vec{k}_B)}}|\vec{k}_A\,\vec{k}_B\rangle_{in}$$

i.e. the function Ψ_B in the transversal direction ($\parallel \vec{b}$) is the Fourier transform of the spatial distribution of the particle B orthogonal to the collision axis (which we take to be the z-axis, for definiteness.

Putting everything together, we obtain for the differential cross-section

$$d\sigma = \prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\sqrt{\omega(\vec{p}_f)}}\int d^2b\left(\prod_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\Psi_i(\vec{k}_i)}{\sqrt{2\omega(\vec{k}_i)}}\int\frac{d^3k_i'}{(2\pi)^3}\frac{\Psi_i^*(\vec{k}_i')}{\sqrt{2\omega(\vec{k}_i')}}\right)$$

$$* e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}\left({}_{out}\langle\{p_f\}|\vec{k}_A\vec{k}_B\rangle_{in}\right)\left({}_{out}\langle\{p_f\}|\vec{k}_A'\vec{k}_B'\rangle_{in}\right)^*$$

$$= \prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\sqrt{\omega(\vec{p}_f)}}\left(\prod_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\Psi_i(\vec{k}_i)}{\sqrt{2\omega(\vec{k}_i)}}\int\frac{d^3k_i'}{(2\pi)^3}\frac{\Psi_i^*(\vec{k}_i')}{\sqrt{2\omega(\vec{k}_i')}}\right)$$

$$*(2\pi)^2\delta^2(\vec{k}_B^{\perp}-\vec{k}_B'^{\perp})\,(2\pi)^4\delta^4\left(\sum_f p_f - k_A - k_B\right)(2\pi)^4\delta^4\left(\sum_f p_f - k_A' - k_B'\right)$$

$$* i\mathcal{M}(\vec{k}_A\vec{k}_B \to \vec{p}_1\vec{p}_2\cdots)\,(-i)\mathcal{M}^*(\vec{k}_A'\vec{k}_B' \to \vec{p}_1\vec{p}_2\cdots)$$

$$= \prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2\sqrt{\omega(\vec{p}_f)}} \left(\prod_{i=A,B} \int \frac{d^3k_i}{(2\pi)^3} \frac{\Psi_i(\vec{k}_i)}{\sqrt{2\omega(\vec{k}_i)}} \int \frac{dk_i'^z}{2\pi} \frac{\Psi_i^*(\vec{k}_i')}{\sqrt{2\omega(\vec{k}_i')}} \right)$$

$\uparrow$
$\vec{k}_B'^{\perp} = \vec{k}_B^{\perp}$
$\vec{k}_A'^{\perp} = \sum_f \vec{p}_f^{\perp} - \vec{k}_B'^{\perp}$

$$* (2\pi)^4 \delta^4\left(\sum_f p_f - k_A' - k_B'\right) 2\pi\, \delta\left(\sum_f \omega_f - \omega_A' - \omega_B'\right)$$

$$* 2\pi\, \delta\left(\sum_f p_f^z - k_A^z - k_B^z\right)$$

$$* i\mathcal{M}(\vec{k}_A \vec{k}_B \to \vec{p}_1 \vec{p}_2 \dots)\, (-i)\mathcal{M}^*(\vec{k}_A' \vec{k}_B' \to \vec{p}_1 \vec{p}_2 \dots)$$

Now,

$$\int \frac{dk_A'^z\, dk_B'^z}{(2\pi)^2}\, 2\pi\, \delta\left(\sum_f \omega_f - \omega_A' - \omega_B'\right) 2\pi\, \delta\left(\sum_f p_f^z - k_A^z - k_B^z\right)$$

$$= \int \frac{dk_A'^z}{2\pi}\, 2\pi\, \delta\left(\sqrt{\vec{k}_A'^2 + m_A^2} + \sqrt{\vec{k}_B'^2 + m_B^2} - \sum_f \omega_f\right)$$

$\uparrow$
$k_B^z = \sum_f p_f^z - k_A^z$

$$= \frac{1}{\left| \frac{k_A'^z}{\omega_A} - \frac{k_B'^z}{\omega_B} \right|} = \frac{1}{|v_A - v_B|}$$

Furthermore, recall that we assume that Ψ_A and Ψ_B are narrowly centered around $\vec{p}_A$ and $\vec{p}_B$. It is therefore good enough to evaluate $\mathcal{M}$ and v_i for these momenta. We obtain the result

$$d\sigma = \frac{(2\pi)^4 \delta^4(\sum_f p_f - k_A - k_B)}{2\omega_A 2\omega_B |v_A - v_B|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2\omega_f} \right) |\mathcal{M}(\vec{k}_A \vec{k}_B \to \vec{p}_1 \vec{p}_2 \dots)|^2$$

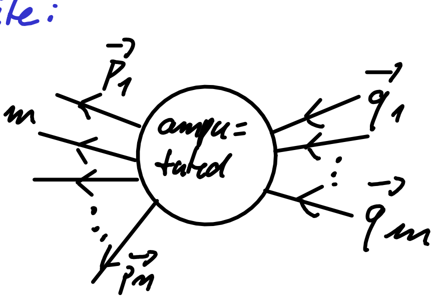
A similar derivation can be performed for the case of the <u>decay rate</u> Γ of a single decaying particle

$$d\Gamma = \frac{1}{2\omega_A} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2\omega_f} \right) |\mathcal{M}(\vec{p}_A \to \vec{p}_1 \vec{p}_2 \dots)|^2 (2\pi)^4 \delta^4\left(p_A - \sum_f p_f\right)$$

The remaining task is to determine the relation between the S-matrix elements and the time-ordered Green functions, that we have considered earlier and that are intuitively related to scattering processes due to their representation by Feynman diagrams.
The desired relation is given by the LSZ (Lehmann–Symanzik-Zimmermann) reduction formula:

$$\prod_{i=1}^{n}\int d^4x_i\, e^{ip_i\cdot x_i}\prod_{j=1}^{m} d^4y_j\, e^{-iq_j\cdot y_j}\langle\Omega|T[\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)]|\Omega\rangle$$
$$=\left(\prod_{i=1}^{n}\frac{\sqrt{Z}\,i}{p_i^2-m^2+i\varepsilon}\right)\left(\frac{\sqrt{Z}\,i}{q_j^2-m^2+i\varepsilon}\right)\langle\vec{p}_1\cdots\vec{p}_n|S|\vec{q}_1\cdots\vec{q}_m\rangle$$

We now recall that in the perturbative expressions for the n-point functions, the external legs are represented by propagators. These are now "amputated" by the factors $\frac{1}{\sqrt{Z}}(q_i^2-m^2+i\varepsilon)$.
$\sqrt{Z}$ is a "field strength renormalisation" which we will calculate explicitly at a later point.
Pictorially, we can therefore write:

$$\langle\vec{p}_1\cdots\vec{p}_n|S|\vec{q}_1\cdots\vec{q}_m\rangle=(\sqrt{Z})^{n+m}$$


This cartoon version of the LSZ formula applies to all particles (in particular, to spin $\frac{1}{2}$-fermions and to gauge bosons as well).
For the proof of the scalar case, we first note that it is possible to "invert" the field operator as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \left(a(\vec{p}) e^{-ip\cdot x} + a^\dagger(\vec{p}) e^{ip\cdot x} \right)\Big|_{p^0 = \omega(\vec{p})}$$

$\Longrightarrow$

$$A \overleftrightarrow{\partial} B = A \partial B - (\partial A) B$$

$$a(\vec{k}) = i \int d^3x \frac{e^{ik\cdot x}}{\sqrt{2\omega(\vec{k})}} \overleftrightarrow{\partial_0} \phi(x)$$

$$a^\dagger(\vec{k}) = -i \int d^3x \frac{e^{-ik\cdot x}}{\sqrt{2\omega(\vec{k})}} \overleftrightarrow{\partial_0} \phi(x)$$

The S-matrix mediates between free asymptotic in and out states. We assume that the free and the interacting fields are related by the field strength renormalisation Z as

$$\sqrt{Z} \langle f | \phi_{out}(x) | i \rangle \xleftarrow[x^0 \to \infty]{} \langle f | \phi(x) | i \rangle \xrightarrow[x^0 \to -\infty]{} \sqrt{Z} \langle f | \phi_{in}(x) | i \rangle$$

So we can express the S-matrix element

$${}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \vec{q}_1 \dots \vec{q}_m \rangle_{in} = \sqrt{2\omega(\vec{q}_1)}\; {}_{out}\langle \vec{p}_1 \dots \vec{p}_n | a^\dagger_{in}(\vec{q}_1) | \vec{q}_2 \dots \vec{q}_m \rangle_{in}$$

$$= -i \int d^3x\, e^{-iq_1\cdot x} \overleftrightarrow{\partial_0}\; {}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \phi(x)_{in} | \vec{q}_2 \dots \vec{q}_m \rangle_{in}$$

In order to obtain a four-dimensional integral, as in the LSZ formula, use the identity:

$$\left(\lim_{t\to\infty} - \lim_{t\to -\infty} \right) \int d^3x\, A(\vec{x}, t) = \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} \int d^3x\, A(\vec{x}, t)$$

Add & subtract the same term at $t \to \infty$

$${}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \vec{q}_1 \dots \vec{q}_n \rangle_{in} = \frac{i}{\sqrt{Z}} \int d^4x\, \partial_0 \left[e^{-iq_1\cdot x} \overleftrightarrow{\partial_0}\; {}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \phi(x) | \vec{q}_2 \dots \vec{q}_m \rangle_{in} \right]$$
$$- \frac{i}{\sqrt{Z}} \lim_{t\to\infty} \int d^3x\, e^{-iq_1\cdot x} \overleftrightarrow{\partial_0}\; {}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \phi(x) | \vec{q}_2 \dots \vec{q}_m \rangle_{in}$$

Now, the second term, as $t \longrightarrow \infty$, can also be expressed as acting on an out state,

$$\sqrt{2\omega(\vec{q}_1)}\ {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | a^{+}_{out}(\vec{q}_1) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

$$= \sum_{i=1}^{n} 2\omega(\vec{q}_1)(2\pi)^3 \delta^3(\vec{p}_i - \vec{q}_1)\ {}_{out}\langle \vec{p}_1 \cdots \vec{p}_{i-1} \vec{p}_{i+1} \cdots \vec{p}_n | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

This corresponds to a disconnected graph, because the first of the "in"-particles is not affected by the scattering.
Back to the first term. Dropping the factor of $\frac{i}{\sqrt{Z}}$ and integrating by parts, we manipulate:

$$-\int d^4x \left(\partial_0^2 e^{-iq_1\cdot x}\right) {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

$$+\int d^4x\ e^{-iq_1\cdot x}\ \partial_0^2\ {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

$$= \int d^4x \left(\left[-\vec{\partial}^2 + m^2\right] e^{-iq_1\cdot x}\right) {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

$$+\int d^4x\ e^{-iq_1\cdot x}\ \partial_0^2\ {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

$$= \int d^4x\ e^{-iq_1\cdot x}\left(\partial^2 + m^2\right)\ {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

Provided the collection of in- & out-states is such that there are no disconnected contributions, we obtain

$${}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{q}_1 \cdots \vec{q}_m \rangle_{in} = \frac{i}{\sqrt{Z}} \int d^4x\ e^{-iq_1\cdot x}\left(\partial^2 + m^2\right) {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in}$$

We have "reduced" an S-matrix element to a matrix element of the operator ϕ. We can proceed to reduce out all external states iteratively. A point of crucial importance occurs when we next reduce out an out state:

$$ {}_{out}\langle \vec{p}_1 \cdots | \phi(x_1) | \vec{q}_2 \cdots \rangle_{in} = \sqrt{2\,\omega(\vec{p}_1)}\; {}_{out}\langle \vec{p}_2 \cdots | a_{out}(\vec{p}_1)\, \phi(x_1) | \vec{q}_2 \cdots \rangle_{in} $$

$$ = \lim_{y_1^0 \to \infty} \frac{i}{\sqrt{Z}} \int d^3 y_1\, e^{i p_1 \cdot y_1}\, \overleftrightarrow{\partial}_{y_1^0}\; {}_{out}\langle \vec{p}_2 \cdots | \phi(y_1)\, \phi(x_1) | \vec{q}_2 \cdots \rangle_{in} $$

At this stage, we may as well replace $\phi(y_1)\phi(x_1)$ by $T\phi(y_1)\phi(x_2)$, since the limit forces $y_1^0 > x_1^0$. For the subsequent manipulation however, we replace the 3-dimensional integral by a Lorentz-invariant 4-dimensional one and we want to express a term

$$ \frac{i}{\sqrt{Z}} \lim_{y_1^0 \to -\infty} \int d^3 y_1\, e^{i p_1 \cdot y_1}\, \overleftrightarrow{\partial}_{y_1^0}\, \phi(y_1) = \sqrt{2\omega(\vec{p}_1)}\; a_{in}(\vec{p}_1) $$

The time ordered product then makes sure that, since in this term, $y_1^0 \to -\infty$, $a_{in}(\vec{p}_1)$ acts directly to the right, such that we only obtain a vanishing or disconnected and irrelevant contribution. Therefore,

$$ {}_{out}\langle \vec{p}_1 \cdots | \phi(x_1) | \vec{q}_2 \cdots \rangle_{in} = $$

$$ = \frac{i}{\sqrt{Z}} \int d^4 y_1\, e^{i p_1 \cdot y_1} (\partial^2_{y_1} + m^2)\; {}_{out}\langle \vec{p}_2 \cdots \vec{p}_n | T\phi(y_1)\phi(x_1) | \vec{q}_2 \cdots \vec{q}_m \rangle_{in} $$

Completing the iterative procedure, we obtain

$$ {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{q}_1 \cdots \vec{q}_m \rangle_{in} = \text{disconnected graphs} $$

$$ + \left(\frac{i}{\sqrt{Z}}\right)^{n+m} \int d^4 y_1 \cdots d^4 x_m\, e^{i p_1 \cdot y_1} \cdots e^{-i q_m \cdot x_m} $$

$$ * (\partial^2_{y_1} + m^2) \cdots (\partial^2_{x_m} + m^2) \langle \Omega | T\phi(y_1) \cdots \phi(x_m) | \Omega \rangle $$

Upon integration by parts and division by $(p_1^2 - m^2) \cdots (q_m^2 - m^2) * \left(\frac{-i}{\sqrt{Z}}\right)^{n+m}$, we obtain the formula quoted above.

The S-matrix is therefore the residue of the poles of the external legs of the time-ordered Green functions. Another remarkable property is that we see directly that when we replace an incoming particle with momentum $\vec{q}$ with an outgoing one with momentum $-\vec{q}$, we obtain the same matrix element. (For charged particles, this reflection also includes charge conjugation). This observation is called crossing symmetry, that we diccuss in detail within the next chapter.

The short story of this chapter is that we can calculate cross sections and decay rates in terms of the invariant matrix element $i\mathcal{M}$. $i\mathcal{M}$ arises from the T-matrix by deleting the overall momentum-conserving δ-function. The T-matrix is obtained from the S-matrix by subtracting an identity matrix, corresponding to "forward scattering" without interactions. The S-matrix is obtained from time-ordered Green functions by amputating the external legs. The Green functions can effectively be calculated using Feynman rules that can be derived using either the canonical or the functional formalism.
In the following chapters, we simply quote the Feynman rules for Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD) and the Weak Interactions. From the discussion in the present Chapter, these rules should be plausible and it should be clear how to derive these.