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Mechanik (Theoretische Physik 1)

Sommersemester 2018

Abgabe bis Freitag, 29.06.18, 12:00 neben PH 3218.
Übungsblatt Nr. 12 Dieses Blatt wird in den Übungen vom 2.07. - 6.07.18 besprochen.

Aufgabe 1: Noether charge with transformations of time — 5 Punkte

As seen in the lectures, if the action is invariant under an infinitesimal transformation of the generalized coordinates $q_i(t) \to q'_i(t) = q_i(t) + \epsilon\delta Q_i[q_i(t)]$, such that

$$dt\left(\mathcal{L}\left[t, q'_i[t], \frac{dq'_i}{dt}[t]\right] - \epsilon\frac{dR}{dt}\right) = dt\mathcal{L}\left[t, q_i[t], \frac{dq_i}{dt}[t]\right], \tag{1}$$

then there is a conserved Noether charge

$$Q = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta Q_i - R. \tag{2}$$

Consider now transformations of both the generalized coordinates and the time variable,

$$\begin{aligned} q_i(t) &\to q'_i(t') = q_i(t) + \epsilon\delta Q[q_i(t)], \\ t &\to t' = t + \epsilon\delta T(t), \end{aligned}$$

which are assumed to leave the action invariant.

1. Show that one has (up to higher order corrections in ϵ)

$$\begin{aligned} dt'\mathcal{L}\left[t', q'_i[t'], \frac{dq'_i}{dt'}[t']\right] &= dt\left(1+\epsilon\frac{d\delta T}{dt}\right)\mathcal{L}\left[t+\epsilon\delta T, q'_i[t'], \frac{dq'_i}{dt}[t'] - \epsilon\frac{d\delta T}{dt}\frac{dq_i}{dt}[t]\right] = \\ & dt\left(\mathcal{L}\left[t, q'_i[t'], \frac{dq'_i}{dt}[t']\right] - \epsilon\frac{d}{dt}\left[\left(\sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L}\right)\delta T\right]\right), \end{aligned}$$

which, according to equations (1) and (2), leads to the conservation of the charge

$$Q = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta Q_i - \left(\sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L}\right)\delta T.$$

[Hints. Use the identity

$$\frac{\partial\mathcal{L}}{\partial t} = \frac{d\mathcal{L}}{dt} - \sum_i\left(\dot{q}_i\frac{\partial\mathcal{L}}{\partial q_i} + \ddot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right),$$

as well as the Euler-Lagrange equations.]

2. Consider the Lagrangian of a particle in a central potential decaying as $1/r^2$:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - \frac{\alpha}{r^2}.$$

Show that the action is invariant under transformations

$$\begin{aligned} r[t] \rightarrow r'[t'] &= (1+\epsilon)r[t], \\ t \rightarrow t' &= (1+\epsilon)^2 t, \end{aligned}$$

and compute the associated Noether charge in terms of energy, radial coordiante and its time derivative.

Hint: Remember that when the time coordinate is affected by the transformation, the measure of integration that appears in the action should change as well, namely:

$$dt \rightarrow \left|\frac{dt}{dt'}\right| dt'$$

Aufgabe 2:
The Steiner theorem **5 Punkte**

Consider a frame K centered in the center of mass of a rigid body of mass M and fixed to the body. In this frame, the tensor of inertia of the rigid body is $J_{\mu\nu}$. If we consider another frame K' which is parallel to K but shifted by a constant vector $\mathbf{a}$, the tensor of inertia in this frame will be $J'_{\mu\nu}$. Show that the two tensors are related by the so called Steiner theorem for parallel axes:

$$J'_{\mu\nu} = J_{\mu\nu} + M[|\mathbf{a}|^2\delta_{\mu\nu} - a_\mu a_\nu]$$

Keeping in mind the latter relation, complete the following tasks:

1. Compute the tensor of inertia of a sphere of radius R and homogeneous mass M in the frame of its center of mass.

2. Consider a system made of two spheres with equal mass $M/2$ and same radius R, welded together in their point of contact. Compute the tensor of inertia in centered in the latter.

3. Compute the tensor of inertia of a regular cone of height h, and base radius R in the frame centered in its center of mass. (The position of the center of mass must be calculated.)
   Hint1: In the frame centered in the vertex of the cone and with one axis along the symmetry axis, the tensor of inertia is already diagonal. Compute $I_i,\ i = 1, 2, 3$ in that frame and then use the Steiner theorem to shift them to the center of mass.
   Hint2: Cylindrical coordinates might help.

4. Consider a homogeneous cuboid of mass M and dimensions $a_1 = a_2$, a_3. Starting from frame K shown in figure, the tensor of inertia is diagonal. Compute it in that frame, and then compute it in the primed frame K' in which the x_3' axis is pointing in the direction of the main diagonal of the cuboid, as shown in figure.
   Hint: Use the fact that $a_1 = a_2$. This implies that the tensor has some symmetry, so that

from the frame K you can reach the tensor in the K' by rotating it with a matrix of the form:

$$R_\phi = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$

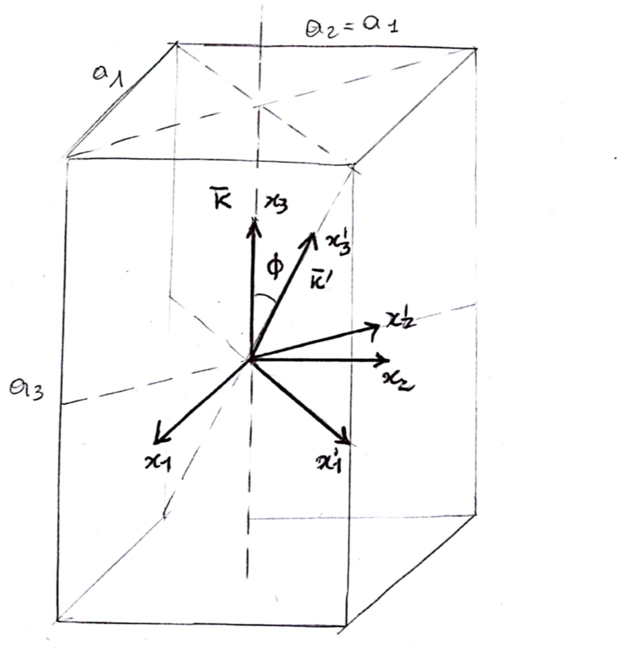

Abbildung 1: The two frames K and K' mentioned above.