Technische Universität München
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Mechanik (Theoretische Physik 1)

Sommersemester 2018

Abgabe bis Freitag, 22.06.18, 12:00 neben PH 3218.

Übungsblatt Nr. 10 Dieses Blatt wird in den Übungen vom 25.06. - 29.06.18 besprochen.

Aufgabe 1:
Natural boundary conditions **3 Punkte**

Consider the functional

$$I = \int_{x_1}^{x_2} F\left(y(x), y'(x), x\right) dx.$$

As you know from the lectures, under the assumption that variations δy vanish on the boundaries x_1 and x_2, the functions F which extremise I have to comply with the Euler-Lagrange equations. Show that vanishing variations at the boundaries are not the only option, and the functional I can be extremised by asking that the integrand satisfies the Euler-Lagrange and that the *generalized momentum* $\partial F/\partial y'$ vanishes on the boundary surface.
Study then the following two examples:

1. Starting from the action of a free falling point-like particle of mass m in a constant gravitational field $\mathbf{g}$ pointing towards negative x, obtain its trajectory with the single boundary condition
$$x(t_1 = 0) = 0,$$
i.e. without constraining $x(t)$ at some final time.
Prove that the solution actually minimizes the action.

2. Find the function $y(x)$ that extremizes the functional
$$I = \int_0^1 dx \left[y'^2 + yy' + y + y'\right]$$
and satisfies the initial condition $y(0) = \frac{1}{2}$.

Aufgabe 2:
Saddle Points **3 Punkte**

Consider a harmonic oscillator described by the action:

$$S[x(t)] = \int_0^{t_2} dt \left[\frac{m}{2}\dot{x}^2 - \frac{D}{2}x^2\right]$$

Show that the trajectory $x(t) = a\sin(\omega t)$ with $\omega = \sqrt{\frac{D}{m}}$ is a saddle point of the action if the final time t_2 is larger than half of the oscillation period T of the oscillator.

Hint: Consider a perturbation of the trajectory, $x(t) \to \hat{x}(t) = x(t) + \epsilon\eta(t)$, and study the sign of the ensuing change ΔS of the action. In order to facilitate this, parametrize the deviation η as follows:

$$\eta(t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{t_2}t\right) \quad , \quad b_k = \text{const} \quad \forall k \geq 1.$$

Aufgabe 3:
Kepler meets Lagrange **4 Punkte**

Consider the motion of a particle in a plane, under the influence of a central potential

$$V(r) = -\frac{\alpha}{r}.$$

1. Use the Euler-Lagrange equations to prove the conservation of angular momentum $l = mr^2\dot{\phi}$ and energy

$$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r).$$

2. Use the above conservation laws to derive a differential equation for $d\phi/dr$, and solve it to show that the trajectories are ellipses of the form

$$\frac{1}{r} = \frac{m\alpha}{l^2}(1 + \epsilon\cos(\phi - \phi_0)).$$

Hint:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-a}}\text{arcos}\frac{2ax + b}{\sqrt{b^2 - 4ac}} + C, \quad a < 0, b^2 - 4ac > 0.$$