Prof: Björn Garbrecht Technische Universität München
TA: Juan S. Cruz* December 13, 2019

Group Theory in Physics WS 2019/2020
Exercise Sheet 8

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

Spherical Harmonics and $SO(3)$

1. Starting from the definition of angular momentum as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ build a functional representation of the $\mathfrak{so}(3)$ algebra with representation space $L^2(S^2)$ that is square-integrable complex-valued functions on the sphere, in other words write down differential operators that implement the commutator relations of the $\mathfrak{so}(3)$ algebra.

2. Build the spherical basis by defining $\mathbf{L}$, L_z and $L_\pm$ so that we have ladder operators, compute the commutators.

3. Find an explicit basis of orthogonal functions, $\{\psi_m^\ell(\theta, \phi)\}$, for the operators $\mathbf{L}^2$, L_z, by solving first the differential equation for the eigenfunctions of L_z and the eigenfunctions of $\mathbf{L}^2$ (Just recognize the differential equation after using spherical coordinates).

4. Using the basis found in the previous item compute the matrix elements for the different representation matrices $D^\ell(\lambda_i)$, defined by

$$P(\lambda_i)\psi_m^\ell(\theta, \phi) = \sum_{m'=-\ell}^{\ell} D^\ell(\lambda_i)_{m'm}\psi_{m'}^\ell(\theta, \phi) \tag{1}$$

   and where $P(\lambda_i)$ are the operators from item 1). Find also the matrices corresponding to the basis built in 2).

5. Let V_ℓ the vector subspace of $L^2(S^2)$ spanned by $\{\psi_m^\ell\}$. Given the constraints coming from the equations above, find the dimensions of the V_ℓ. Is there a subspace of dimension 2?

6. Exponentiating the representations that we have, $D^\ell(\lambda_i)$, obtain the matrix corresponding to a finite rotation around an axis $\hat{n}$ and an angle α for the case of $\ell = 1$. Compare with the results from the previous sheet.

7. Find a relation between Clebsch-Gordan coefficients and the $\{\psi_m^\ell\}$.

*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de