Prof: Björn Garbrecht Technische Universität München
TA: Juan S. Cruz* December 6, 2019

Group Theory in Physics WS 2019/2020
Exercise Sheet 7

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

1 The groups $SO(3)$, $SO(4)$ and $SO(3,1)$ and their algebras

Recall the definition of the orthogonal group, $O(p,q)$, as the set of linear transformations that leave the standard quadratic form,

$$q(\mathbf{x}) = g_{ij}x^i x^j \tag{1}$$

where the summation over is implied, they run over $1, 2, \ldots, p+q$ and

$$g_{ij} = \begin{cases} \delta_{ij} & i,j \leq p \\ -\delta_{ij} & i,j > p \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

If additionally one asks for the transformations to have determinant 1, one obtains the group $SO(p,q)$. We will study a few specific examples that are relevant for particle physics.

a) Build *surjective*, i.e. they must cover the whole target, homomorphisms between

   i) $SU(2)$ and $SO(3)$

   ii) $SU(2) \times SU(2)$ and $SO(4)$

   iii) $SL(2,\mathbb{C})$ and $SO(3,1)$

   and prove they are homomorphisms.

b) Determine whether the homomorphisms found above are isomorphisms, if they are not, find their kernel.

The domains of the homomorphisms above are called the Spin groups of the target groups. They are simply connected double covers of the target groups, that is they are the universal covers and are relevant because of the projective nature of quantum mechanics. Let us consider the case of classical non-relativistic quantum mechanics.

c) Find an example of a path within $SO(3)$, that is a map $f : [a,b] \to SO(3)$, such that it cannot be continuously deformed to a constant path, that is after the deformation one should have $\tilde{f}(x) = \mathbb{1}$ for $x \in [a,b]$.

d) By identifying the elements of $SO(3)$ with a 3-ball with some identification of points, display examples of paths that can be deformed to be constant and paths that cannot.

*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de

2 Weyl Basis for Euclidean Spinors

Here we focus on a representation of the the Euclidean 4-rotations $SO(4)$, specifically through its algebra $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Recall the commutation relations:

$$[M_i, M_j] = \epsilon_{ijk} M_k, \qquad [M_i, N_j] = \epsilon_{ijk} N_k, \qquad [N_i, N_j] = \epsilon_{ijk} M_k, \tag{3}$$

in the previous homework we redefined our basis to explicitly split into two copies of $\mathfrak{su}(2)$ by defining:

$$J_i = \frac{1}{2}(M_i + N_i) \quad \text{and} \quad K_i = \frac{1}{2}(M_i - N_i) \tag{4}$$

which made the two copies of $\mathfrak{su}(2)$ evident. We will now deal with a specific example of physical relevance. Consider the free Euclidean Dirac operator for a massless fermion

$$(i\gamma^m \partial_m)\Psi = 0 \tag{5}$$

where the summation is implied and $m = 1, 2, 3, 4$ with a metric with signature $(+, +, +, +)$, so raising and lowering indices is immaterial for Euclidean space indices. The γ^m are an Euclidean version of the Dirac matrices:

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \tag{6}$$

with $\sigma = (i\vec{\tau}, \mathbb{1})$ and $\bar{\sigma} = (-i\vec{\tau}, \mathbb{1}) = \sigma^\dagger$, where $\vec{\tau}$ are the Pauli matrices.

a) Prove that the γ^m satisfy the Dirac algebra (Clifford algebra) relation, namely:

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} \tag{7}$$

b) Consider Ψ to be an vector of $\mathbb{R}^4$ described by two two component vectors ψ, χ and obtain the equations that each component should satisfy from (5).

In order to make contact with the group, we can define yet another basis, S_{mn} that contains all the information of the M_m and N_n above by:

$$S \equiv \begin{pmatrix} 0 & -M_3 & M_2 & N_1 \\ M_3 & 0 & -M_1 & N_2 \\ -M_2 & M_1 & 0 & N_3 \\ -N_1 & -N_2 & -N_3 & 0 \end{pmatrix} \tag{8}$$

c) Prove that the S_{mn} satisfies the $\mathfrak{so}(4)$ relation

$$[S_{mn}, S_{ij}] = S_{mi}\delta_{nj} + S_{nj}\delta_{mi} - S_{mj}\delta_{ni} - S_{ni}\delta_{mj} \tag{9}$$

d) Define $\Sigma^{mn} \equiv [\gamma^m, \gamma^n]/4$ and compute the commutation relations for Σ_{mn}.

By now you should be convinced that the $\mathfrak{so}(4)$ Lie algebra can be represented through two copies of Pauli matrices. We can write the components of Σ_{mn} in terms of $\sigma^{mn} \equiv (\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)/4$ and $\dot{\sigma}^{mn} = (\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^n)/4$ for brevity and in order to relate to the Left/Right traditional Minkowski spinors. It can also be proven that σ^{mn} and $\bar{\sigma}^{mn}$ satisfy the same commutation relations as S_{mn}. We can build representations for the 4-Euclidean rotations by exponentiating said generators (times $-i$ by convention):

$$R(\omega^{mn}) = \exp\left(-i\omega^{mn}\sigma^{mn}\right) \in GL(2) \tag{10}$$

$$\dot{R}(\omega^{mn}) = \exp\left(-i\omega^{mn}\dot{\sigma}^{mn}\right) \in GL(2) \tag{11}$$

e) Determine if the two representations constructed, one with σ^{mn} the other with $\dot{\sigma}^{mn}$ are equivalent and or faithful. Check quickly that indeed using $\exp(i\omega^{mn}\Sigma_{mn})$ as transformation rule for Ψ you get that each component is transforming according to the rules given in the previous item.