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# Group Theory in Physics WS 2019/2020
# Exercise Sheet 4

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

## 1 Theorem on Equivalence of Characters and Representations

Let $G$ be either a finite or a compact Lie group and prove the following statement. Two representations, $\Gamma$ and $\Gamma'$ for $G$ are equivalent if only if the character systems $\chi_\Gamma$ and $\chi_{\Gamma'}$ are the same.

## 2 Building the character table for $S_4$

Consider the permutation group of four elements. We will build the table of characters for $S_4$ for its irreducible representations. Following these steps:

a) Find out the conjugacy classes for $S_4$. First recall that for permutation groups $S_n$ the conjugacy classes are in 1-1 correspondence with the partitions of $n$. In other words there are as many conjugacy classes as there are ways to add up $n$ (with positive integers). For $n = 4$ one has

- $4 = 1 + 1 + 1 + 1$
- $4 = 1 + 1 + 2$
- $4 = 1 + 3$
- $4 = 4$
- $4 = 2 + 2$

Recall the number of irreducible representations (irreps.) is less than or equal to the number of conjugacy classes, so our table will have at most 5 irreps. Compute the classes and find the number of elements in each class, remember the total number of elements is 24.

b) Besides the trivial representation which sends all elements to 1. There exists another easy representation for permutation groups called *the alternating* representation, which sends an element $g \in S_4$ into

$$\operatorname{sgn}(g) = \begin{cases} 1 & \text{if } g \text{ is a permutation with an even number of transpositions} \\ -1 & \text{if } g \text{ is a permutation with an odd number of transpositions} \end{cases} \tag{1}$$

Compute the characters for the trivial and the alternating representations of $S_4$.

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c) Notice that the standard representation, where elements of $S_4$ act on vectors of four elements, is not irreducible by finding a subspace of $\mathbb{R}^4$ fixed by all elements of $S_4$. However one can see that the decomposition $\mathbb{R}^4 = T \oplus V$ of $\mathbb{R}^4$ into the fixed subspace $T$ and a complementary subspace, $V$, does produce an irrep. over $V$. Use the property $\chi_{\mathbb{R}^4} = \chi_T + \chi_V$ to compute the characters of another irrep. of $S_4$, namely that on $V$.

Check that adding the squares of the dimensions of the representation spaces that we have, do not sum up to 24. So we are still missing representations whose squares of their dimension adds up properly, i.e. such that the following formula is satisfied:

$$\sum_{W \text{ irreps. of } S_4} \dim^2(\chi_W) = 24. \tag{2}$$

Check that the result of $24 - \sum_W \dim^2 \chi$ (for $W$ irreps. found so far) is not a square therefore there must be more than one irrep. left to find. Since we cannot have more than 5 in total, we conclude at this point there are exactly 2 irreps. remaining.

d) Recall tensor representations satisfy $\chi_{V\otimes W} = \chi_V \cdot \chi_W$, for known representation spaces $V, W$. One can produce new representations by non-trivial *tensor* products.† Build a new representation of dimension 3, by taking the tensor product of the alternating representation and the representation on $V$ of item c). Compute its character and check that it is indeed irreducible.

e) Using all the information so far and the orthogonality properties of the characters, obtain the last representation for $S_4$.

†Do not confuse tensor product with direct product or direct sum, those produce reducible representations, although the symbol $\otimes$ is used interchangeably in the literature.