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Group Theory in Physics WS 2019/2020
Exercise Sheet 3

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

1 The Haar Measure

A measure μ is a function defined on a subset of the power set† of a set S, $\mathcal{P}(S)$, of subsets of S, into $\mathbb{R}^+$. So that for a given subset $A \in \mathcal{P}(S)$, $\mu(A)$ represents the size of A. An example of a measure is de Lebesgue measure, which allows one to define formally integration over $\mathbb{R}$. Our objective here is to generalize the idea of measure and study the case of $SU(N)$ in order to be able to integrate functions over $SU(N)$.

A measure by definition must fulfill certain properties. One of such properties is the invariance with respect to translation. It is useful to draw some intuition from the case of $\mathbb{R}$. *Measurable* subsets can be for example intervals (a, b), where the Lebesgue measure, λ, assigns $\lambda((a, b)) = |b - a|$, it is then intuitive to demand that $\lambda((a + c, b + c)) = \lambda((a, b)) = |b - a|$. In the above statement we require that subsets *translated* by means of the group operation (addition in the previous case) are assigned the same size.

Specializing for the case of a linear Lie group, G, let us take left multiplication as the group operation (the same can be done using right multiplication). We want to build a measure that is **left-invariant**, that is we want to define μ such that for a measurable subset $A \subset G$ and any element $g \in G$, $\mu(gA) = \mu(A)$, with $gA = \{ga \mid a \in A\}$ in analogy with the case of $\mathbb{R}$ discussed above.

1.1 The general Construction

Since a Lie group has a manifold structure one can define 1-forms‡ which can be thought of as dual entities to vector-fields. The Maurer-Cartan 1-Form is a Lie algebra-valued 1-form defined as:

$$\Theta(g) = g^{-1}\mathrm{d}g \quad \text{with} \quad \mathrm{d}g = \sum_{i,j=1}^{N} \frac{\partial g}{\partial x^{ij}} \mathrm{d}x^{ij} \tag{1}$$

for $g \in G$ and a parameterization of $g = g(x^{ij}) \in GL(N, \mathbb{R})$.

a) Prove that for $SU(N)$, the Maurer-Cartan 1-form is anti-hermitian.

b) Prove it is also traceless to conclude it is indeed Lie algebra-valued, by considering $g = \exp(iA)$ for some $A \in \mathfrak{g}$ the Lie algebra of G.

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†See *Borel set* for more details.

‡A mapping defined on $g \in G$, resulting in an element of the cotagent space at that point, T_gG which in its own is a linear mapping from the tangent space to $\mathbb{R}$.

c) Prove that Θ is indeed left-invariant when acted by the pullback of left multiplication, that is when the point at which the form is evaluated is multiplied by a fixed element from the left.

To be able to define integration one needs a form of top-level, which means a form of the order of the dimension of the group. If our group is $SU(N)$, its dimension is $N^2 - 1$ which requires us to find a left-invariant $N^2 - 1$-form. Define

$$d\mu(g) = \mathrm{Tr}(\underbrace{\Theta(g) \wedge \Theta(g) \wedge \cdots \wedge \Theta(g)}_{N^2-1-times}) \tag{2}$$

1.2 Some specific interesting specific cases

d) Compute Θ for $SU(2)$ (*Hint*: Use the usual parametrization given by the mapping of S^3 into $SU(2)$).

e) Compute $\mathrm{d}\mu(g)$ for $SU(2)$.

f) Repeat the construction for $SL(2, \mathbb{R})$, using the parametrization

$$U = \begin{pmatrix} \cosh \rho e^{it} & \sinh \rho e^{-i\phi} \\ \sinh \rho e^{i\phi} & \cosh \rho e^{-it} \end{pmatrix}. \tag{3}$$

1.3 Alternative method

Integration using the Haar measure coincides with usual integration on a manifold if the group has also a manifold structure. This means that one can arrive to a proportional measure by considering a group G as a submanifold of another space and using the metric to build the volume form.

g) Compute the volume form of S^3 in spherical coordinates, to do this, use

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta\cos\psi \\ z &= \sin\theta\sin\psi\cos\phi \\ w &= \sin\theta\sin\psi\sin\phi \end{aligned}$$

to build the metric components g_{ij} for $i, j \in \theta, \psi, \phi$ in this hyperspherical coordinates. A conventionally normalized[§] volume form is taken to be the square root of the determinant. Compare your result with e).

h) Compute the Haar measure $\mathrm{d}\mu$ for $SU(3)$ using this method.
Hint: An $SU(3)$ matrix can be written as the product SR, where R is a $SU(2)$ matrix extended by one row and column with a 1 in the diagonal and S is the following 5 parameter $SU(3)$ matrix

$$S = \begin{pmatrix} e^{i\alpha_{23}}\cos\theta & 0 & e^{i\alpha_{23}}\sin\theta \\ -e^{i\alpha_{22}}\sin\theta\sin\phi_2 & e^{-i\alpha_{21}-i\alpha_{23}}\cos\phi_2 & e^{i\alpha_{22}}\cos\theta\sin\phi_2 \\ -e^{i\alpha_{21}}\sin\theta\cos\phi_2 & -e^{-i\alpha_{22}-i\alpha_{23}}\sin\phi_2 & e^{i\alpha_{21}}\cos\theta\cos\phi_2 \end{pmatrix}. \tag{4}$$

[§] See tensor densities for more details.