Prof: Björn Garbrecht Technische Universität München
TA: Juan S. Cruz* October 24, 2019

Group Theory in Physics WS 2019/2020
Exercise Sheet 2

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

---

1 The subgroups of $\mathbb{Z}$

Consider the subgroups $n\mathbb{Z} = \{nm \,|\, m \in \mathbb{Z}\}^{\dagger}$ with $n \in \mathbb{N}^*$ (natural number without zero) of the group $\mathbb{Z}$ with *addition* as the group operation.

a) What are the cosets of $n\mathbb{Z}$ in $\mathbb{Z}$? Exhibit possible choices for representatives?

b) What are the elements of $\mathbb{Z}/n\mathbb{Z}$ (usually written as $\mathbb{Z}_n$)?

c) Can you find $\mathbb{Z}_n$ as a subgroup of $\mathbb{Z}$? Sketch a proof.

2 Cosets, factor groups and conjugacy classes

We will study the group with the following presentation:

$$\mathcal{G} = \langle \bar{e}, i, j, k \,|\, \bar{e}^2 = e, i^2 = j^2 = k^2 = ijk = \bar{e} \rangle \tag{1}$$

a) Find the order (size) of this group and build its multiplication table.

b) Determine all subgroups of $\mathcal{G}$. Which of these are invariant (normal, in mathematical terms)?

c) For the subgroups, $\mathcal{H}$, found to be invariant, find $\mathcal{G}/\mathcal{H}$. Is any of these factor groups isomorphic to any of the subgroups of $\mathcal{G}$?

d) Obtain the conjugacy classes of $\mathcal{G}$.

3 Invariant subgroups, factor groups and direct product

Let G be a group (not necessarily finite) and M and N be invariant subgroups of G.

a) Prove that the intersection set $M \cap N := \{x \,|\, x \in M \text{ and } x \in N\}$ is an invariant subgroup of G.

b) Prove that $G/(M \cap N)$ is isomorphic to some subgroup of $G/M \otimes G/N$. (Here $\otimes$ means direct product, don't confuse with the tensor product for vector spaces)

---

*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de

†Here nm means repeat the group operation n times, I.e. $nm = m + m + \cdots + m$ with n terms in the sum; recall there is formally no multiplication in the group.

4 Central series

Define the center of a group G to be the set $Z(G) = \{x \in G \,|\, \text{for all } g \in G \; xg = gx\}$.

a) Prove that $Z(G)$ is an invariant subgroup of G.

b) Consider the subgroup G of $SL(N)$ of $N \times N$ real upper triangular matrices with determinant equal to 1 together with matrix multiplication. Build a descending chain of proper subgroups G_i for $i \in \{1, 2, ..., n-1\}$ of G, such that $G_i \triangleleft G_{i+1}$, that is G_i is an invariant subgroup of G_{i+1} and such that the factor group G_{i+1}/G_i is a subgroup of $Z(G/G_i)$. This chain

$$\{e\} \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G \tag{2}$$

is called the central series of a group.

Hint: Consider generalizing the Heisenberg group to matrices of dimension $N \times N$.