Prof: Björn Garbrecht
TA: Juan S. Cruz*

Technische Universität München
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Group Theory in Physics WS 2019/2020
Exercise Sheet 12

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

1 Real, compact forms of simple complex Lie algebras

1. Prove that $\mathfrak{su}(2,\mathbb{C})$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.
2. Prove that $\mathfrak{su}(2,\mathbb{R})$ is not isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.
3. Prove that $\mathfrak{su}(3,\mathbb{R})$ is the compact real form of $\mathfrak{sl}(3,\mathbb{C})$
4. Let $\mathfrak{g}$ be a complex simple Lie Algebra. Given a Chevalley basis, that is, elements of a Cartan subalgebra $\mathcal{H} = \{H_i, i = 1, ..., r\}$ and corresponding ladder operators $E_i^{\pm}$ satisfying

$$[H_i, H_j] = 0, \tag{1}$$
$$[H_i, E_j^{\pm}] = \pm A_{ji} E_j^{\pm}, \tag{2}$$
$$[E_i^{+}, E_j^{-}] = \delta_{ij} H_j, \tag{3}$$
$$(\mathrm{ad}_{E_i^{\pm}})^{1-A^{ji}} E_j^{\pm} = 0, \tag{4}$$

prove that the **real** vector space spanned by them together with the bracket from $\mathfrak{g}$ form a **real** Lie algebra. Consider a new vector space spanned by real linear combinations of:

$$\{\mathrm{i}H_j \mid \text{such that } H_j \in \mathcal{H}\} \tag{5}$$

together with

$$\left\{ \frac{\mathrm{i}}{2}\sqrt{\langle\alpha,\alpha\rangle}(E^{\alpha} \pm E^{-\alpha}) \mid \alpha \in \Phi \right\} \tag{6}$$

where Φ is the set of roots of $\mathfrak{g}$, prove that this is also a real Lie Algebra.

Compute the Killing form in both cases, are these Lie algebras different? How? Which are their associated groups?

*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de