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Technische Universität München
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Group Theory in Physics WS 2019/2020
Exercise Sheet 11

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

1 Adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$ in "Weight" language

Let us denote by V the vector space spanned by $\{T^a = \frac{\lambda_a}{2}, a = 1, ..., 8\}$ where the λ_a's are real Gell-Mann matrices as used in Sheet 9. Recall the adjoint representation is defined by

$$\Gamma[X](Y) = ad_X(Y) = [X, Y] \quad \text{for } X, Y \in V = \mathfrak{sl}(3,\mathbb{C}) \tag{1}$$

which indeed takes an element from the algebra, X, and associates to it a linear transformation $\text{ad}_X \in GL(V)$. Weight spaces are defined in general as:

$$V_\omega = \{Y \in V \mid \text{for all } h \in H \subseteq \mathfrak{sl}(3,\mathbb{C}), \Gamma[h]Y = \omega(h)Y\} \tag{2}$$

where H is a Cartan subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ and ω is a linear functional of H.

a) Using the definition, find weight spaces for $\mathfrak{sl}(3,\mathbb{C})$ such that V is decomposed into

$$V = \bigoplus_\omega V_\omega \tag{3}$$

where the sum goes over some finite number of $\omega \in H^*$, (the dual of the Cartan subalgebra).

Hint: Most of the work has been done in Sheet 9, the language is just new and there is still some freedom in the choice of ω's.

b) As mentioned in the lecture, the linear functionals ω labeling the V_ω's are called the weights of a representation. What are the weights of the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$?

2 Fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$ in "Weight" language

The fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$ is obtained by taking $V = \mathbb{C}^3$ and acting by matrix multiplication. Repeat steps a) and b) from exercise 1 for this representation.

3 Anti-fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$ in "Weight" language

The anti-fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$ is obtained by taking $V = \mathbb{C}^3$ and acting by matrix multiplication using minus the hermitian conjugate, namely for $X \in \mathfrak{sl}(3,\mathbb{C})$

$$\Gamma[X]Y = -X^\dagger Y \tag{4}$$

Repeat steps a) and b) from exercise 1 for this representation.

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4 Fundamental $\otimes$ Anti-fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$ in "Weight" language

The anti-fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$ is obtained by taking $V = \mathbb{C}^3 \otimes \mathbb{C}^3$ and acting on each component according to the (anti-)fundamental representation, namely for $X \in \mathfrak{sl}(3,\mathbb{C})$

$$\Gamma[X](Y \otimes Z) = XY \otimes -X^\dagger Z \tag{5}$$

Repeat steps a) and b) from exercise 1 for this representation.
Is this representation reducible? How does it connect to physics?