Prof: Björn Garbrecht Technische Universität München
TA: Juan S. Cruz* October 9, 2019

Group Theory in Physics WS 2019/2020
Exercise Sheet 1

Problems will be discussed in the tutorial sessions every Friday at 2:00p.m. in the Minkowski Room

1 Group presentations and infinite groups

A *group presentation* (do not confuse with representation) is a way of specifying a group. It is comprised by generators, or symbols, from a set S by concatenating its elements†, and subject to a set of relations R and is normally denoted $\langle S|R\rangle$. For example the group $\mathbb{Z}_2$ consisting of $1, -1$ with the usual multiplication can be *presented* as:

$$\langle a|a^2 = e\rangle \equiv \langle a|a^2\rangle. \tag{1}$$

The multiplicative cyclic group of rotations by $e^{2\pi i/n}$ with $n \in \mathbb{Z}$, can be presented as:

$$\langle a|a^n = e\rangle \equiv \langle a|a^n\rangle, \tag{2}$$

where e denotes the identity, it is customary to just write the left hand side of the expressions for the relations, in case they are equal to the identity. Above examples have had one generating element, $S = \{a\}$, and are therefore automatically Abelian, i.e. the multiplication is commutative, but this does not need to be the case:

a) Write down the multiplication table of the following group:

$$\langle a, b|a^4, b^2, (ab)^2\rangle \tag{3}$$

b) Write down the multiplication table for the Dihedral group of degree 4, D_4, of symmetries of the square.

c) Compare the two tables, what is the relation between the group from item a) and D_4? Is one a subgroup of the other, is there no relation? What is the **order** (the size) of these groups?

d) Consider the following presentation

$$\langle x, y|x^3, y^3, (xy)^3\rangle \tag{4}$$

What is the order of this group? You can argue informally.

2 Representation of the permutation group

A group can be represented in matrix spaces. Consider the permutation group S_n consisting of all possible permutations of n elements, operating among each other through composition.

a) Build its trivial/fundamental representation over the space of n-tuples, or in other words find a representation of S_n on $GL(n)$.

b) Can you find a representation of D_4 within the one of S_4? What is the order of S_4?

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†One assumes inverse symbols exist within the set S

3 Symmetries of a Hamiltonian on a Ring

Consider the one-dimensional Hamiltonian for a spin-chain on a ring with interactions only through nearest-neighbor interaction, with no external field

$$H = -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1}, \tag{5}$$

where $\sigma_i = \pm 1$, and $N + 1 \equiv 1$,

a) Find the symmetries associated with this system, for the case of N particles.

b) Consider the same Hamiltonian for the case of an infinite two-dimensional regular lattice with N lattice points in each direction and with the same spacing in both directions together with periodic boundary conditions. That is the set of spins $\{\sigma_{ij} \,|\, 1 \leq i, j \leq N+1, \sigma_{N+i,j} = \sigma_{i,j} = \sigma_{i,N+j}\}$. What are the symmetries of the Hamiltonian for this case?