Theoretische Physik 2 (Elektrodynamik) WS 2018/2019
Technische Universität München
January 18, 2019

Exercise Sheet 11*

Deadline: Sheet to be turned in by Friday 25th of January 2019 by 12 pm in the mailbox next to PH3218.

Exercise 1:
Circular antenna **6 Points**

Let a thin circular wire of radius R lie on the plane $z = 0$, and let its center be at the origin. Furthermore, assume the current $I(t) = I_0 \sin(\omega t)$ flows through it.

(a) (0.5 P.) Write down the current density $\mathbf{j}(\mathbf{x}, t)$ for the limiting case of an infinitesimally thin wire by using the Dirac δ function.

(b) (0.5 P.) How large is the magnetic moment $\boldsymbol{\mu}$ of this antenna in the case of a constant current $I = I_0$?

(c) (1 P.) Let us consider now the time-dependent case $I(t) = I_0 \sin(\omega t)$. Determine the vector potential $\mathbf{A}(\mathbf{x}, t)$ in Lorenz gauge. Carry out the time integration, while the volume integral does not need to be evaluated for now.

(d) (1 P.) Assume that $c/\omega \gg R$ and consider the radiation field in the far zone $|\mathbf{x}| \gg c/\omega \gg R$. Evaluate now the remaining integrations for the vector potential to leading order in the inverse distance $1/|\mathbf{x}|$.

(e) (1 P.) Determine the field $\mathbf{B}(\mathbf{x}, t)$ in the above approximation to leading order.

(f) (1 P.) Express the result for $\mathbf{B}$ by using the magnetic moment $\boldsymbol{\mu}$ from part (b), i.e. show that $\mathbf{B}(\mathbf{x}, t) = k^2(\mathbf{x} \times \boldsymbol{\mu}) \times \mathbf{x} \sin(\omega t - k|\mathbf{x}|)/|\mathbf{x}|^3$, where $k = \omega/c$.

(g) (1 P.) Express the electric field $\mathbf{E}(\mathbf{x}, t)$ in terms of the magnetic moment $\boldsymbol{\mu}$ from item (b) up to leading order in $1/|\mathbf{x}|$. (You answer this question by using the result for $\mathbf{B}$ given in part (f), even without having obtained the solutions to (a)-(f).)

Exercise 2:
Dirac Monopole **4 Points**

In this problem we model a magnetic monopole. Assume there is a magnetic point charge located at the origin with strength g. Maxwell's equation then becomes:

$$\nabla \cdot \vec{B} = 4\pi g \delta^3(\vec{r}) \tag{1}$$

(a) Solve for $\vec{B}$ and find its flux over a sphere of radius R.

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(b) Assume there is a unique vector potential $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$ and use Gauß's law to prove that this contradicts the result in (a).

(c) Assume a slightly different geometric setting instead, where we have an infinitely thin and long cable, L, located at $x = 0, y = 0, z < 0$, carrying a current with magnitude g. Use the definition for the vector potential:

$$\vec{A}(\vec{x}) = \frac{g}{4\pi} \int_L \frac{\mathrm{d}\vec{l}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}, \tag{2}$$

to calculate $\vec{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\phi = \frac{g(1 - \cos\theta)}{4\pi r \sin\theta} = \left(\frac{g}{4\pi r}\right) \tan\frac{\theta}{2}. \tag{3}$$

(d) Verify that $\vec{B} = \nabla \times \vec{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta = \pi$ (I.e. almost the field we expected to describe in (a)).

(e) With the $\vec{B}$ determined in part (d), evaluate the total magnetic flux passing through the circular loop of radius $R \sin\theta$ shown in the figure 1. Consider $\theta < \pi/2$ and $\theta > \pi/2$ separately, but always calculate the upward flux.

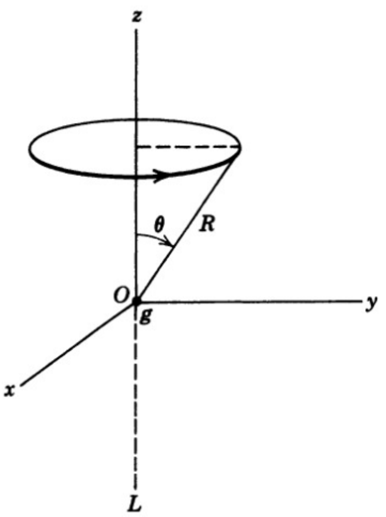

Figure 1: Loop for problem 2.

(f) From $\oint \vec{A} \cdot \mathrm{d}\vec{l}$ around the loop, determine the total magnetic flux through the loop. Compare the result with that found in part (e). Show that they are equal in the upper hemisphere, but have a constant difference at the lower hemisphere. Interpret this difference.