Theoretische Physik 2 (Elektrodynamik) WS 2018/2019
Technische Universität München
October 19, 2018

Introduction Problems
Harmonic oscillator Revisited*

In this sheet we study the harmonic oscillator equation and present different methods commonly used to solve this type of problems.

1 Direct solution

The harmonic oscillator differential equation in 1-dimension, subject to a friction term and an applied force reads:

$$m\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+2\gamma\frac{\mathrm{d}}{\mathrm{d}t}+\omega_0^2\right)x(t)=F(t), \tag{1}$$

where x can be thought of as position, t as time, γ as the friction strength, ω_0 as the natural frequency of the oscillator and $F(t)$ as some external force which can be dependent on time.

- Solve the equation (1) for the case where $F(t) = I\delta(t)$ directly by using exponential Ansätze and implementing matching conditions appropriately.
- Study the cases $\gamma > \omega_0$,$\gamma = \omega_0$ and $\gamma > \omega_0$.
- Impose homogeneous boundary conditions, namely $x(0) = 0$ and $\lim_{t\to\infty} x(t) = 0$.

2 Fourier Transform

Solve the same problem as in the previous section by using the Fourier transform.

3 Green's Functions

Using the solutions from previous sections, write down a Green's function for each case studied. That is, find $G(t, t')$ such that

$$m\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+2\gamma\frac{\mathrm{d}}{\mathrm{d}t}+\omega_0^2\right)G(t,t')=\delta(t-t'). \tag{2}$$

3.1 Examples

i) **Harmonic external force**, study again the harmonic oscillator problem this time with $F(t) = F\cos(kt)$, that is: some force which is applied externally and harmonically. Find the particular solution to this case, subject to the same homogeneous boundary conditions for the weakly-damped case ($\gamma < \omega_0$) by using a Green's function.

ii) Study the case where the external force is linear and is applied for a finite period of time (T_i, T_f), that is $F(t) = Ft\Theta(T_f - t)\Theta(t - T_i)$, again for the weakly-damped case $\gamma < \omega_0$.

*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de

4 Review on Complex Analysis

4.1 Definitions

We review in this section some of the definitions coming from complex analysis that are needed to be able to use the residue theorem.

Def: A function $f : \mathbb{C} \to \mathbb{C}$ is *analytic* on a domain D if it is complex differentiable at all points within D. (I.e. for $x, y \in \mathbb{R}$ it follows $\frac{\partial f(x+iy)}{\partial x} = -i\frac{\partial f(x+iy)}{\partial y}$.)

Def: A point, $z_0 \in \mathbb{C}$ is called an isolated singular point iff $f(z) : \mathbb{C} \to \mathbb{C}$ fails to be analytic at that point but there exists a neighborhood U of z_0 such that $f(z)$ is analytic for all $z \in U \backslash z_0$.

Def: A singular point, z_0, of a complex variable function $f(z)$ is called a pole of order m iff $f(z)$ can be written in the form.

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}, \tag{3}$$

where $\phi(z)$ is analytic and non-zero at z_0.

Residue: Given $f(z)$ a complex function with a singular point z_0, and C any closed contour enclosing z_0 and no other singular point, we call the quantity:

$$\operatorname*{Res}_{z=z_0}(f) = \frac{1}{2\pi i} \oint_C f(z)\mathrm{d}z \tag{4}$$

the residue of f at z_0.

4.2 Residue formulas

The definition above is however not very useful for the computation of residues in practice, therefore we state some commonly used formulas. Assume $f(z)$ has a singular point z_0

- It z_0 is a pole of order m, then

$$\operatorname*{Res}_{z=z_0}(f) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) \tag{5}$$

- Alternatively for order 1, one can expand $f(z)$ in a Laurent series around z_0 and read-off the residue directly from coefficient of the term proportional to z^{-1}.
- If the singular point is at infinity:

$$\operatorname*{Res}_{z=\infty}(f) = \operatorname*{Res}_{z=0}\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) \tag{6}$$

4.3 Cauchy's Residue Theorem

Theorem: Let C be a simple closed path which is positively oriented, if a function is analytic inside C except at a finite number of singular points z_k, $(k = 1, ..., n)$, then

$$\int_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k}(f(z)) \tag{7}$$

4.4 Jordan's Lemma

Theorem: Assume that:

i.) a function $f(z)$ is analytic at all points z of the upper-half plane that satisfy $|z| > R_0$, for some $R_0 > 0$.

ii.) Let C_R denote the semicircular path on the upper-half of the plane, parametrized by $z = Re^{i\theta}$, with $0 \geq \theta \geq \pi$ and $R > R_0$.

iii.) There exists a positive constant M_R such that for all points $z \in C_R$, $|f(z)| \geq M_R$ and $\lim_{R\to\infty} M_R = 0$.

Then, for any $a \in \mathbb{R}$,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\mathrm{d}z = 0. \tag{8}$$